

Errors in 2D beam emittance measurement in FODO channel

Purpose: minimization of errors in beam emittance measurements through beam profile measurements

1. Intro

4D beam is described by $\vec{\sigma}$ - matrix:

$$\vec{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} \quad (1-1)$$

Matrix (1-1) is symmetric, $\sigma_{ij} = \sigma_{ji}$, so only 10 elements of matrix are independent. Measurements are provided for beam sizes

$$\langle x^2 \rangle = \sigma_{11}, \langle y^2 \rangle = \sigma_{33}, \langle xy \rangle = \sigma_{13} \quad (1-2)$$

at different locations . Explicit transformation for $\sigma_{11}, \sigma_{33}, \sigma_{13}$ is:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \end{pmatrix} = \begin{pmatrix} R_{11}^2 & 2R_{11}R_{12} & 2R_{11}R_{13} & 2R_{11}R_{14} & R_{12}^2 & 2R_{12}R_{13} & 2R_{12}R_{14} & R_{13}^2 & 2R_{13}R_{14} & R_{14}^2 \\ R_{31}^2 & 2R_{31}R_{32} & 2R_{31}R_{33} & 2R_{31}R_{34} & R_{32}^2 & 2R_{32}R_{33} & 2R_{32}R_{34} & R_{33}^2 & 2R_{33}R_{34} & R_{34}^2 \\ R_{11}R_{31} & R_{12}R_{31}+R_{11}R_{32} & R_{13}R_{31}+R_{11}R_{33} & R_{31}R_{14}+R_{11}R_{34} & R_{12}R_{32} & R_{13}R_{32}+R_{12}R_{33} & R_{32}R_{14}+R_{12}R_{34} & R_{13}R_{33} & R_{14}R_{33}+R_{13}R_{34} & R_{14}R_{34} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{24} \\ \sigma_{33} \\ \sigma_{34} \\ \sigma_{44} \end{pmatrix} \quad (1-3)$$

To determine 10 independent values of σ -matrix, we need $3 \times 3 + 1 = 10$ equations from at least 4 independent measurement stations. To be physically correct, the following conditions for matrix elements have to be fulfilled (for details see BDS meeting note of YB, 09/12/2006):

$$\boxed{\sigma_{ii} > 0}, \quad \boxed{\begin{vmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ij} & \sigma_{jj} \end{vmatrix} > 0}, \quad i = 1, 2, 3, 4; \quad j > i \quad (1-4)$$

Complete analysis of errors in 4D beam emittance measurements requires solution of 10x10 linear system with variable parameters. In this note we consider errors in 2D beam emittance measurements (uncoupled beam), where beam at each plane (x-x'), (y-y') is determined by 3 parameters α , β , \exists . Special case of 2D beam with $\alpha = 0$ was considered in BDS meeting note of YB, 10/24/2006.

2. Errors in 2D beam emittance measurement

Consider 2D beam emittance measurement problem for the beam propagating in FODO channel. Single particle transformation matrix

$$\begin{vmatrix} x \\ x' \end{vmatrix} = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \begin{vmatrix} x_o \\ x'_o \end{vmatrix} \quad (2-1)$$

Beam ellipse transformation:

$$\begin{vmatrix} \beta \\ \alpha \\ \gamma \end{vmatrix} = \begin{vmatrix} C^2 & -2CS & S^2 \\ -CC' & CS'+SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{vmatrix} \begin{vmatrix} \beta_o \\ \alpha_o \\ \gamma_o \end{vmatrix} \quad (2-2)$$

As far as $\beta\vartheta = R^2$, where R is the beam size and ϑ is the beam emittance, equation for unknown beam parameters $\alpha_o, \beta_o, \gamma_o$, is

$$\begin{vmatrix} R_1^2 \\ R_2^2 \\ R_3^2 \end{vmatrix} = \begin{vmatrix} C_1^2 & -2C_1S_1 & S_1^2 \\ C_2^2 & -2C_2S_2 & S_2^2 \\ C_3^2 & -2C_3S_3 & S_3^2 \end{vmatrix} \begin{vmatrix} \beta_o\vartheta \\ \alpha_o\vartheta \\ \gamma_o\vartheta \end{vmatrix} \quad (2-3)$$

Solution of Eq. (2-3) is

$$\alpha_{o\vartheta} = \frac{C_3^2(R_2^2S_1^2 - R_1^2S_2^2) + C_1^2(R_3^2S_2^2 - R_2^2S_3^2) + C_2^2(R_1^2S_3^2 - R_3^2S_1^2)}{2(C_2S_1 - C_1S_2)(C_3S_1 - C_1S_3)(C_2S_3 - C_3S_2)} \quad (2-4)$$

$$\beta_{o\vartheta} = \frac{-R_3^2S_1S_2(C_2S_1 - C_1S_2) + C_3S_3(R_2^2S_1^2 - R_1^2S_2^2) - S_3^2(C_1S_1R_2^2 - C_2S_2R_1^2)}{(C_2S_1 - C_1S_2)(C_3S_1 - C_1S_3)(C_2S_3 - C_3S_2)} \quad (2-5)$$

$$\gamma_{o\vartheta} = \frac{R_1^2C_2S_3(C_2S_3 - C_3S_2) + C_1S_1(R_2^2C_3^2 - R_3^2C_2^2) + C_1^2(C_2S_2R_3^2 - C_3S_3R_2^2)}{(C_2S_1 - C_1S_2)(C_3S_1 - C_1S_3)(C_2S_3 - C_3S_2)} \quad (2-6)$$

Beam emittance

$$\vartheta = \sqrt{(\beta_{o\vartheta})(\gamma_{o\vartheta}) - (\alpha_{o\vartheta})^2} \quad (2-7)$$

Error in beam emittance

$$\begin{aligned}
\frac{d\mathfrak{E}}{\mathfrak{E}} = & -\{2(C_2S_1 - C_1S_2)^2(C_3S_1 - C_1S_3)^2(C_3S_2 - C_2S_3)^2 \\
& \left[-\frac{(C_1^2R_2^2 - C_2^2R_1^2)(C_1^2R_2^2 \frac{dR_2}{R_2} - C_2^2R_1^2 \frac{dR_1}{R_1})}{C_1^2C_2^2(C_2S_1 - C_1S_2)^2} - \right. \\
& -\frac{(C_1^2R_3^2 - C_3^2R_1^2)(C_1^2R_3^2 \frac{dR_3}{R_3} - C_3^2R_1^2 \frac{dR_1}{R_1})}{C_1^2C_3^2(C_3S_1 - C_1S_3)^2} + \\
& +\frac{(C_2^2R_3^2 - C_3^2R_2^2)(C_2^2R_3^2 \frac{dR_3}{R_3} - C_3^2R_2^2 \frac{dR_2}{R_2})}{C_2^2C_3^2(C_3S_2 - C_2S_3)^2} + \\
& +\frac{2C_2(C_3^4R_1^4 \frac{dR_1}{R_1} - C_1^4R_3^4 \frac{dR_3}{R_3})}{C_1^2C_3^3(C_2S_1 - C_1S_2)(C_3S_1 - C_1S_3)} + \\
& \left. \frac{2C_1(C_3^4R_2^4 \frac{dR_2}{R_2} - C_2^4R_3^4 \frac{dR_3}{R_3})}{C_2^2C_3^3(C_2S_1 - C_1S_2)(C_3S_3 - C_3S_2)} \right\} / D
\end{aligned} \tag{2-8}$$

