Four-loop QCD corrections to the ρ parameter

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In collaboration with: K.G. Chetyrkin, M. Faisst, J.H. Kühn, P. Maierhöfer

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II. Strategy & Methods
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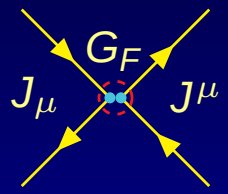
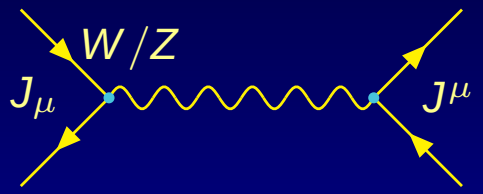
Introduction

ρ-parameter: Parametrizes relative strength of charged (CC) and neutral current (NC)

Fermi Model $\longleftrightarrow$ **Standard Model**





CC: $\mathcal{M}_{CC} = \frac{4\,G_F}{\sqrt{2}}\,J_\mu J^\mu \longleftrightarrow \mathcal{M}_{CC} = -\frac{g^2}{2}\,J_\mu\,\frac{1}{q^2-M_W^2}\,J^\mu$

NC: $\mathcal{M}_{NC} = \rho\,\frac{8\,G_F}{\sqrt{2}}\,J_\mu J^\mu \longleftrightarrow \mathcal{M}_{NC} = -\frac{g^2}{\cos^2\theta_W}\,J_\mu\,\frac{1}{q^2-M_Z^2}\,J^\mu$

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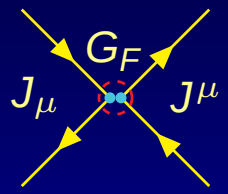

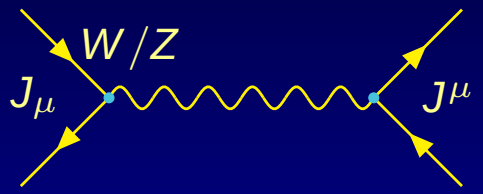

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$$\rho = \frac{M_W^2}{\cos^2\theta_W M_Z^2} = 1 \quad \text{at tree-level in SM}$$

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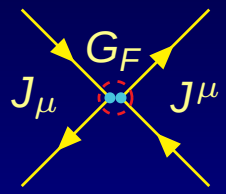


CC: $\mathcal{M}_{CC} = \frac{4\,G_F}{\sqrt{2}}\, J_\mu J^\mu \longleftrightarrow \mathcal{M}_{CC} = +\frac{g^2}{2}\, J_\mu \frac{1}{M_W^2 - \Pi_T^W(0)}\, J^\mu$

NC: $\mathcal{M}_{NC} = \rho\, \frac{8\,G_F}{\sqrt{2}}\, J_\mu J^\mu \longleftrightarrow \mathcal{M}_{NC} = +\frac{g^2}{\cos^2\theta_W}\, J_\mu \frac{1}{M_Z^2 - \Pi_T^Z(0)}\, J^\mu$

$$\rho = \frac{M_W^2}{\cos^2\theta_W M_Z^2} = 1 \qquad \text{at tree-level in SM}$$

Higher order corrections: $\rho = 1 + \delta\rho \qquad \delta\rho = \frac{\Pi_T^Z(0)}{M_Z^2} - \frac{\Pi_T^W(0)}{M_W^2}$

$\Rightarrow$ Self-energies of W- and Z-propagators for vanishing external momentum needed $\Rightarrow$ tadpoles

Introduction

- Fermion-doublet:

  1-loop $\delta\rho = \frac{3\,G_F}{8\sqrt{2}\pi^2}\left(m_t^2 + m_b^2 + 2\,\frac{m_b^2\,m_t^2}{m_t^2 - m_b^2}\,\log\left(\frac{m_b^2}{m_t^2}\right)\right)$ M.J.G. Veltman

  $\rightsquigarrow$ Establish limit on the mass splitting within one fermion doublet

- Consider $m_t \gg m_b$ $\rightsquigarrow$ leading m_t behaviour:

$$\delta\rho = 3\,\frac{G_F m_t^2}{8\sqrt{2}\pi^2} \equiv 3\,x_t$$

2-loop QCD corrections A. Djouadi, C. Verzegnassi; B.A. Kniehl, J.H. Kühn, G. Stuart

3-loop QCD corrections L. Avdeev, J. Fleischer, S. Mikhailov, O. Tarasov; K.G. Chetyrkin, J.H. Kühn, M. Steinhauser

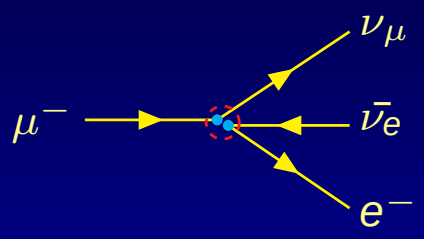
2-loop electroweak corr. J.J. van der Bij F. Hoogeveen; R. Barbieri, M. Beccaria, P. Ciafaloni, G. Curci, A. Vicere; J. Fleischer, O. Tarasov, F. Jegerlehner

