Light quark mass effects in the on-shell renormalization constants

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June 2nd, 2007

Why on–shell renormalisation constants?

current and future experiments

high accuracy for quark masses needed

- Br$(B \to X e \bar{\nu}) \propto m_b^5$
- pole mass can only be extracted with ambiguity of order Λ_{QCD} $\rightsquigarrow$ Renormalons
- trade the pole–mass in for $\overline{\text{MS}}$–mass: still large perturbative corrections
- $\rightsquigarrow$ use "short distance" mass definitions:
  - potential subtracted mass [Beneke '98]
  - 1S mass [Hoang, Teubner '98]
  - kinetic mass [Bigi et al. '97]

Potential Subtracted Mass

perturbative series of Coulomb potential is better behaved in momentum space than in coordinate space

$\rightsquigarrow$ use cut-off for Fourier transformation

subtracted potential

$$\begin{aligned} V(r,\mu_f) &= V(r) + 2\,\delta m(\mu_f) \\ \delta m(\mu_f) &= -\frac{1}{2}\int\limits_{|q|<\mu_f} \frac{\mathrm{d}^3 q}{(2\pi)^3}\,\tilde{V}(q) \end{aligned}$$

$\rightsquigarrow$ potential subtracted mass

$$m_{\mathsf{PS}}(\mu_f) = M_{\mathsf{pole}} - \delta m(\mu_f)$$

Relation between PS and $\overline{\text{MS}}$ mass

$$
\begin{aligned}
m_{\text{PS}}(\mu_f) &= M_{\text{pole}} - \delta m(\mu_f) \\
&= \left(\frac{M_{\text{pole}}}{\bar{m}(\bar{m})}\right) \bar{m}(\bar{m}) - \delta m(\mu_f) \\
&= \bar{m}(\bar{m}) \left[1 + \frac{\alpha_s(\bar{m}(\bar{m}))}{\pi} C_F \left(1 - \frac{\mu_f}{\bar{m}(\bar{m})}\right) + \mathcal{O}\left(\alpha_s^2\right)\right]
\end{aligned}
$$

- large perturbative corrections in $\frac{M_{\text{pole}}}{\bar{m}}$ cancel with $\delta m(\mu_f)$
  $\rightsquigarrow$ precise determination of $\overline{\text{MS}}$ mass
- relation between pole and $\overline{\text{MS}}$ mass is necessary

Outline

Renormalized propagator

$$S_F(q) = \frac{-i Z_2^{\mathrm{OS}}}{\not{q} - m_{q,0} + \Sigma(q, M_q)} \xrightarrow{q^2 \to M_q^2} \frac{-i}{\not{q} - M_q}$$

with

$$m_{q,0} = Z_m^{\mathrm{OS}} M_q \,, \qquad \psi_0 = \sqrt{Z_2^{\mathrm{OS}}}\, \psi$$

$$\Sigma(q, m_q) = M_q \, \Sigma_1(q^2, M_q) + (\not{q} - M_q)\, \Sigma_2(q^2, M_q)$$

- Z_m^{OS} is IR–finite, gauge–invariant quantity, Z_2^{OS} is not

Mass relation

$$m_{q,0} = Z_m^{\overline{\mathrm{MS}}} \bar{m}_q \quad \longrightarrow \quad \frac{Z_m^{\mathrm{OS}}}{Z_m^{\overline{\mathrm{MS}}}} = \frac{\bar{m}_q(\mu)}{M_q}$$