Denominator:

$$D = G^4 \left[\left(\frac{A+B}{G} \right)^2 - 1 \right] \left[\left(\frac{-A+B}{G} \right)^2 - 1 \right] \quad (2-9)$$

$$A = R_1(C_2S_3 - C_3S_2) \quad (2-10)$$

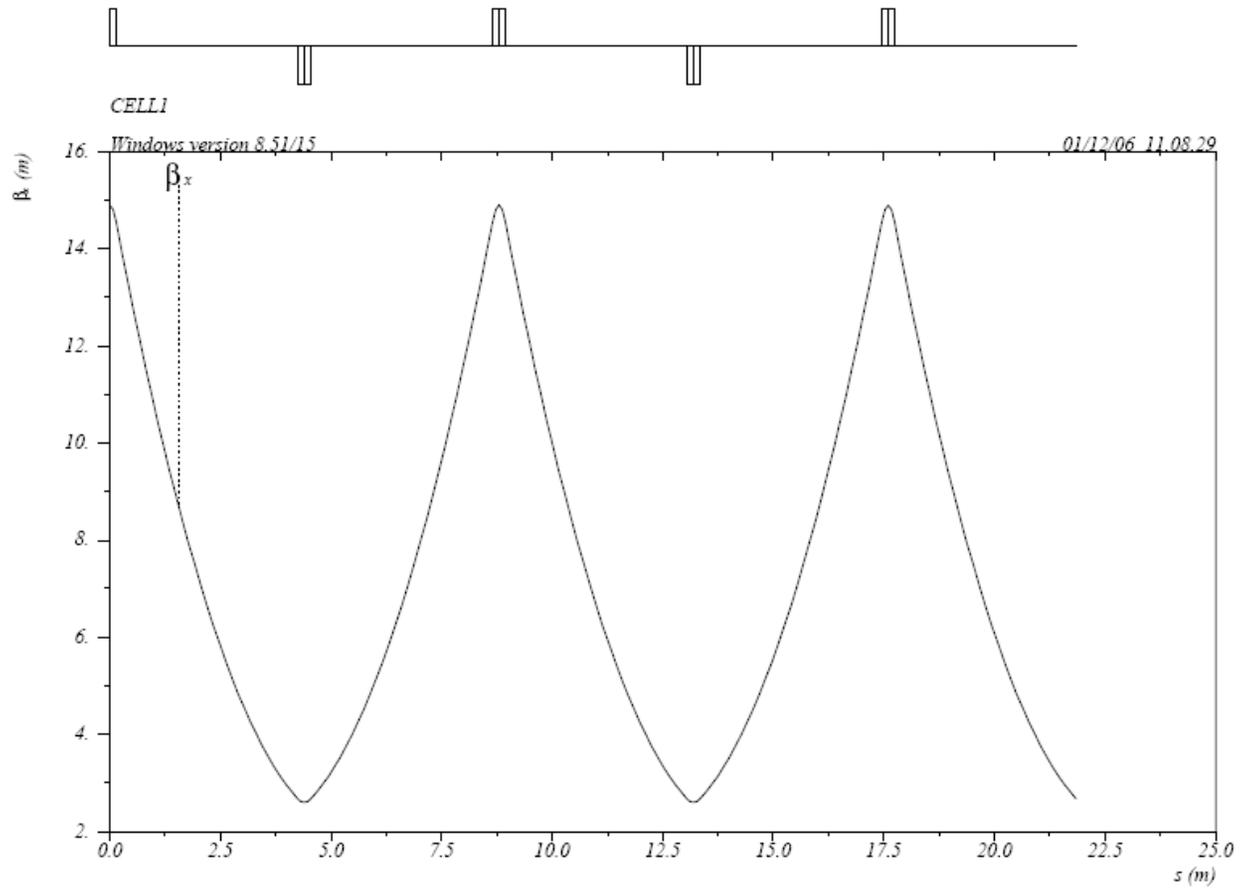
$$B = R_2(C_3S_1 - C_1S_3) \quad (2-11)$$

$$G = R_3(C_2S_1 - C_1S_2) \quad (2-12)$$

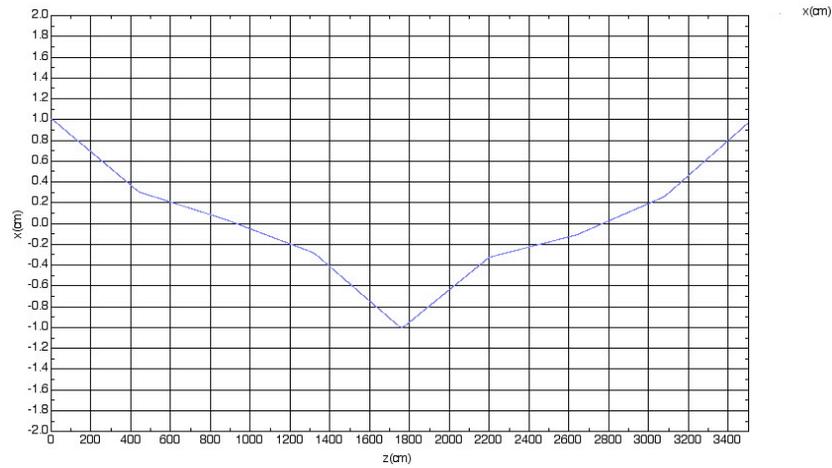
Large error in beam emittance is expected if

$$a = \frac{A+B}{G} \approx 1 \quad \text{or} \quad b = \frac{B-A}{G} \approx 1 \quad (2-13)$$

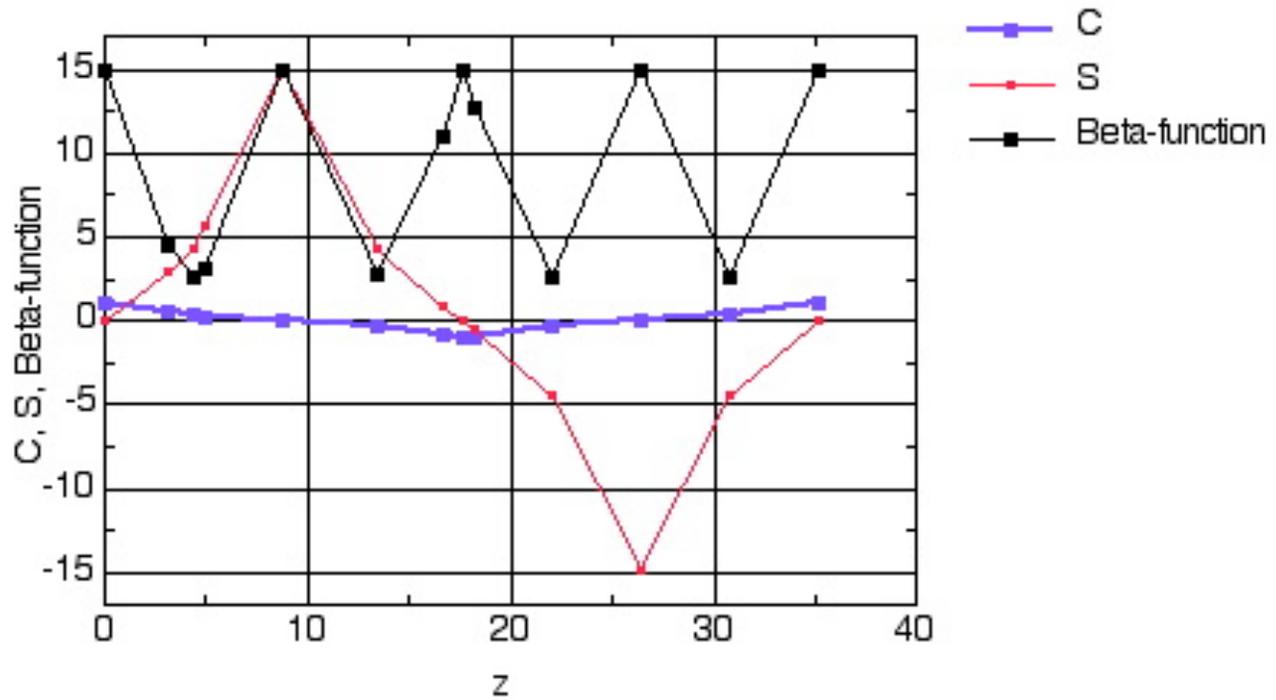
Consider 90° phase advance FODO channel:



Beam energy	5 GeV
Lens Gradient	17.8 T/m
Quad length	0.3 m
FODO period	8.8 m
Distance between quads	4.4 m