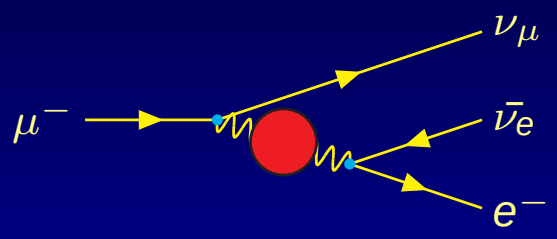
3-loop mixed EW/QCD corr. J.J. van der Bij, K.G. Chetyrkin, M. Faisst, G. Jikia, T. Seidensticker; M. Faisst, J.H. Kühn, T. Seidensticker, O. Veretin

Motivation

appears e. g. in muon decay:

Fermi Model $\longleftrightarrow$ **Standard Model**





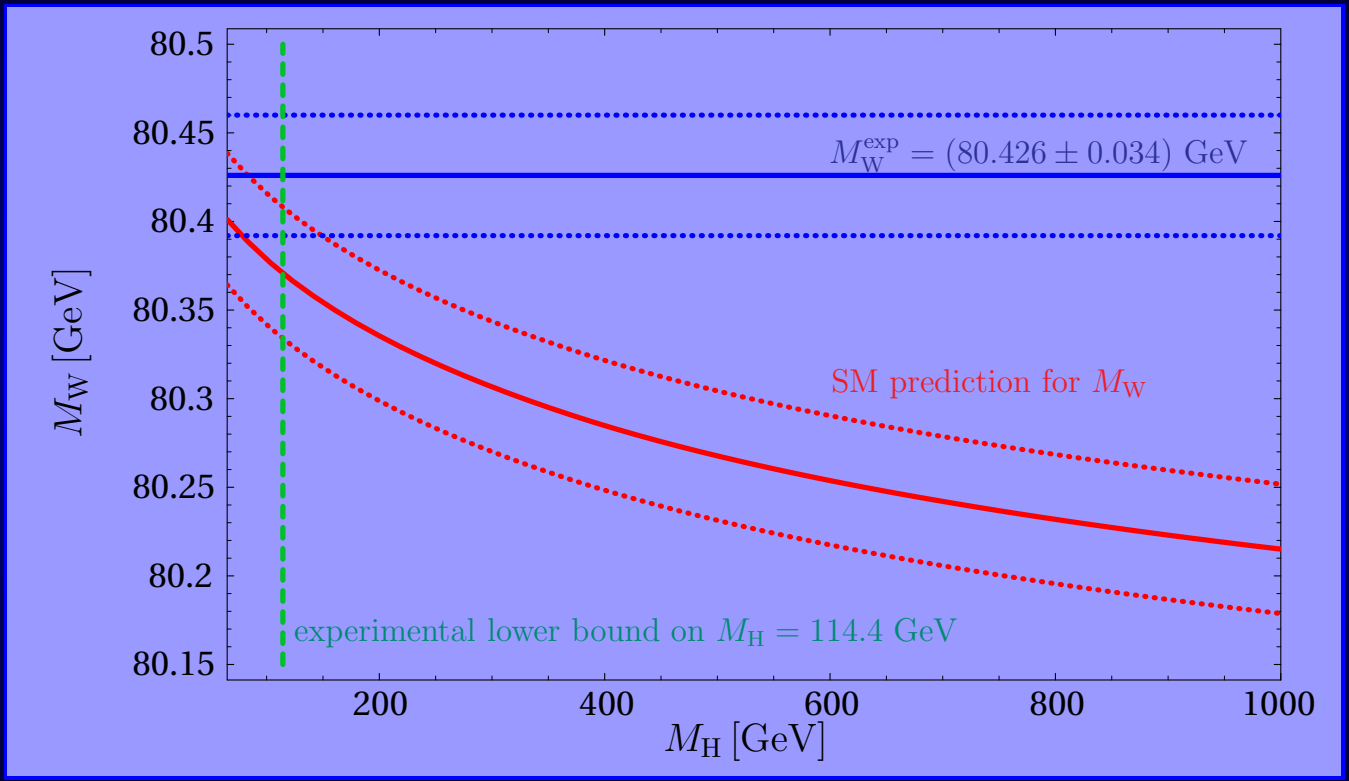
$$\frac{4\,G_F}{\sqrt{2}} = \frac{2\,\alpha\,\pi}{\left(1-\frac{M_W^2}{M_Z^2}\right) M_W^2 \left(1-\frac{\cos^2\theta_W}{\sin^2\theta_W}\,\delta\rho + \textit{other corrections}\right)}$$