- no ϵ–poles left

Computation of the renormalization constants

- introduce 4–vector Q with $q = Q(1+t)$ and $Q^2 = M_q^2$

Projector

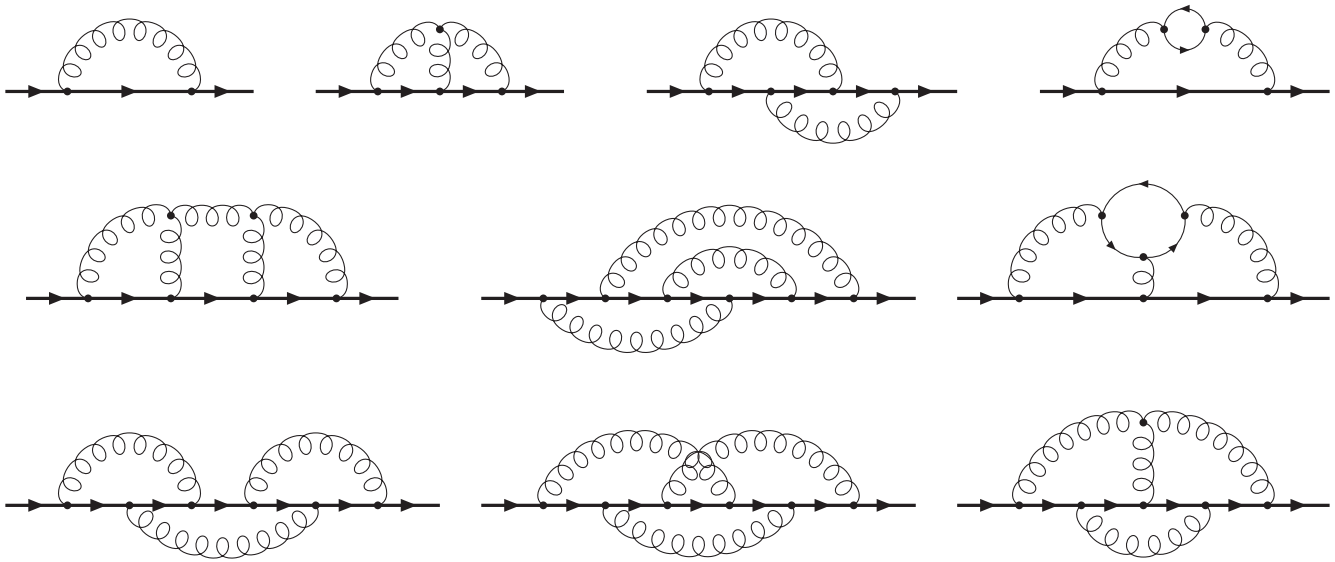
$$\mathrm{Tr}\left\{\frac{\not{Q} + M_q}{4M_q^2}\Sigma(q, M_q)\right\} =$$

$$\underbrace{\Sigma_1(M_q^2, M_q)}_{Z_m^{\mathrm{OS}}-1} + \Bigg(\underbrace{2M_q^2 \frac{\partial}{\partial q^2}\Sigma_1(q^2, M_q)\Big|_{q^2=M_q^2} + \Sigma_2(M_q^2, M_q)}_{(Z_2^{\mathrm{OS}})^{-1}-1}\Bigg)t$$

- Renormalisation: insertion of mass counterterms $\propto M_q\Sigma_1(M_q^2, M_q)$ in lower order diagrams

On-Shell Quark Mass and Wave Function Renormalisation

- necessary input for (multi-) loop calculations
- Z_m determined from on-shell self-energy diagrams
- Z_2 determined from derivative of self-energies