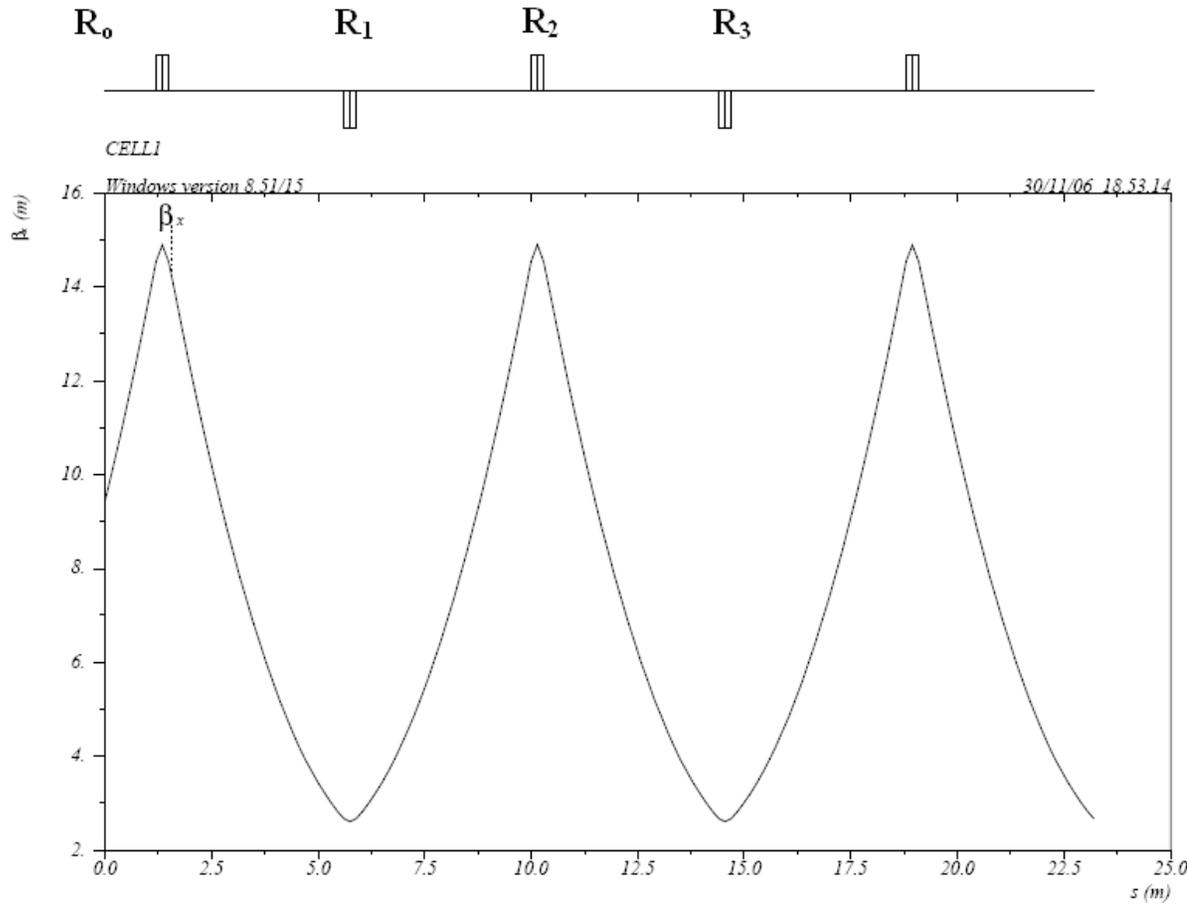


Single particle trajectory

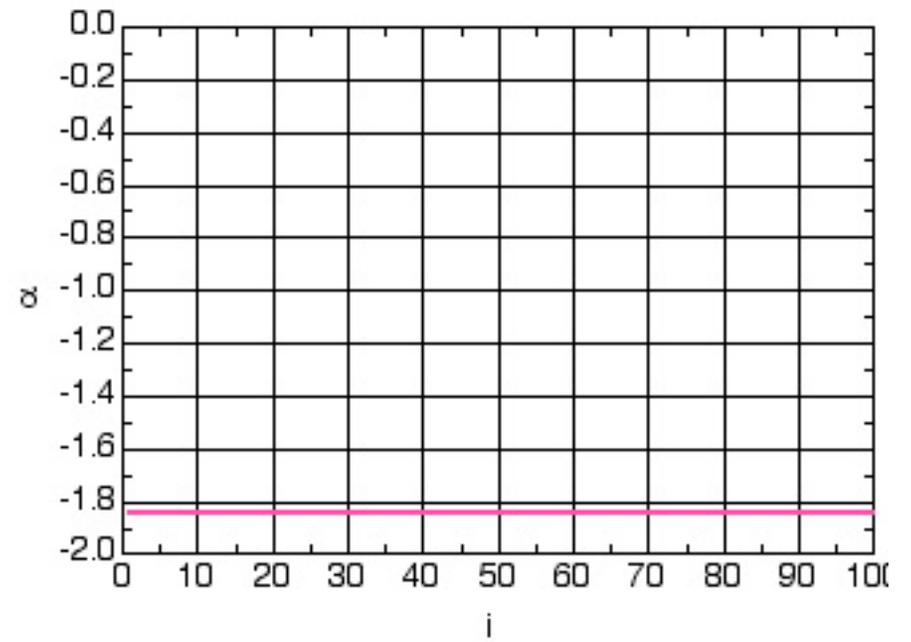
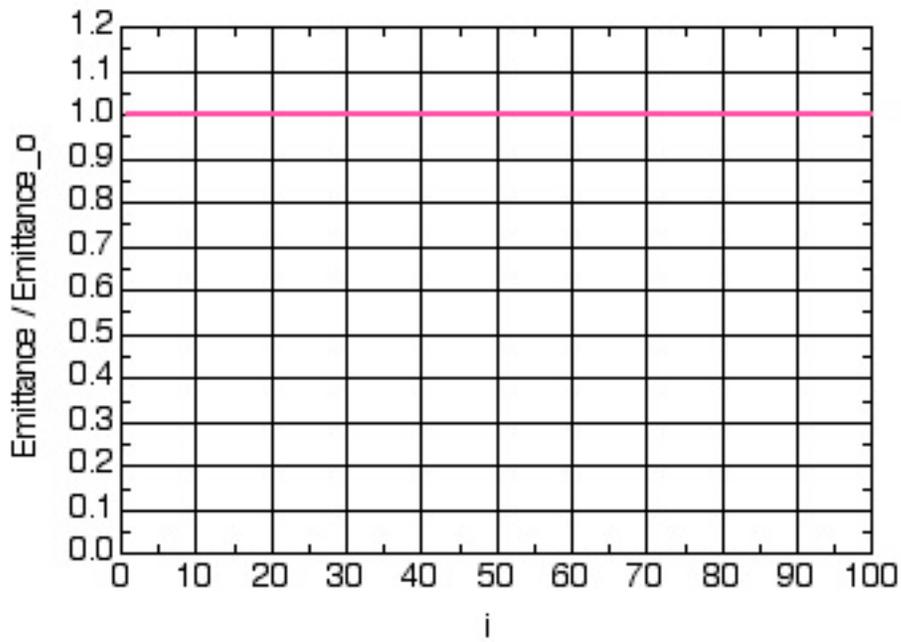
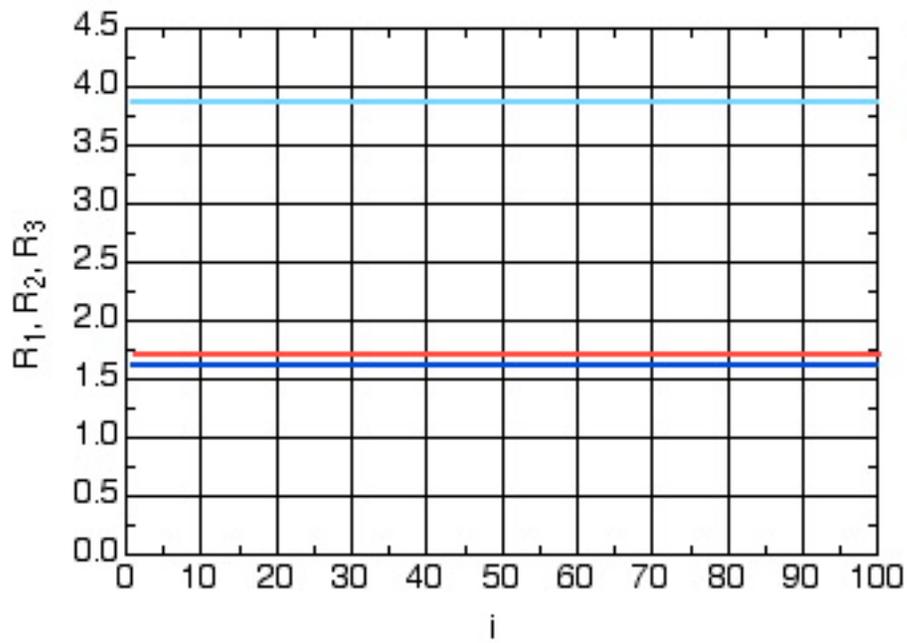


Matrix parameters and β -function of the structure.