- ρ-parameter depends on SM parameters: $M_t, M_H, \alpha_s, \alpha, \ldots$

  $\Longrightarrow$ prediction of $M_W^{theory} \leftrightarrow M_W^{experiment}$

- Constraint on the Higgs mass

Motivation



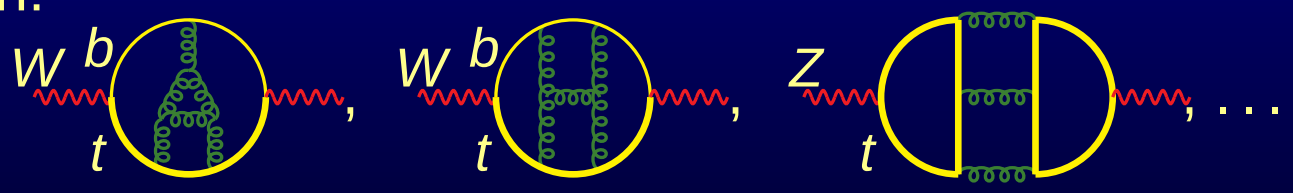
M. Awramik, M. Czakon, A. Freitas, G. Weiglein

Motivation

- Higher order corr. induce shift: $\delta M_W = \frac{M_W}{2}\frac{c_W^2}{c_W^2 - s_W^2}\delta\rho$
- Current uncertainty of M_W: $\delta M_W^{\text{exp}} \sim 35$ MeV
- Anticipated precision at ILC: $\delta M_W^{\text{exp,LC}} \sim 6$ MeV
- 3-loop QCD-correction shift: $\delta M_W^{\text{3-loop}} \sim -11$ MeV

$\Longrightarrow$ Study four-loop QCD corrections

In this talk: four-loop QCD-corrections: $\mathcal{O}(G_F m_t^2 \alpha_s^3)$
Classification:



non-singlet singlet

singlet contribution $\longrightarrow$ Y. Schröder, M. Steinhauser
at $\mathcal{O}(G_F m_t^2 \alpha_s^2)$ in $\overline{\text{MS}}$-scheme singlet-contribution is numerically dominant

Strategy & Methods

Strategy: I.) Reduce all integrals to master integrals
II.) Solve master integrals

I.) Reduction:

Integration-by-parts: K.G. Chetyrkin, F.V. Tkachov

$$0 = \int [d^D k_1] \ldots [d^D k_4] \quad \partial_{(k_j)_\mu} \left(k_l^\mu \, I_{\alpha\beta} \right) \ , \qquad j, l = 1, \ldots, \text{loops=4}$$

$I_{\alpha\beta}$: Generic integrand with propagator powers $\alpha = \{\alpha_1, \ldots\}$
and scalar-product powers $\beta = \{\beta_1, \ldots\}$

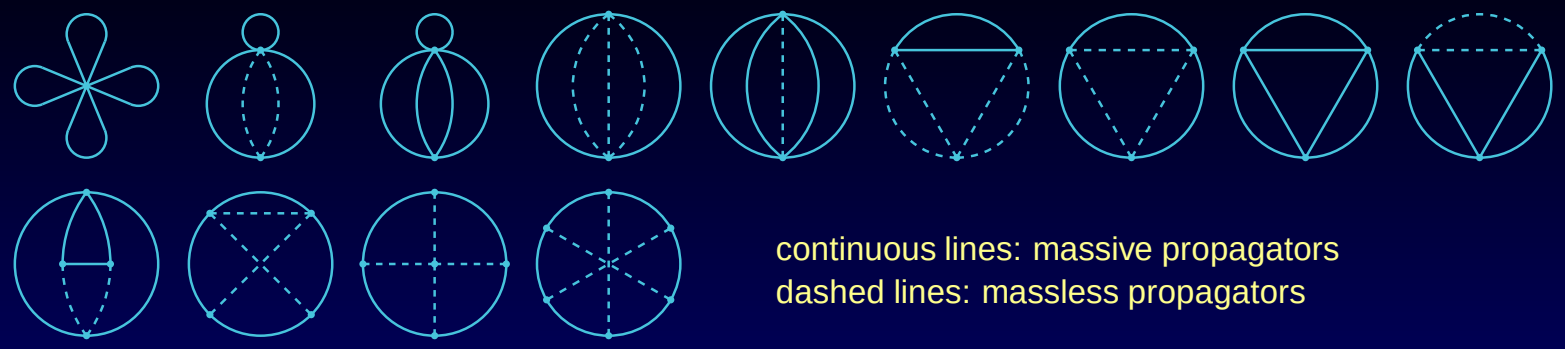
Laporta-Algorithm: S. Laporta, E. Remiddi

Method:
– IBP-identities for explicit numerical values of α, β
– Introduction of an order among the integrals
– Solving a linear system of equations

Here: About 7 million IBP-identities generated and solved

Problem: Sizable number of master integrals: 63 !