$\rightsquigarrow$ subset of on-shell self energy diagrams

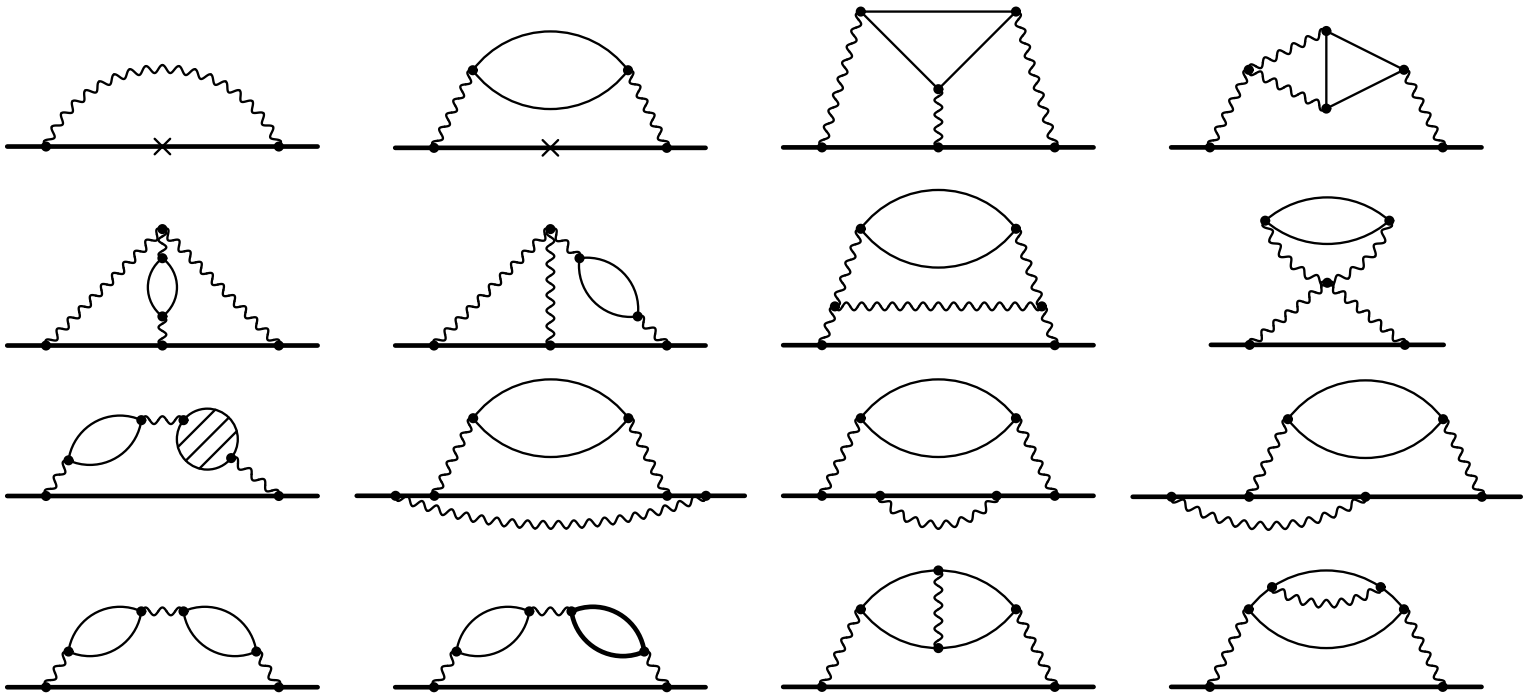
Dimensional Regularisation['t Hooft, Veltman '72]

- complex number of dimensions: $d = 4 - 2\epsilon$
- regulates UV and IR divergencies

Z_m and Z_2 are known up to $\mathcal{O}\left(\alpha_s^3\right)$ in massless approx.

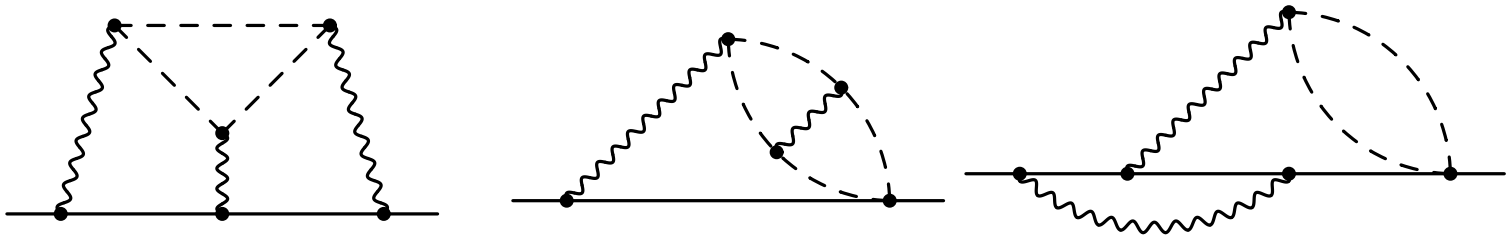
- $\mathcal{O}\left(\alpha_s\right)$: Z_m [Tarrach '80]
- $\mathcal{O}\left(\alpha_s^2\right)$:
  - Z_m [Gray, Broadhurst, Grafe, Schilcher '90]
  - Z_m and Z_2 [Broadhurst, Gray, Schilcher '91]
- $\mathcal{O}\left(\alpha_s^3\right)$:
  - Z_m: semi-numerical [Chetyrkin, Steinhauser '99, '00], estimation of charm–mass effects [Hoang '00]
  - Z_m and Z_2: analytical [Melnikov, van Ritbergen '00]
  - independent confirmation of analytical results [Marquard, Mihaila, Piclum, Steinhauser '07]