Example1: Stable beam emittance measurement



$C1 = -0.428950$
 $S1 = 3.87470$
 $C2 = -2.44000$
 $S2 = 11.7880$
 $R1 = 1.61200$
 $R2 = 3.86000$
 $C3 = -1.01256$
 $S3 = 3.08830$
 $R3 = 1.61200$
 $a = 0.17$
 $b = 5.78$
 $R_0 = 3.07$
 $\vartheta = 1$
 $\alpha = -1.8$



Determination of beam parameters ε , R_0 , α for error in beam sizes R_1 , R_2 , $R = 0$.

Variation of measured beam sizes:

$$R_1 = R_1^{(0)} (1 + f)$$

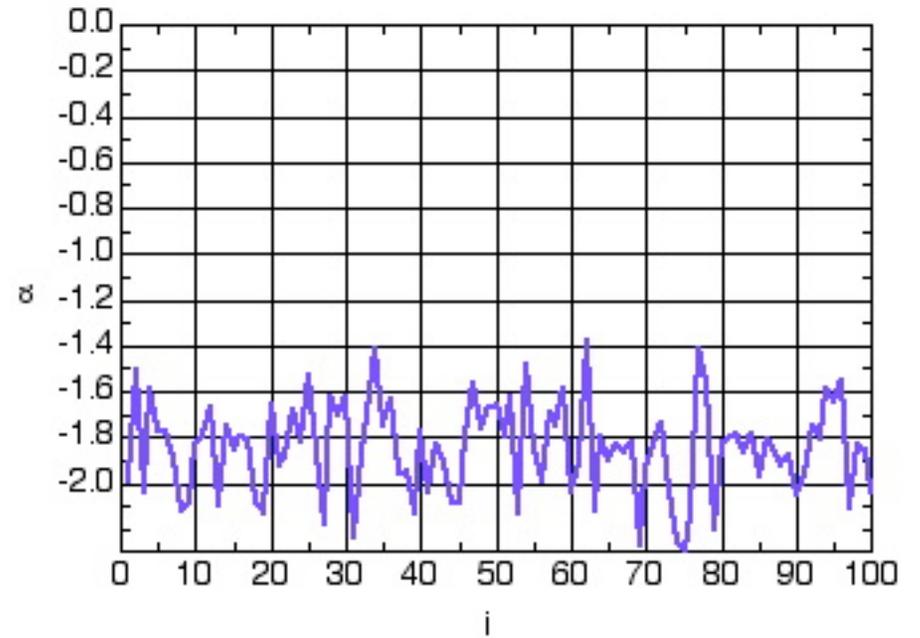
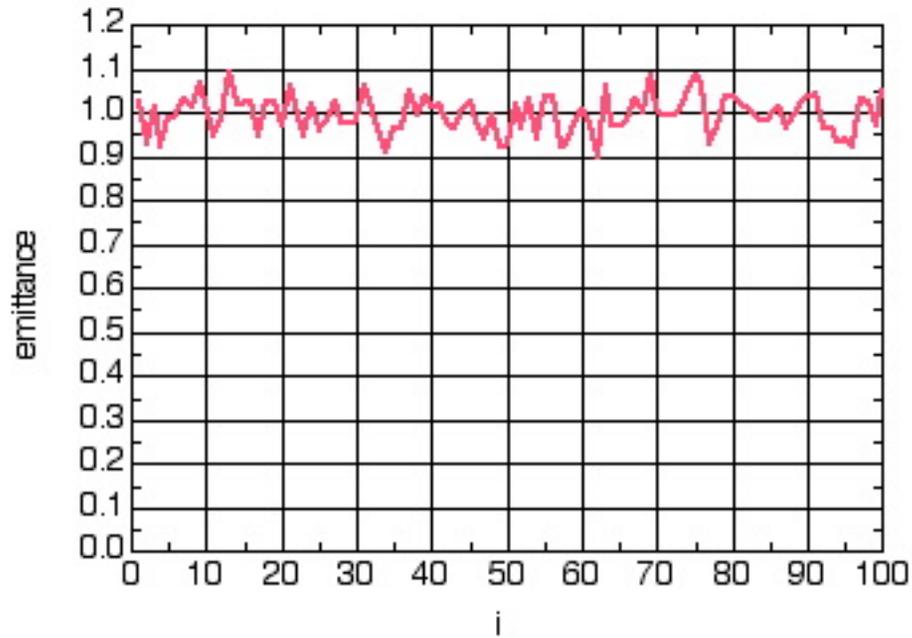
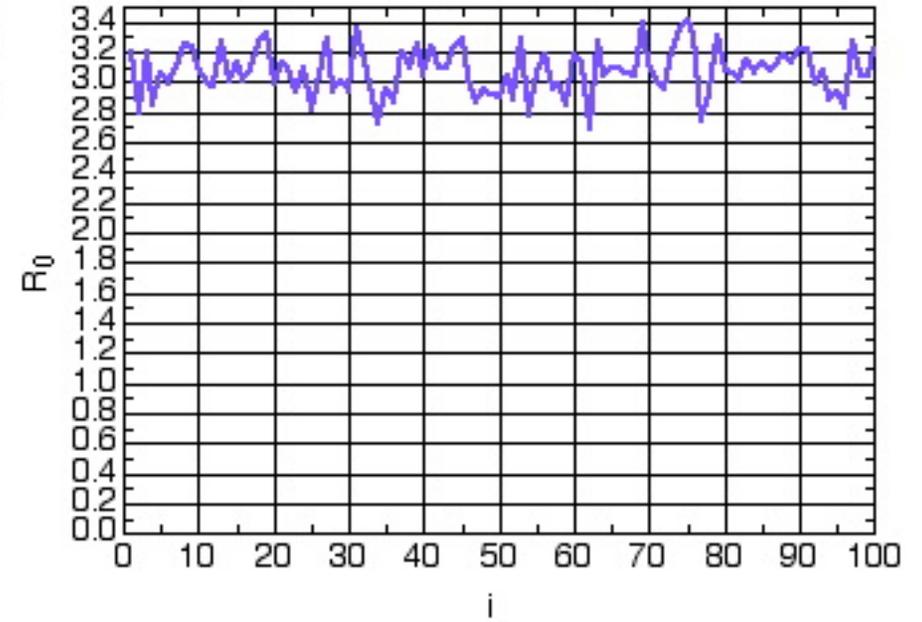
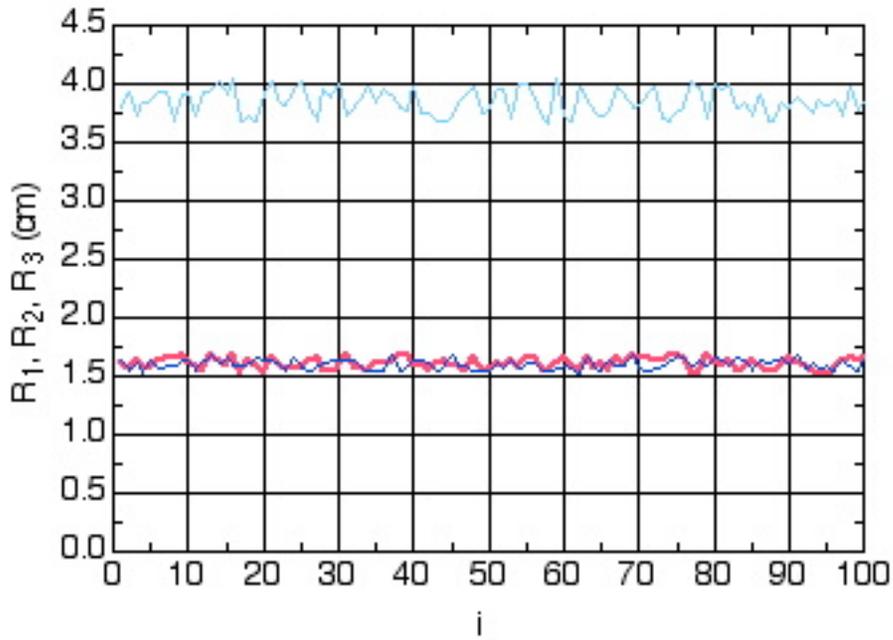
$$R_2 = R_2^{(0)} (1 + g)$$

$$R_3 = R_3^{(0)} (1 + h)$$

where $R_1^{(0)}$, $R_2^{(0)}$, $R_3^{(0)}$ - unperturbed values of measured beam sizes,
f, g, h – generators of random numbers uniformly distributed within interval $[-a, a]$