Master integrals in the Standard Basis

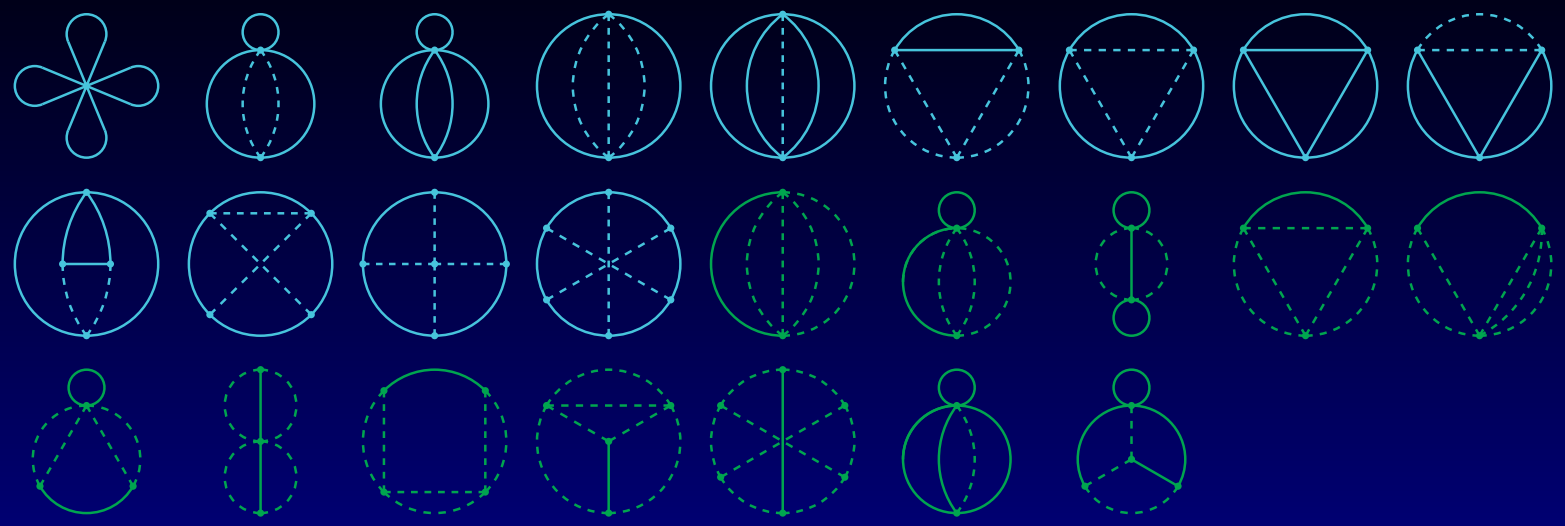


continuous lines: massive propagators
dashed lines: massless propagators

$\rightsquigarrow$ 13 of master integrals known from previous calculations:

- Solution with high precision numerics Y. Schröder, A. Vuorinen
  with difference equation method S. Laporta
- Solution with the method of ε-finite basis
  K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov
- other contributions:
  D.J. Broadhurst; S. Laporta; Y. Schröder, M. Steinhauser; B.A. Kniehl, A.V. Kotikov, A.I. Onishchenko, O.L. Veretin

