The Georgi Algorithms of Jet Clustering

Shao-Feng Ge

(gesf02@gmail.com)

KEK Theory Center

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Howard Georgi, arXiv:1408.1161
SFG, arXiv:1408.3823

Conventional Jet Algorithms [implemented in FastJet]

- **Longitudinally invariant k_t algorithm**

$$d_{ij} = \min(p_{ti}^2, p_{tj}^2)\Delta R_{ij}^2/R^2\,.$$

- **Cambridge/Aachen (C/A) algorithm**

$$d_{ij} = \Delta R_{ij}^2/R^2\,, \qquad d_{iB} = 1\,.$$

- **Anti-k_t algorithm**

$$d_{ij} = \min\left(1/p_{ti}^2, 1/p_{tj}^2\right)\Delta R_{ij}^2/R^2, \qquad d_{iB} = 1/p_{ti}^2\,.$$

- **Generalized k_t algorithm**

$$\begin{aligned} d_{ij} &= \min(E_i^{2p}, E_j^{2p})(1-\cos\theta_{ij})/(1-\cos R)\,, \\ d_{iB} &= E_i^{2p}\,. \end{aligned}$$

- **Durham (e^+e^- k_t) algorithm**

$$d_{ij} = 2\times\min(E_i^2, E_j^2)(1-\cos\theta_{ij})\,.$$

The Georgi Algorithms [arXiv:1408.1161]

Jet function:

$$J_\beta(P_\alpha) \equiv E_\alpha - \beta \frac{P_\alpha^2}{E_\alpha} = E_\alpha \left[(1-\beta) + \beta v_\alpha^2 \right] ,$$

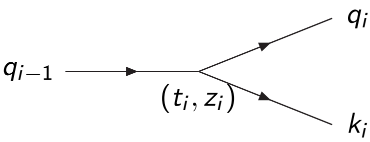
with **jet momentum** $P_\alpha = (E_\alpha, \mathbf{P}_\alpha) \equiv \sum_{i \in \alpha} p_i$ & **velocity** $v_\alpha \equiv \frac{|\mathbf{P}_\alpha|}{E_\alpha}$ where α is a set of subjets.

- J_β **increases** when clustering:
  - E_α increases due to energy conservation;
  - Jet virtuality (mass) P_α^2 doesn't increase that much.
- Not only **pair-wisely**, but also **globally**.
- **Cone** implemented implicitly:

$$J_\beta(P_\alpha + p_j) = (E_\alpha + E_j) \left[(1-\beta) + \beta \frac{|\mathbf{P}_\alpha|^2 + 2|\mathbf{P}_\alpha||\mathbf{p}_j|\cos\theta + |\mathbf{p}_j|^2}{(E_\alpha + E_j)^2} \right]$$

Link to Parton Shower

- $1 \to 2$ **Splitting:**



- **Sudakov Basis:**

$$q_i \equiv \alpha_i p + \frac{q_i^2 - P_{\perp i}^2}{2\alpha_i} n + P_{\perp i} \,,$$

where p & n are light-like, $p \cdot P_{\perp i} = n \cdot P_{\perp i} = 0$ & $p \cdot n = 2$.

- **Energy fraction:**

$$z_i \equiv \frac{\alpha_i}{\alpha_{i-1}} \approx \frac{E_i}{E_{i-1}} \,.$$

Link to Parton Shower

- **Sudakov Factor** for Parton $i = q, g$:

$$\mathbf{\Delta_i(t)} = \exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int_{z_{min}(t')}^{z_{max}(t')} dz \sum_j \mathbf{P_{ji}}(t', z)\right] .$$

- **Analogy to Decay:**

$$\mathbb{P}(\mathbf{t}) = e^{-t\Gamma} = \exp\left(-\int_0^t dt'\Gamma\right) = \exp\left(-\int_0^t dt' \int d\Omega_2 |\mathcal{M}|^2\right) .$$

  - **Survival Probability:**

$$\mathbb{P}(\mathbf{t}) = e^{-t\int_0^t dt'\Gamma} \sim \mathbf{\Delta_i(t)} .$$

  - **Decay Width:**

$$\Gamma(t) \equiv \int d\Omega_2 |\mathcal{M}|^2 \sim \int_{z_{min}(t)}^{z_{max}(t)} dz \sum_j P_{ji}(t, z) .$$