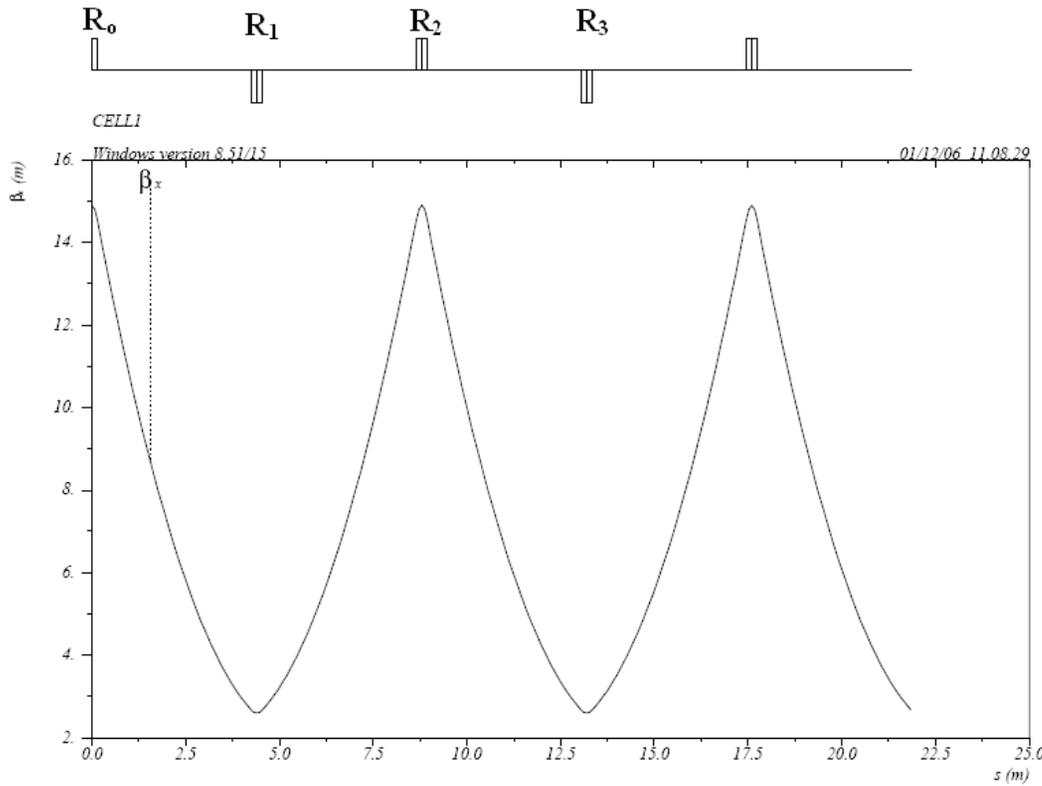
Values of R_1 , R_2 , R_3 are distributed with

$$\sigma = \frac{a}{\sqrt{3}}$$

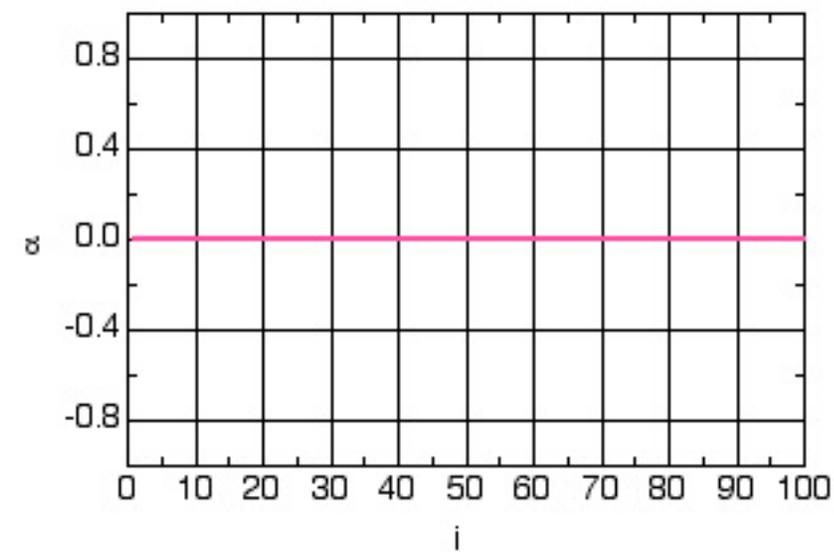
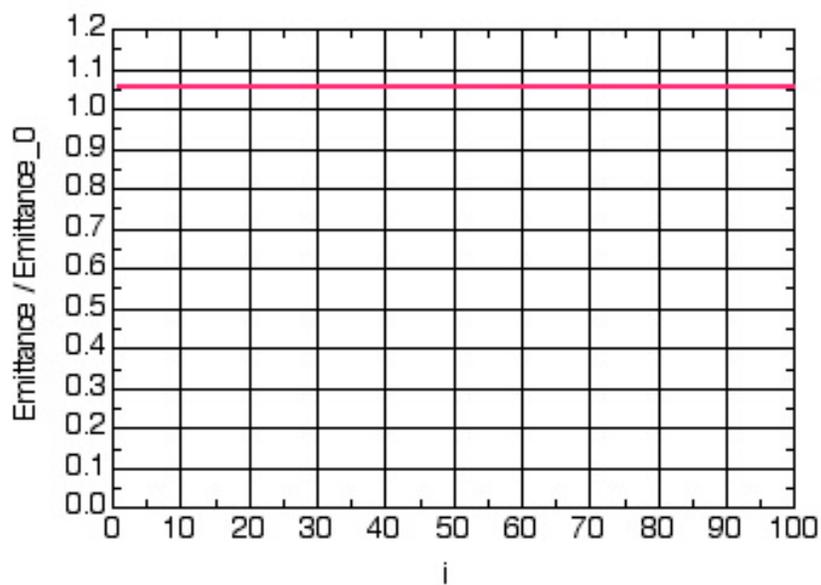
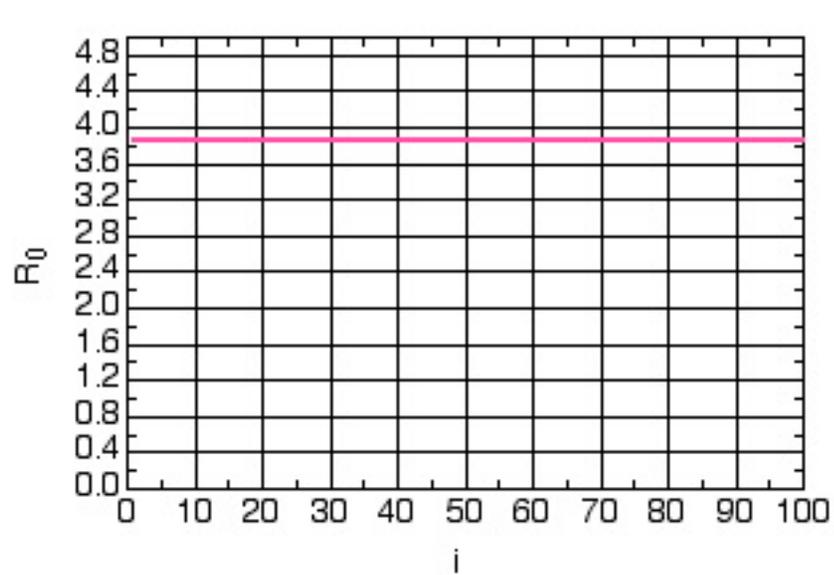
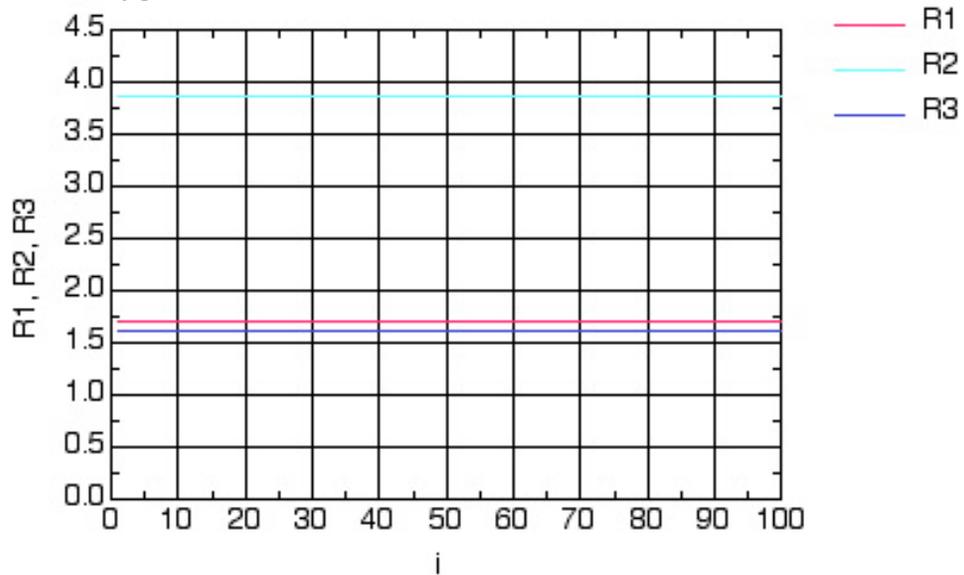


Determination of beam parameters with errors in beam sizes $R_1, R_2, R_3 = \pm 5\%$ ($\sigma_R/R = 2.88\%$). Error in measured value of beam emittance is approximately $\pm 10\%$ ($\sigma_\epsilon/\epsilon = 4.3\%$).