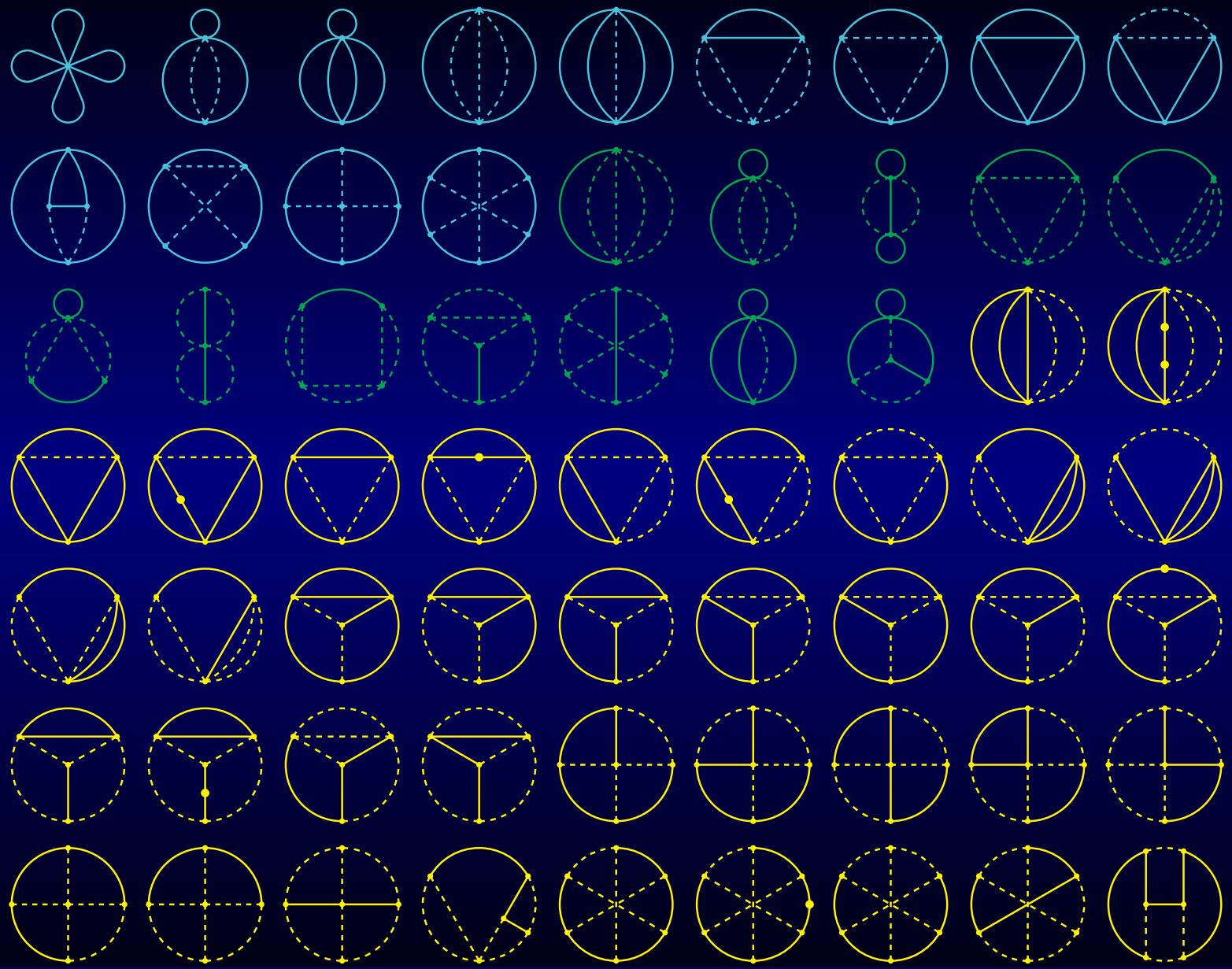
Master integrals in the Standard Basis



... additional 12 master integrals are “simple”,
they are factorized or can be obtained by combining results from literature D.J. Broadhurst; D.I. Kazakov; I. Bierenbaum, S. Weinzierl; K.G. Chetyrkin, P.B. Baikov; S. Bekavac

$\rightsquigarrow$ On remains with solving 38 master integrals:

Master integrals in the Standard Basis



II. Method: Master integrals with difference equations

S. Laporta

Raise one propagator to symbolic power x:

e. g. $M =$  $\rightarrow$ $M(x) =$  $= \int \frac{[dk_1]\ldots[dk_4]}{D_1^x D_2 \ldots D_8}$

Use IPB and Laporta alg. to construct difference equations

$$\sum_{j=0}^{R} p_j(d,x)\, M(x-j) = \text{combination of subtopologies of } M(x)$$

Ansatz: factorial series $M(x) = \mu^x \sum_{s=0}^{\infty} a_s \frac{\Gamma(x+1)}{\Gamma(x+1+s-K)}$

- recursion formula for a_s $\rightarrow$ sum up to specified $s_{max} \sim 1000 - 2000$
- better convergence for large x $\rightarrow$ use DE to get M(1)

Example result

$$
\begin{aligned}
&= 0.125000000000000000000000000000000000000000000000\,\epsilon^{-4} \\
&+ 1.083333333333333333333333333333333333333333333333\,\epsilon^{-3} \\
&+ 7.793062335049300303387513789073577220630038905408 7\,\epsilon^{-2} \\
&+ 42.42293112998116024698856959098483142365099866755 3\,\epsilon^{-1} \\
&+ 172.46863933841121267322728955028150963387029631037 \\
&+ 895.26722303450779233654593474697190309007417717856\,\epsilon \\
&+ 2851.24487745129920226056389168977151286787950149 53\,\epsilon^{2} \\
&+ \mathcal{O}(\epsilon^{3})
\end{aligned}
$$

- High numerical precision (usually $>$ 30 digits)
- However, construction of DEs increasingly tedious for masters with many lines

Method: ε-finite basis K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov

Problem: Coefficient functions c_{ij} in $I_i = \sum_j c_{ij} M_j$ can have "spurious" poles

Arise while solving IBP-identities through division by $(d-4)$.