  - **Energy fraction:** $|\mathcal{M}|^2 \sim \sum_j \mathbf{P_{ji}}(t, z).$

Link to Parton Shower

- **Splitting Functions:**
  - $q \to qg$:

$$P_{qq}(z) = C_F \frac{1+(1-z)^2}{z}\,.$$

  - $g \to gg$:

$$P_{gg}(z) = C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right]\,.$$

  - $g \to qq$:

$$P_{qg}(z) = T_R \left[z^2 + (1-z)^2\right]\,.$$

- **Stopping Conditions:**
  - Perturbativity for $\alpha_s[z^2(1-z)^2 t] \Rightarrow \mathbf{z^2(1-z)^2 t > \Lambda^2}$.
  - **Larger phase space @ higher scale:**

$$\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\Lambda}{\sqrt{t}}} < z < \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\Lambda}{\sqrt{t}}}\,.$$

Link to Parton Shower

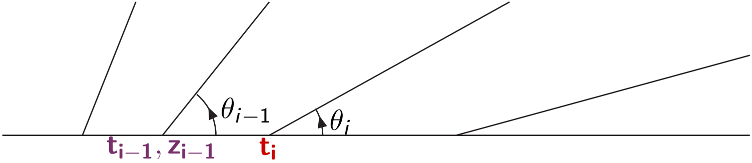
- Tends to emit one **soft parton**,

$$z \to 0\,.$$

- Soft parton takes **less fraction of energy @ higher scale**.

$$\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\Lambda}{\sqrt{t}}} < z < \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\Lambda}{\sqrt{t}}}\,.$$

- **Angular ordering**



$$\theta_i \approx \frac{t_i}{2\alpha_i^2} = \frac{t_i}{2z_i^2\alpha_{i-1}^2}, \quad \theta_i < \theta_{i-1} \quad \Rightarrow \quad \mathbf{t_i} < (\mathbf{1} - \mathbf{z_{i-1}})^2 \mathbf{t_{i-1}}\,.$$

Link to Parton Shower

Consider the clustering of a soft p_j with $P_{\alpha-j} \to P_\alpha$:

$$J_\beta(P_{\alpha-j}) \quad \Rightarrow \quad J_\beta(P_\alpha) \equiv \mathbf{E}_{\boldsymbol{\alpha}} - \beta \frac{\mathbf{P}_{\boldsymbol{\alpha}}^2}{\mathbf{E}_{\boldsymbol{\alpha}}} = E_\alpha \left[(1-\beta) + \beta v_\alpha^2 \right] ,$$

- **Energy increase is z:**

$$z = \frac{E_j}{E_\alpha} = \frac{\mathbf{E}_{\boldsymbol{\alpha}} - E_{\alpha-j}}{E_\alpha} .$$

- **Jet virtuality (mass) increase is less than z:**
  Virtualtiy Reconstruction:

$$P_\alpha^2 = \frac{p_j^2}{z} + \frac{P_{\alpha-j}^2}{1-z} + z(1-z)t \approx \frac{P_{\alpha-j}^2}{1-z} + z(1-z)t ,$$

$\Rightarrow$

$$\frac{1}{E_\alpha} \left[\frac{\mathbf{P}_{\boldsymbol{\alpha}}^2}{\mathbf{E}_{\boldsymbol{\alpha}}} - \frac{P_{\alpha-j}^2}{E_{\alpha-j}} \right] \approx z(1-z) \frac{t}{E_\alpha^2} .$$

Generalized Georgi Algorithms

- The **original** jet function:

$$J_\beta(P_\alpha) \equiv E_\alpha - \beta \frac{P_\alpha^2}{E_\alpha} = E_\alpha \left[(1-\beta) + \beta v_\alpha^2 \right] ,$$

  - E_α increases fast;
  - v_α decreases a little bit.
  - For positive J_β,

$$1 \geq v_\alpha^2 \geq 1 - \frac{1}{\beta} \equiv v_{min}^2 .$$

  $\boldsymbol{\beta \geq 1}$ to ensure a positive v_{min}^2, which is **NOT necessary**.

- **Generalized** version:

$$J_\beta^{(n)}(P_\alpha) \equiv E_\alpha^n \left[(1-\beta) + \beta v_\alpha^2 \right] , \quad \mathbf{n > 0}$$

  - $J_\beta = J_\beta^{(1)}$.
  - As long as n is not too small, $J_\beta^{(n)}$ can increase.
  - More d.o.f for kinematics.