Diagrams at $\mathcal{O}\left(\alpha_s^3\right)$ with nonzero light quark mass



- great challenge: two–scale three loop diagrams

Generic Topologies



- after taking trace: scalar integrals
- Z_2 includes derivative of self–energy and is gauge–dependent
  $\rightsquigarrow$ up to 5 dots and 5 powers of scalar products needed

Integration By Parts

$$\int d^{3d}\ell_{1,2,3}\,\frac{d}{d\ell_i^\mu}\,v^\mu\,I\left(q^2=M_q^2,M_q,M_f\right)=0$$

- use relations to reduce integrals to master integrals

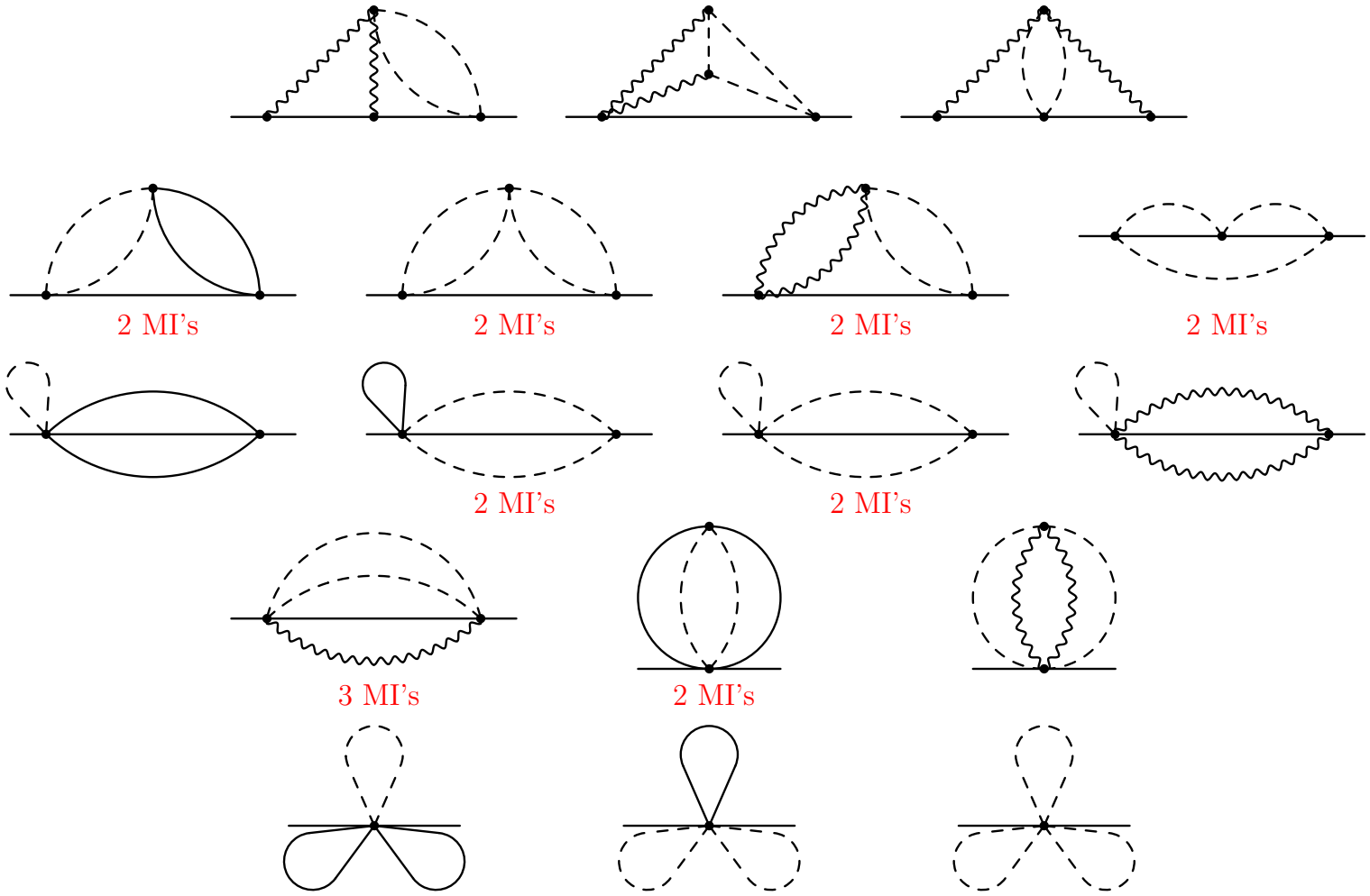
Feynman diagrams

- generated with `QGRAF` [Nogueira '91]
- various topologies are identified with
  `q2e` and `exp` [Harlander '97, Seidensticker '99]

Laporta Algorithm [Laporta '96]

- `Crusher`: Implementation written in C++ [Marquard, DS '06]
- uses `GiNaC` for simple manipulations
- coefficient simplification done with `Fermat`
  $\rightsquigarrow$ interface from [Tentyukov '06]
- automated generation of the IBP identities
- complete symmetrization of the diagrams
- use of multiprocessor environment

Master Integrals



Calculation of Master Integrals

- two independent methods used