$\rightsquigarrow$ Master integrals with spurious poles as coefficient need to be known in higher order in ε

But: Choice of master integrals is not unique

Idea: Select a new basis with finite coefficient functions

Solution: "ε-finite basis"

$\rightsquigarrow$ Advantage: Members need only be evaluated up to order ε^0

Method: Constructing an ε-finite basis

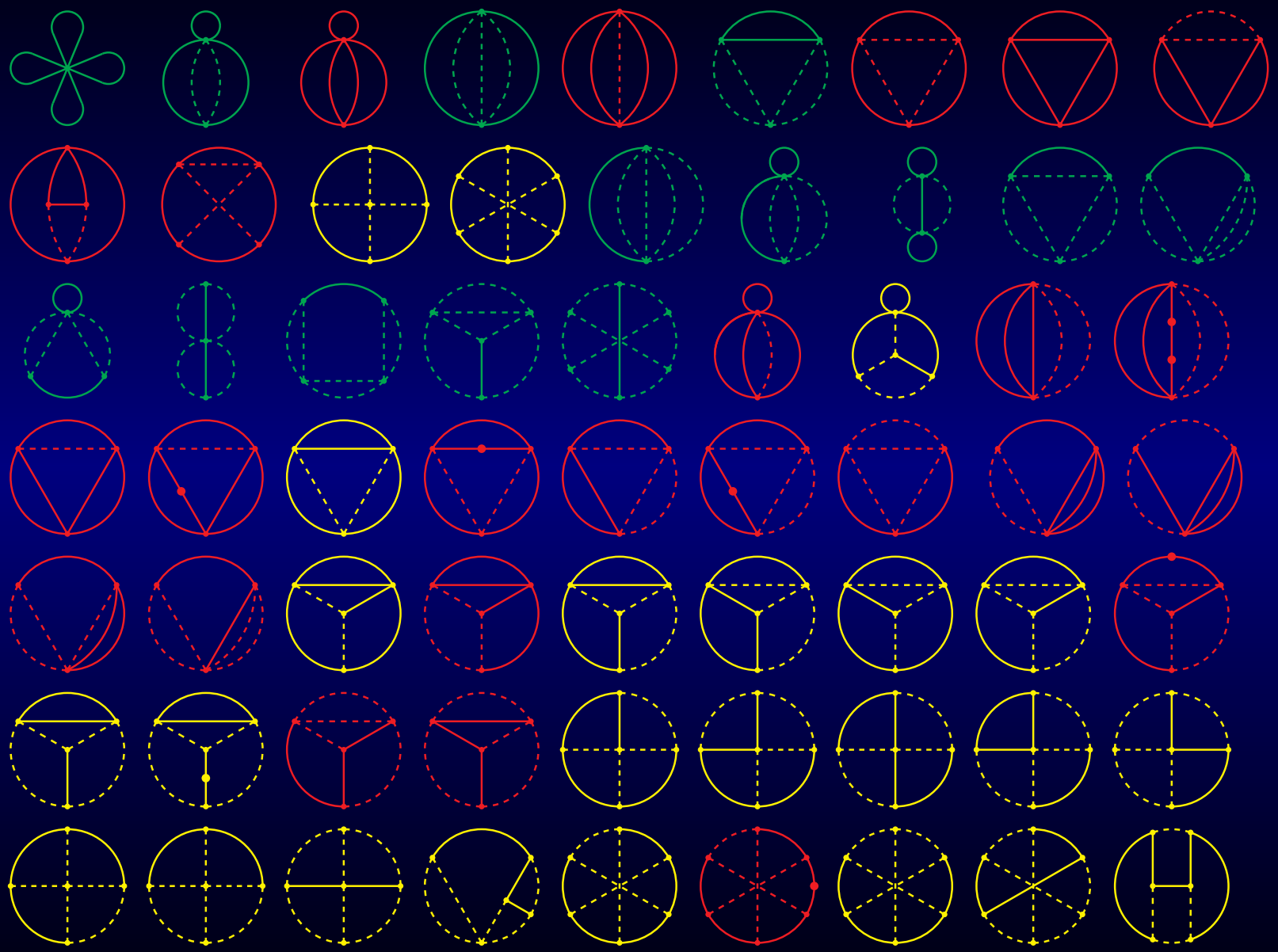
K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov