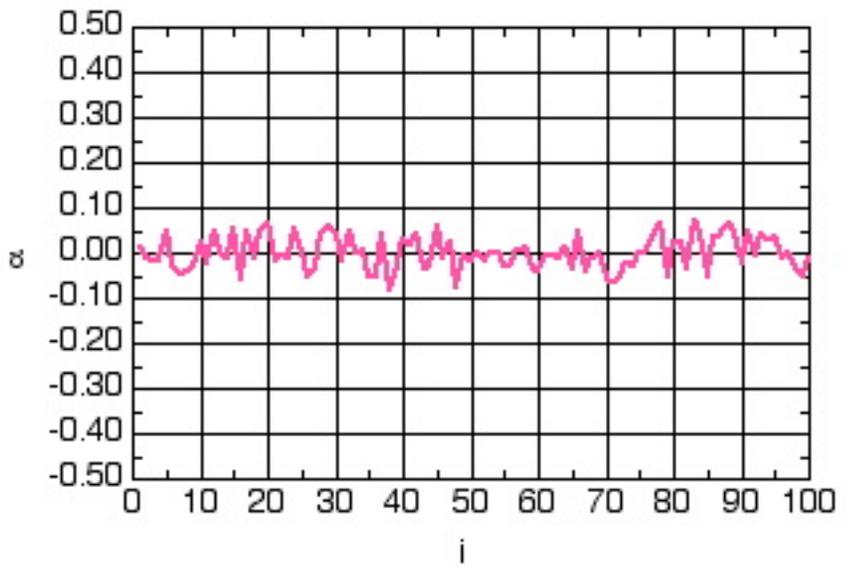
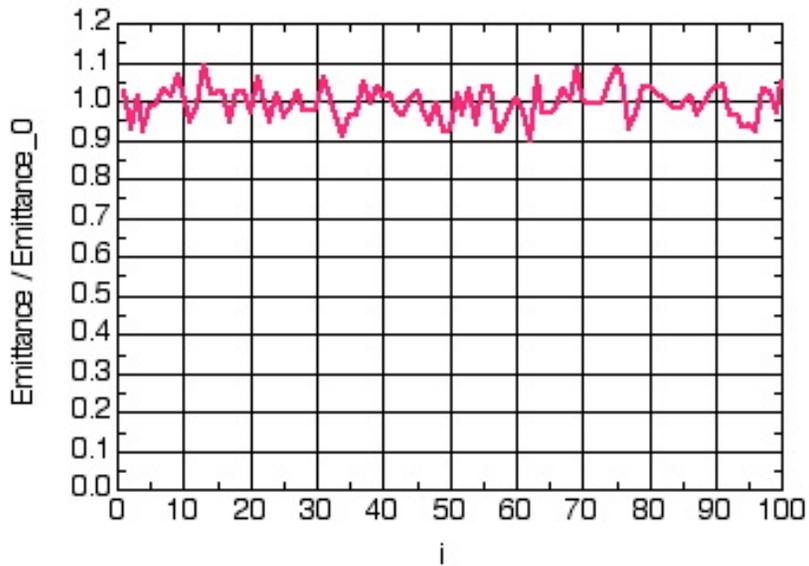
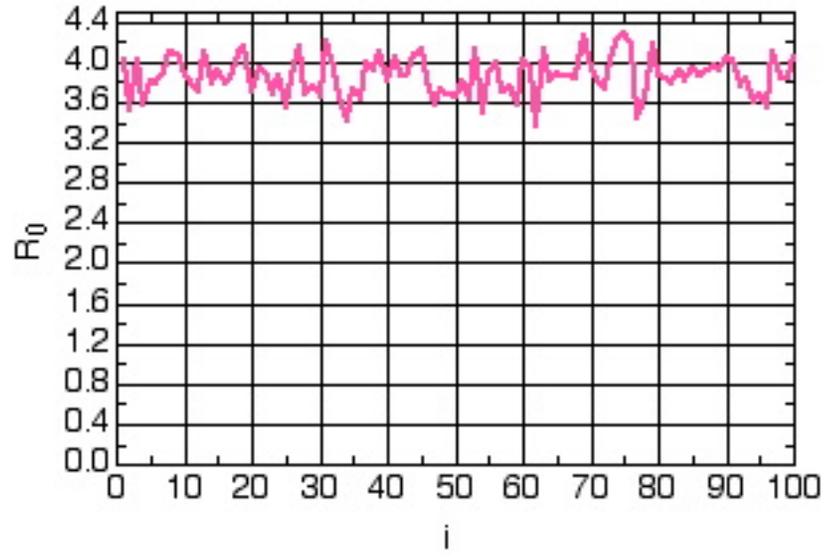
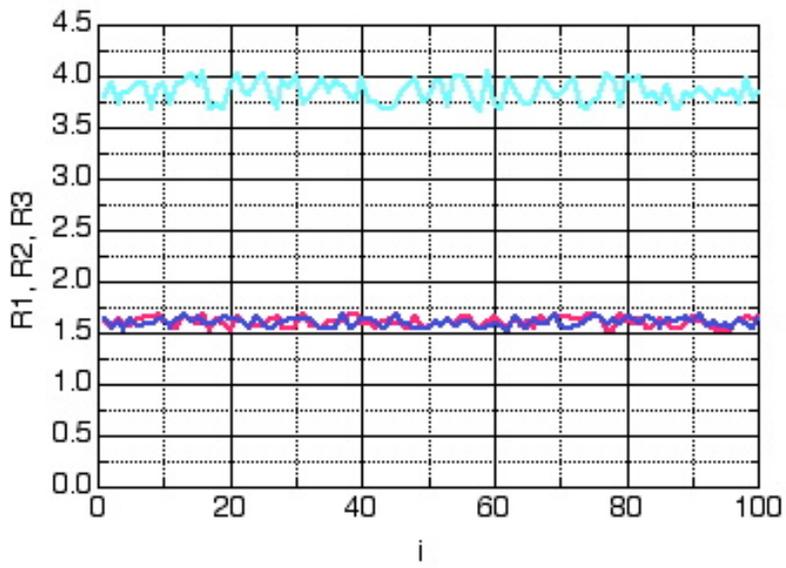
Example 2: Stable beam emittance measurements (phase shift 45° , 90° , 135°)



$$\begin{aligned}
 C1 &= 0.295340 \\
 S1 &= 4.40000 \\
 C2 &= 0.000000 \\
 S2 &= 14.9000 \\
 R1 &= 1.70000 \\
 R2 &= 3.86000 \\
 C3 &= -0.295340 \\
 S3 &= 4.40000 \\
 R3 &= 1.61200 \\
 a &= 0.17 \\
 b &= 5.28 \\
 R_0 &= 3.86 \\
 \vartheta &= 1 \\
 \alpha &= 0
 \end{aligned}$$