Mellin–Barnes

$$\frac{1}{(K-M)^\lambda} = \frac{1}{(K)^\lambda}\frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_\gamma ds \left(-\frac{M}{K}\right)^s \Gamma(-s)\Gamma(\lambda+s)$$

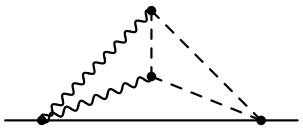
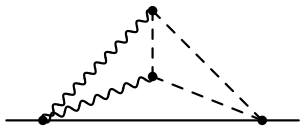
- trade massive propagator for massless one
- simplify Feynman integral representations

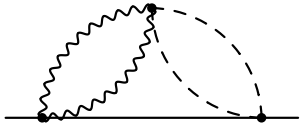
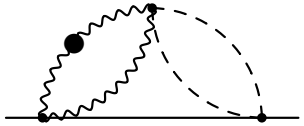
$\rightsquigarrow$ at most 4-dimensional representation for complicated integrals (calculated with `MB.m` [Czakon '05])

- partially checked with `AMBRE` [Gluza, Kajda, Riemann '07]

Differential Equation Method

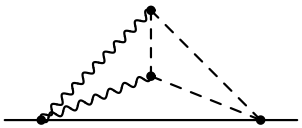
$$\frac{d}{dz}\int d^{3d}\ell_i \frac{1}{\left(\ell_1^2 - z^2\right)\dots} = 2z\int d^{3d}\ell_i \frac{1}{\left(\ell_1^2 - z^2\right)^2\dots} + \dots$$

$\frac{d}{dz}$  $= \frac{(d-4)}{z}$ 

$-\frac{2(d-3)}{M_q^2(z-1)z(z+1)}$  $-\frac{4(d-4)}{(d-3)(z-1)z(z+1)}$ 

$+\dots$

- homogenious part trivial
- solution expressable in terms of "standard" HPL's
- in principle up to any desired order in ϵ
- initial conditions known from $n_m = 0$ calculation