Members of ε-finite basis can be found among the set of initial integrals I_i

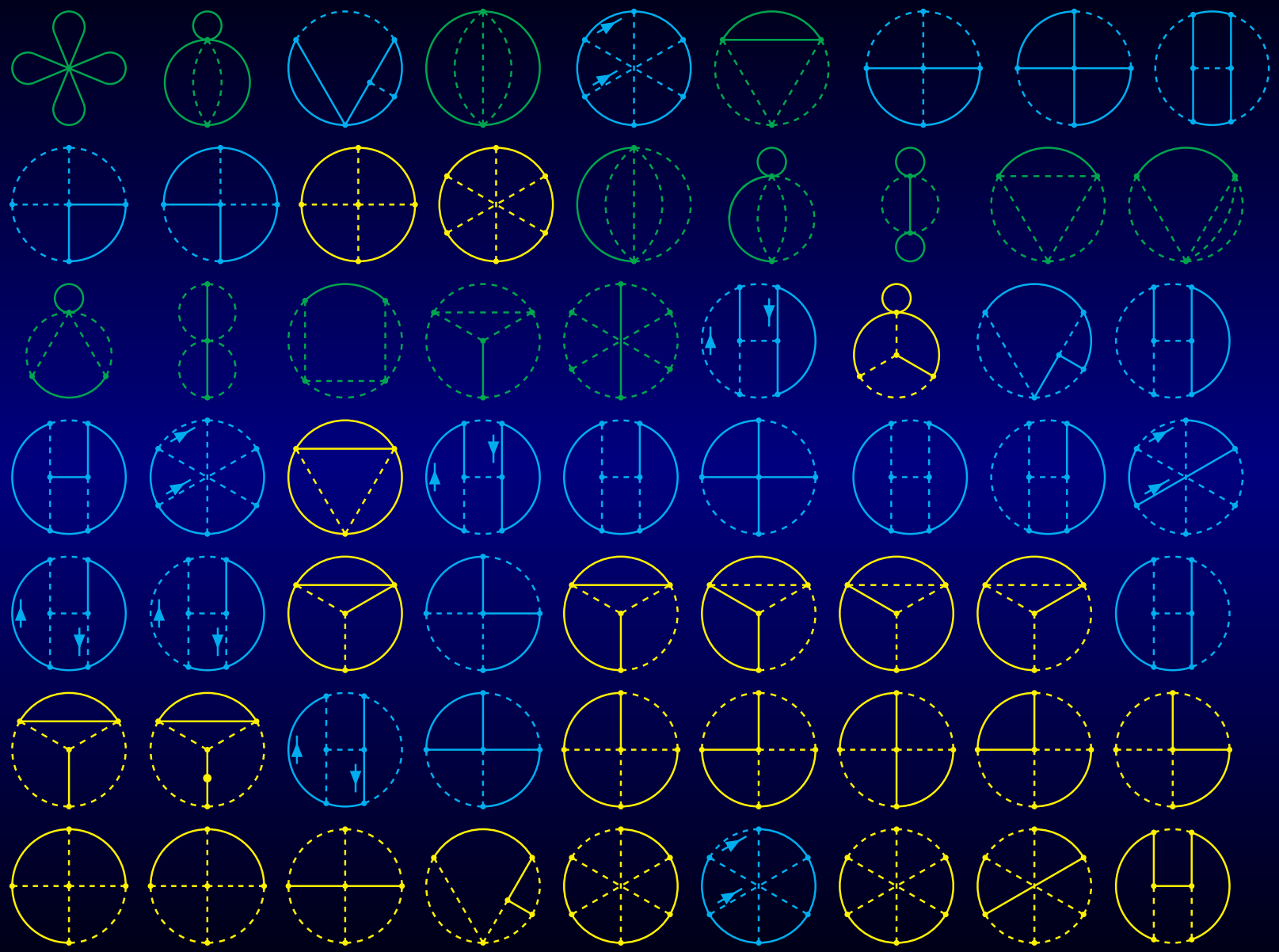
Algorithm:

1. Express all initial integrals in terms of standard masters $I_i = \sum_j c_{ij} M_j$
2. Choose equation with highest ϵ-pole in a coefficient c_{ij}
3. Solve it for M_j
   $\longrightarrow$ All coefficients in the new equation are finite
4. Replace M_j in all equations and treat I_i as new master integral
5. Repeat steps 2–4 until all coefficients are finite

Master integrals in the ε-finite Basis



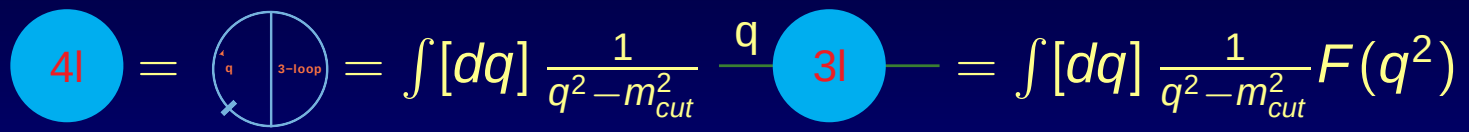
Master integrals in the ε-finite Basis



Method: Padé-method

Fleischer, Tarasov; Broadhurst, Baikov

<u>Idea</u>: Perform integration over 3 loop momenta
"semi-analytically" and the 4th numerically:



$$= \int [dq] \frac{1}{q^2 - m_{cut}^2} = \int [dq] \frac{1}{q^2 - m_{cut}^2} F(q^2)$$