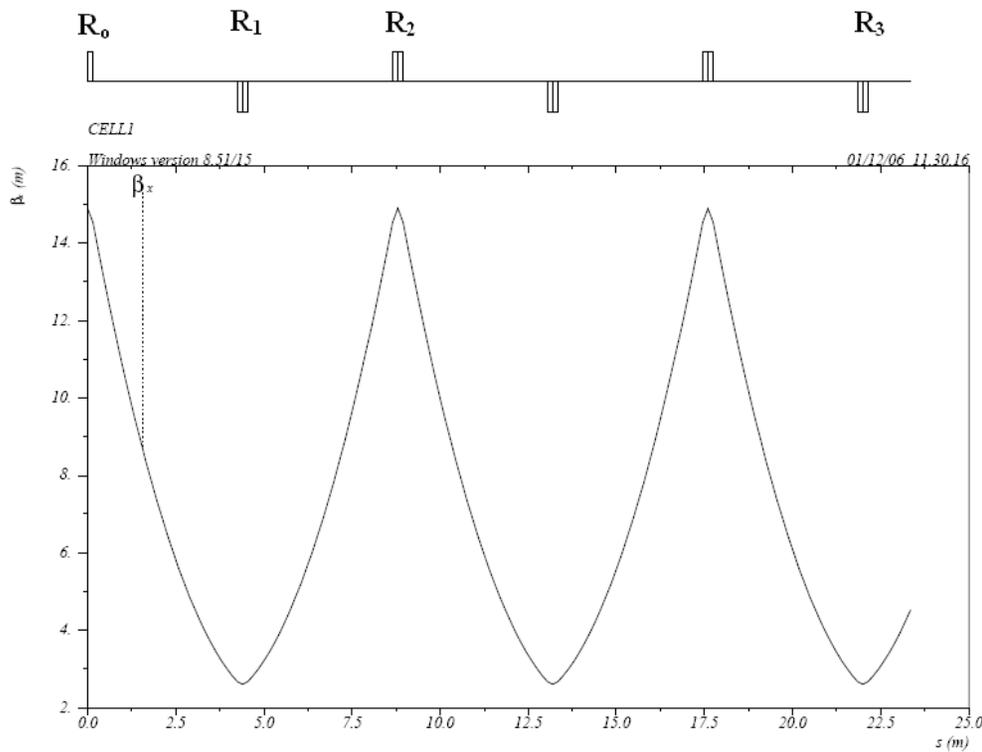


Determination of beam parameters ε , R_0 , α with error in beam sizes $R_1, R_2, R_3 = 0$.



Determination of beam parameters with errors in beam sizes $R_1, R_2, R_3 = \pm 5\%$ ($\sigma_R/R = 2.88\%$). Error in measured value of beam emittance $\approx \pm 10\%$ ($\sigma_\epsilon/\epsilon = 4.29\%$).

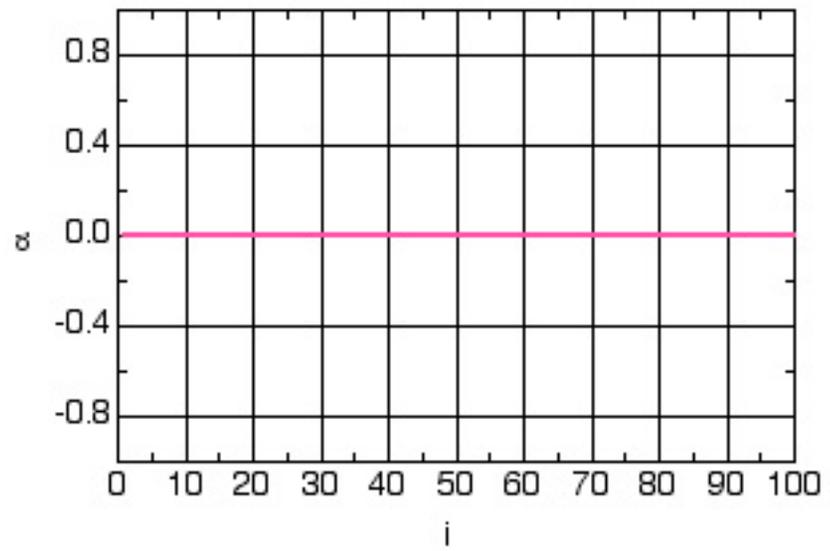
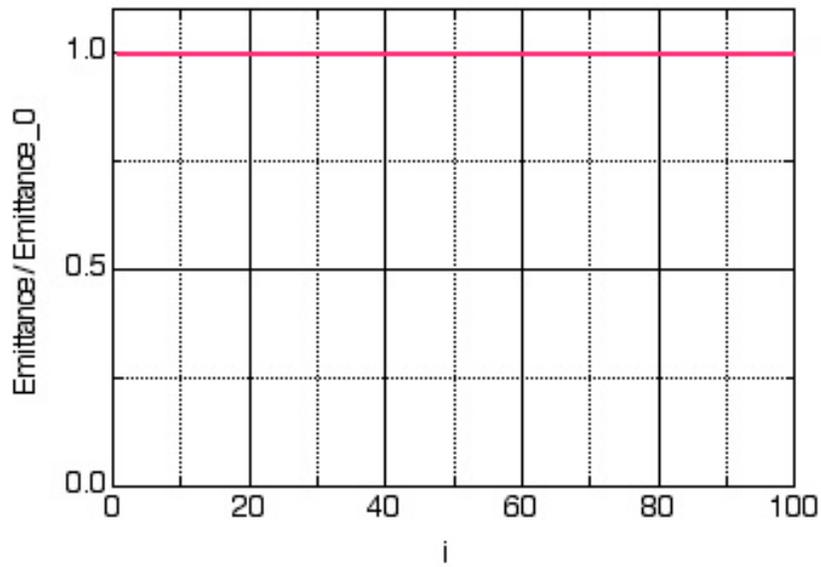
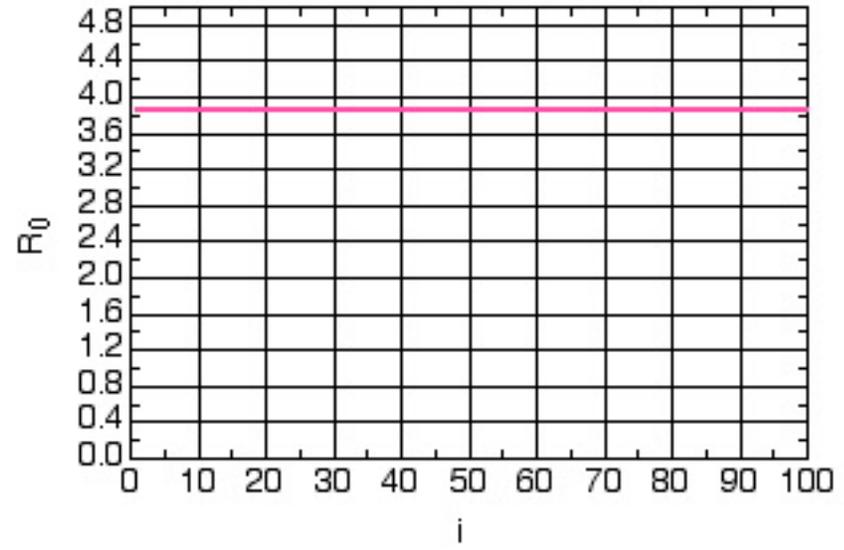
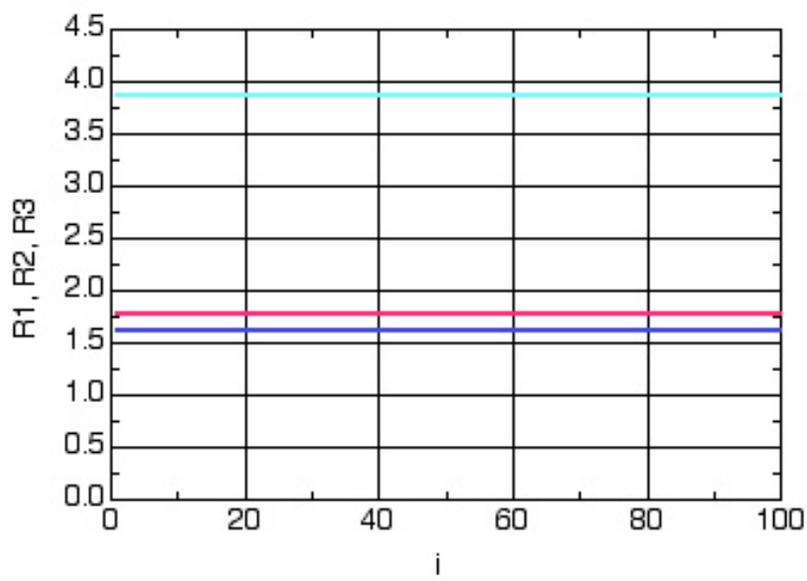
Example 3: Unstable beam emittance measurements (45°, 90°, 225°)