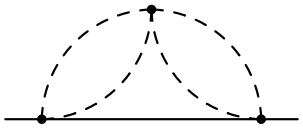


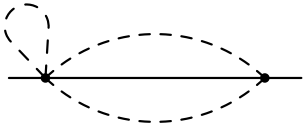
$$= -\frac{2\mathrm{Zeta}(3)}{\epsilon}$$

$$+\frac{4}{3}\pi^2\mathrm{HPL}(\{0\},z)z^2 - 12\mathrm{HPL}(\{0,0\},z)z^2 + 16\mathrm{HPL}(\{0,0,0\},z)z^2$$
$$-\pi^2(z-2)z + \frac{10}{3}\pi^2\mathrm{HPL}(\{-2\},z) - \frac{5}{3}\pi^2\left(z^2-1\right)\mathrm{HPL}(\{-1\},z)$$
$$-\frac{1}{3}\pi^2\left(z^2-1\right)\mathrm{HPL}(\{1\},z) + \frac{2}{3}\pi^2\mathrm{HPL}(\{2\},z) - 8\mathrm{HPL}(\{-3,0\},z)$$
$$-4\left(z^2+1\right)\mathrm{HPL}(\{-2,0\},z) + 2(z+1)(3z-1)\mathrm{HPL}(\{-1,0\},z)$$
$$+\left(-6z^2+4z+2\right)\mathrm{HPL}(\{1,0\},z) + 4\left(z^2+1\right)\mathrm{HPL}(\{2,0\},z) + 8\mathrm{HPL}(\{3,0\},z)$$
$$+16\mathrm{HPL}(\{-2,0,0\},z) + 8\mathrm{HPL}(\{-2,1,0\},z) + \left(8-8z^2\right)\mathrm{HPL}(\{-1,0,0\},z)$$
$$+\left(4-4z^2\right)\mathrm{HPL}(\{-1,1,0\},z) + \left(4-4z^2\right)\mathrm{HPL}(\{1,-1,0\},z)$$
$$+8\left(z^2-1\right)\mathrm{HPL}(\{1,0,0\},z) + 8\mathrm{HPL}(\{2,-1,0\},z) - 16\mathrm{HPL}(\{2,0,0\},z)$$
$$+6\left(z^2-2\right)\zeta(3) - \frac{\pi^4}{30}$$

- Simplification and numerical evaluation done with `HPL.m` [Maitre '05,'07]

no analytic solution for 2 topologies

$\frac{d}{dz}$  $= \ldots$

$$+\frac{4}{M_q^2}\left(\frac{2}{z}+\frac{1}{z+1}-\frac{4}{2z-1}-\frac{4}{2z+1}+\frac{1}{z-1}\right)$$ 

$+\ldots$

- "wrong" pole structure $\rightsquigarrow$ no transformation found
- analytic results up to ϵ^{-1}
- Mellin–Barnes: 1-dimensional sums left after z–expansion