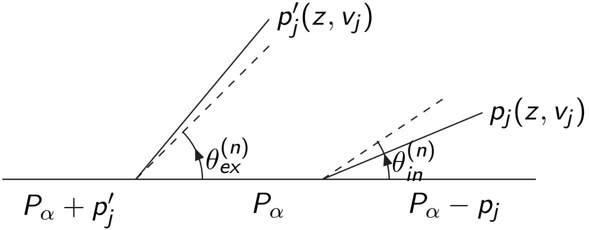
Kinematic Properties

- **Cone should't shrink**, otherwise:
  - Not self-consistent.
    - For 2 subjets $\mathbf{p_j}(E, |\mathbf{p}|, \theta)$ & $\mathbf{p'_j}(E, |\mathbf{p}|, \theta)$;
    - The 1st subjet p_j is within the cone & can be clustered;
    - Cone shrinks;
    - The 2nd subjet p'_j may be excluded;
    - vice versa
    - The result of jet clustering is procedure-dependent.
  - Cannot comply with angular ordering.
- **Larger cone for smaller z**:
  - Soft emission tends to have larger openning angle.

$$\theta_i \approx \frac{t_i}{2\alpha_i^2} = \frac{t_i}{2z_i^2\alpha_{i-1}^2}\,.$$

- **Cone is bounded from above**:
  - To avoid mixing up different jets.
  - For $e^+e^- \to jj$, cone $<$ half sphere.

Property 1: Cone shouldn't shrink



- **Inclusion cone**:

$$J^{(n)}_\beta(P_\alpha) > \max\left\{ J^{(n)}_\beta(P_\alpha - p_j), J^{(n)}_\beta(p_j) \right\} .$$

- **Exclusion cone**:

$$J^{(n)}_\beta(P_\alpha) > \max\left\{ J^{(n)}_\beta(P_\alpha + p'_j), J^{(n)}_\beta(p'_j) \right\} .$$

Property 1: Cone shouldn't shrink

When **expanded**, the conditions on $\boldsymbol{\theta^{(n)}_{in}}$ $(-)$ & $\boldsymbol{\theta^{(n)}_{ex}}$ $(+)$,

$$\bar\beta + \beta v_\alpha^2 > z^n \left[\bar\beta + \beta v_j^2 \right] , \qquad \bar\beta \equiv 1 - \beta \, , z_\pm \equiv 1 \pm z .$$

$$\bar\beta + \beta v_\alpha^2 > z_\mp^n \left[\bar\beta + \frac{\beta}{z_\mp^2} (v_\alpha^2 + z^2 v_j^2 \mp 2z \cos\theta v_\alpha v_j) \right] ,$$

- The **1st** is **phase space broadening**:

$$v_\alpha^2 - v_{min}^2 > z^n \left(v_j^2 - v_{min}^2 \right) \ .$$

- The **2nd**:

$$\cos \theta^{(n)}_{in} \equiv \frac{\left(1 - z_-^{2-n}\right) \left(v_\alpha^2 - v_{min}^2\right) + z^2 \left(v_j^2 - v_{min}^2\right) + 2 z v_{min}^2}{2 z v_\alpha v_j} ,$$

$$\cos \theta^{(n)}_{ex} \equiv \frac{\left(z_+^{2-n} - 1\right) \left(v_\alpha^2 - v_{min}^2\right) - z^2 \left(v_j^2 - v_{min}^2\right) + 2 z v_{min}^2}{2 z v_\alpha v_j} .$$

Property 1: Cone shouldn't shrink

- **Distance** between inclusion & exclusion cones:

$$\cos\theta^{(n)}_{in} - \cos\theta^{(n)}_{ex} = \frac{2 - z_-^{2-n} - z_+^{2-n}}{2 z v_\alpha v_j}(v_\alpha^2 - v_{min}^2) + \frac{z}{v_\alpha v_j}(v_j^2 - v_{min}^2)$$

To ensure self-consistency, $\mathbf{n} \leq \mathbf{2}$.

- **Expansion** for $z \to 0$,

$$\cos\theta^{(n)}_{in} - \cos\theta^{(n)}_{ex} \approx -\frac{(2-n)(1-n)(v_\alpha^2 - v_{min}^2) + 2(v_j^2 - v_{min}^2)}{2 v_\alpha v_j} z$$

Together with **phase space broadening**,

$$\left[\frac{1}{z^n} - \frac{(2-n)(1-n)}{2}\right](v_\alpha^2 - v_{min}^2) \geq 0\,,$$

which is always true for $\mathbf{n} \geq \mathbf{0}$.

Property 2: Larger cone for smaller z

$$\cos\theta^{(n)}_{in} \equiv \frac{\left(1 - z_-^{2-n}\right)\left(v_\alpha^2 - v_{min}^2\right) + z^2\left(v_j^2 - v_{min}^2\right) + 2zv_{min}^2}{2zv_\alpha v_j},$$

$$\cos\theta^{(n)}_{ex} \equiv \frac{\left(z_+^{2-n} - 1\right)\left(v_\alpha^2 - v_{min}^2\right) - z^2\left(v_j^2 - v_{min}^2\right) + 2zv_{min}^2}{2zv_\alpha v_j}.$$