- Perform low and high-energy expansion of $F(q^2)$
- Reconstruction of $F(q^2)$ through Padé-Approximation

$$F(q^2) \longrightarrow [i/j](q^2) = \frac{a_0 + a_1 q^2 \ldots a_n q^{2n}}{b_0 + b_1 q^2 \ldots b_m q^{2m}}$$

$[i/j](q^2)$: Same low- and high-energy behavior like $F(p^2)$

- Numerical integration over Padé-Approximation
- Choose convenient integrals for evaluation
  by exploiting freedom in choice of ϵ-finite basis
- Pole part analytically
  $\Rightarrow$ Allows analytical cancellation of (UV-)ϵ-poles

Results for master integrals

- Result for ε-finite master obtained through Padé-method:

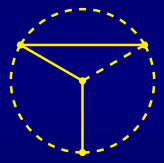
$$= \frac{5\zeta_5}{\varepsilon} - 24.8172810 + \mathcal{O}(\varepsilon)$$

- Get analytical information also for standard basis:

$$\boxed{\text{standard basis}} \xleftrightarrow{\text{IBP-relations}} \boxed{\varepsilon\text{-finite basis}}$$

$\rightsquigarrow$ Express standard basis through ε-finite one, compare coefficients of ε^n

- Combination of the two methods:



$$= \frac{1}{8\varepsilon^4} + \frac{13}{12\varepsilon^3} + \frac{1}{2\varepsilon^2}\left(\frac{143}{12} + \frac{\pi^2}{4} + \zeta_3\right) + \frac{1}{6\varepsilon}\left(\frac{317}{2} + \frac{43}{6}\pi^2 + \frac{\pi^4}{20} + 17\zeta_3\right)$$
$$+ \frac{2455}{24} + \frac{545}{72}\pi^2 + \frac{\pi^4}{90} + \frac{169}{18}\zeta_3 - \frac{\pi^2\zeta_3}{6} - \frac{59}{6}\zeta_5 + 2\sqrt{3}s_2 - 8s_2^2$$
$$+ 895.2672230345077923365459347469719030900741771785 63\varepsilon$$
$$+ 2851.2448774512992022605638916897715128678795014953 1\varepsilon^2$$

- "Special relations" among particular orders of different masters:

$$M^{(0)}_{6,13} - M^{(0)}_{6,18} = -\frac{11561}{128} - \frac{1685\pi^2}{288} - \frac{\pi^4}{4} + \frac{9\sqrt{3}s_2}{2} + \frac{166\zeta_3}{9}$$

Compute one (e.g. $M^{(0)}_{6,13}$) $\rightsquigarrow$ get one for free ($M^{(0)}_{6,18}$)

Four-loop Result

$$\delta\rho^{\overline{\mathrm{MS}}} = 3x_t\left(1 - \frac{\alpha_s}{\pi}\,0.19325 + \left(\frac{\alpha_s}{\pi}\right)^2\left(\underset{\sim\sim\sim}{-4.2072}\underline{+0.23764}\right) + \boxed{\left(\frac{\alpha_s}{\pi}\right)^3\left(\underset{\sim\sim\sim}{-3.2866}\underline{+1.6067}\right)}\right)$$

$\sim\sim\sim$: singlet, $\underline{\quad}$: non-singlet
4-loop singlet: Y. Schröder, M.Steinhauser

New result induces shift of the W-mass: $\sim 2\,\mathrm{MeV}$

Anticipated experimental precision of future colliders:

ILC: 6 MeV

$\Rightarrow$ theoretical uncertainty well below the experimental precision

Confirmed by an independent calculation R. Boughezal, M. Czakon

Four-loop Result

$$\delta\rho^{\mathrm{OS}} = 3X_t\Big(1 - \frac{\alpha_s}{\pi}\,2.8599 + \left(\frac{\alpha_s}{\pi}\right)^2(\underset{\sim\sim\sim}{-4.2072}\underline{-10.387}) + \boxed{\left(\frac{\alpha_s}{\pi}\right)^3(\underset{\sim\sim\sim}{+7.9326}\underline{-101.0827})}\Big)$$

$\sim\sim\sim$: singlet, $\underline{\quad}$: non-singlet

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Summary & Conclusion

- Four-loop contribution of $\mathcal{O}(G_F m_t^2 \alpha_s^3)$ from top- and bottom-quarks to the ρ-parameter in pQCD has been computed
- All appearing integrals have been reduced to master integrals
- The master integrals have been calculated using 2 different methods:
  Standard basis : Method of difference equations
  ε-finite basis : Padé-method
  New analytical information at least for the pole-part of the four-loop masters has been obtained
- The four-loop result induces a shift of 2 MeV into M_W
  $\Longrightarrow$ theoretical uncertainty well below the anticipated accuracy of M_W at ILC ($\delta M_W^{exp.} \sim$6 MeV)