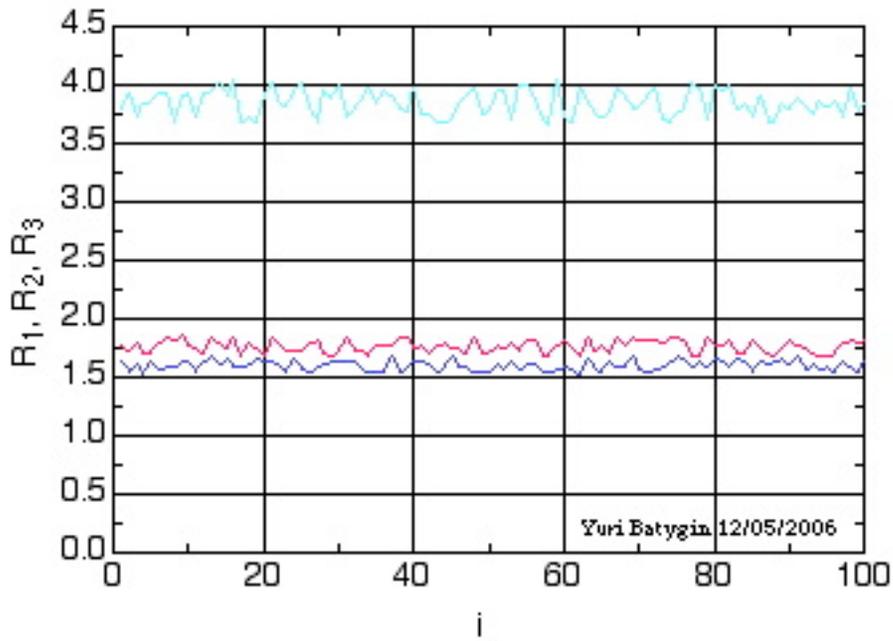
$C1 = 0.255000$
 $S1 = 5.68000$
 $C2 = 0.000000$
 $S2 = 14.9000$
 $R1 = 1.77220$
 $R2 = 3.86000$
 $C3 = -0.295340$
 $S3 = -4.40000$
 $R3 = 1.61200$

$a = 0.85$
 $b = 2.63$

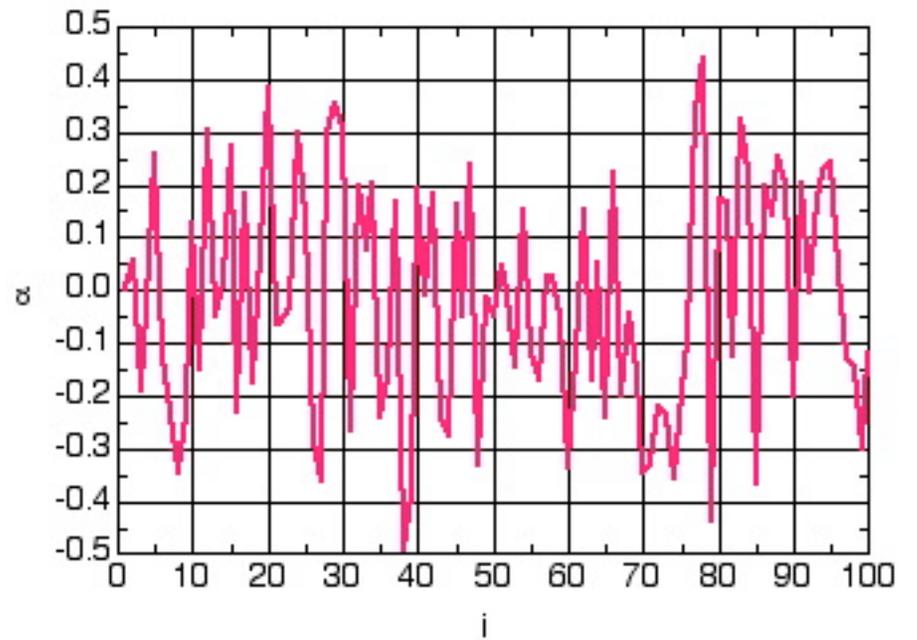
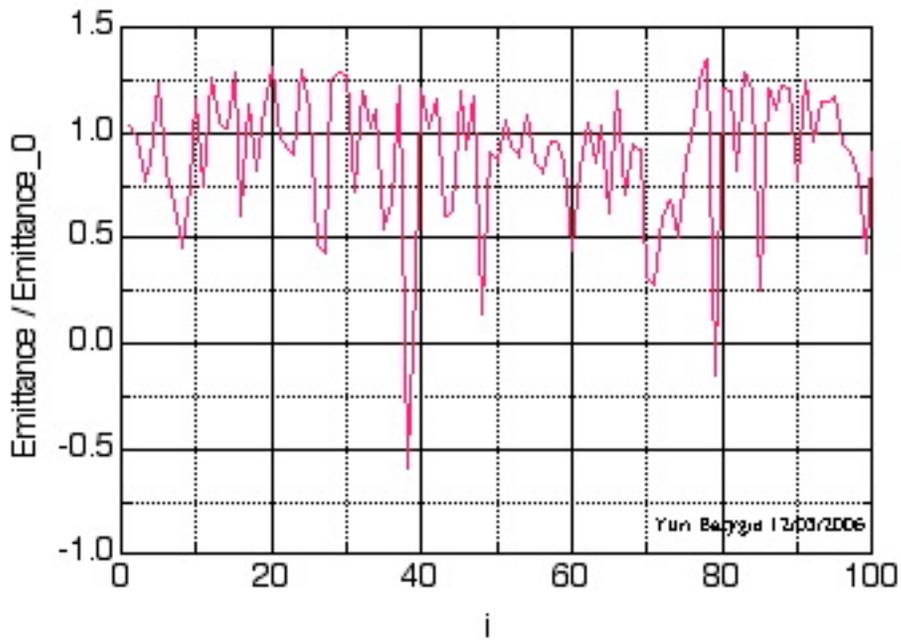
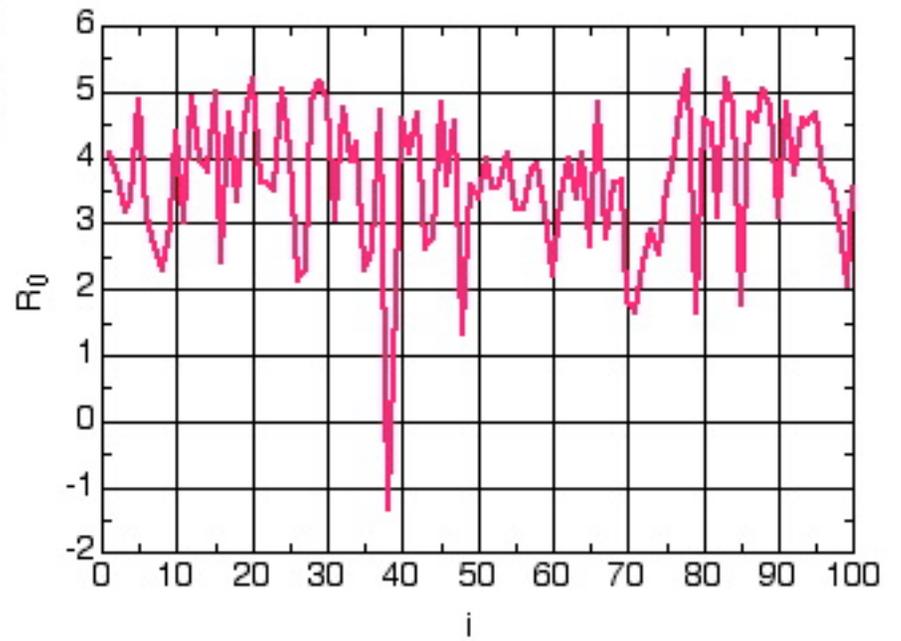
$R_0 = 3.86$
 $\vartheta = 1$
 $\alpha = 0$



Determination of beam parameters ε , R_0 , α with error in beam sizes $R_1, R_2, R_3 = 0$.



— R_1
— R_2
— R_3



Unstable determination of beam parameters with errors in beam sizes $R_1, R_2, R_3 = \pm 5\%$.

3. Summary

1. Error in determination of beam emittance is larger than error in measured beam sizes.
2. Determination of emittance through beam size measurements at different locations

R_1, R_2, R_3 is performed with significant error if the following conditions are fulfilled:

$$A = R_1(C_2S_3 - C_3S_2)$$

$$B = R_2(C_3S_1 - C_1S_3)$$

$$G = R_3(C_2S_1 - C_1S_2)$$

$$a = \frac{A + B}{G} \approx 1$$

or

$$b = \frac{B - A}{G} \approx 1$$

where $S_1, S_2, S_3, C_1, C_2, C_3$ are single particle matrix transformation elements.

3. Next step: error analysis for 4D beam.