$\rightsquigarrow$ sufficient numerical precision for phenomenology

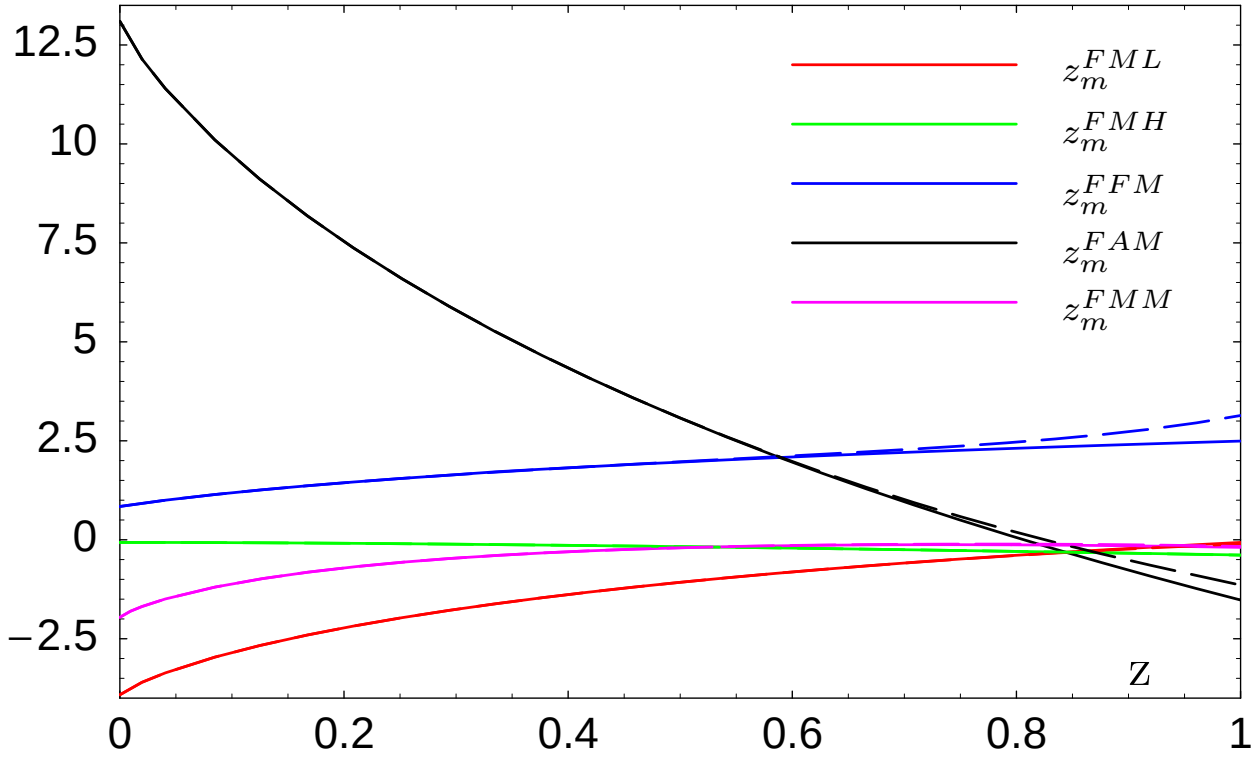
Mass relation

$$z_m(\mu) = \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}} = \frac{\bar{m}_q(\mu)}{M_q} = 1 + \frac{\alpha_s(\mu)}{\pi}\,\delta z_m^{(1)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \delta z_m^{(2)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \delta z_m^{(3)} + \mathcal{O}\left(\alpha_s^4\right)$$

$$\delta z_m^{(2)} = C_F^2\, z_m^{FF} + C_F C_A\, z_m^{FA} + C_F T_F n_l z_m^{FL} + C_F T_F n_h z_m^{FH} + C_F T_F n_m z_m^{FM}$$

$$\begin{aligned}\delta z_m^{(3)} &= C_F^3\, z_m^{FFF} + C_F^2 C_A\, z_m^{FFA} + C_F C_A^2\, z_m^{FAA} \\ &+ C_F T_F n_l \left(C_F\, z_m^{FFL} + C_A\, z_m^{FAL} + T_F n_l\, z_m^{FLL} + T_F n_h\, z_m^{FHL} + T_F n_m\, z_m^{FML}\right) \\ &+ C_F T_F n_h \left(C_F\, z_m^{FFH} + C_A\, z_m^{FAH} + T_F n_h\, z_m^{FHH} + T_F n_m\, z_m^{FMH}\right) \\ &+ C_F T_F n_m \left(C_F\, z_m^{FFM} + C_A\, z_m^{FAM} + T_F n_m\, z_m^{FMM}\right)\end{aligned}$$

- z_m^i containing factor n_m depend on ratio of OS-quark masses $z = M_f/M_q$
- all z_m^i depend on $\ln \mu/M_q$



- preliminary results at $\mu = M_q$
- known results at $z = 0, 1$ reproduced
- z_m^{FAM} enhanced by color factor

- last missing piece of 3–loop $\overline{\text{MS}}$–on–shell relation calculated
- Laporta–implementation capable of dealing with bigger problems
- combination of analytical and numerical results sufficient for phenomenological treatment

- results for Z_2 have to be worked out
- mass shift in b–mass determination