- **Mirror Symmetry**:

$$\cos\theta^{(n)}_{ex}(z) = \cos\theta^{(n)}_{in}(-z)\,.$$

- **Property 1 = Property 2**:

$$\cos\theta^{(n)}_{ex} - \cos\theta^{(n)}_{in} > 0 \quad \Rightarrow \quad \theta^{(n)}_{in}(z) \text{ increases with z}$$

Property 3: Cone is bounded from above

- The **largest opening angle** is associated with the **first emission** & usually the **smallest** z.
- In the **expanded** form,

$$\cos\theta_{ex}^{(n)} \approx \frac{1}{2v_\alpha v_j}\left[(2-n)(v_\alpha^2 - v_{min}^2) + 2v_{min}^2\right] + \frac{1}{2v_\alpha v_j}\left[\frac{(2-n)(1-n)}{2}(v_\alpha^2 - v_{min}^2) - (v_j^2 - v_{min}^2)\right] z\,.$$

There is an **upper limit** if **$1 \leq n \leq 2$**,

$$\cos\theta_{ex}^{(n)} \gtrsim \frac{1}{2v_\alpha v_j}\left[(2-n)(v_\alpha^2 - v_{min}^2) + 2v_{min}^2\right] + \frac{1}{2v_\alpha v_j}\left[\frac{(2-n)(1-n)}{2}(v_\alpha^2 - v_{min}^2) - (v_j^2 - v_{min}^2)\right] \left(\frac{v_\alpha^2 - v_{min}^2}{v_j^2 - v_{min}^2}\right)^{\frac{1}{n}}\,.$$

Property 3: Cone is bounded from above

- There is an **upper limit** if $\mathbf{1 \leq n \leq 2}$,

$$\cos\theta^{(n)}_{ex} \gtrsim \frac{1}{2v_\alpha v_j}\left[(2-n)(v^2_\alpha - v^2_{min}) + 2v^2_{min}\right] + \frac{1}{2v_\alpha v_j}\left[\frac{(2-n)(1-n)}{2}(v^2_\alpha - v^2_{min}) - (v^2_j - v^2_{min})\right] \left(\frac{v^2_\alpha - v^2_{min}}{v^2_j - v^2_{min}}\right)^{\frac{1}{n}} .$$

- In the relativistic limit $\mathbf{v_\alpha \approx v_j \approx 1}$,

$$2\sin^2\left(\frac{1}{2}\theta^{(n)}_{ex}\right) \lesssim \frac{n(5-n)}{4}\frac{1}{\beta}, \quad \Rightarrow \quad \boldsymbol{\beta} > \frac{\mathbf{4}}{\mathbf{n(5-n)}} .$$

- β & n v.s. **phase space boundary/maximal cone size**.
  - $n = 1$ (original), $\beta > 1$, $v^2_{min} > 0$.
  - $1 < n \leq 2$ (generalized), $v^2_{min} < 0$.

Lorentz Invariance

$$J^{(n)}_{\beta}(P_\alpha) \equiv E^n_\alpha \left(1 - \beta \frac{P^2_\alpha}{E^2_\alpha}\right) .$$

- Jet virtuality P^2_α is invariant.
- Change only comes from jet energy,

$$E_\alpha \to \gamma_B E_\alpha$$

- The effect of boost on the dimensionless part can be compensated by,

$$\beta \to \gamma^2_B \beta$$

  jet by jet.
- Jet function should be redefined,

$$J^{(n)}_{\beta} \to \gamma_B^{-n} J^{(n)}_{\gamma^2_B \beta}$$

  to retain the jet function value.
- **Jet sequence not affected!**

Summary

$$J_{\beta}^{(n)}(P_{\alpha}) \equiv E_{\alpha}^{n}\left[(1-\beta)+\beta v_{\alpha}^{2}\right] .$$

- J_β **increases** when clustering:
  - E_α increases due to energy conservation;
  - Jet virtuality P_α^2 doesn't increase that much.
- Not only **pair-wise**, but also can be defined **globally**.
- **Cone** implemented implicitly:

$$J_{\beta}^{(n)}(P_{\alpha}+p_{j}) = (E_{\alpha}+E_{j})^{n}\left[\bar{\beta}+\beta\frac{|\mathbf{P}_{\alpha}|^{2}+2|\mathbf{P}_{\alpha}||\mathbf{p}_{j}|\cos\boldsymbol{\theta}+|\mathbf{p}_{j}|^{2}}{(E_{\alpha}+E_{j})^{2}}\right]$$

- **Kinematic Properties**:
  - Cone shouldn't shrink;
  - Larger cone for smaller z;
  - Cone is bounded from above.
- **Parameter space**: $\mathbf{1 \leq n \leq 2}$, $\boldsymbol{\beta > 4/n(5-n)}$.
- Lorentz invariance.

Thank You!