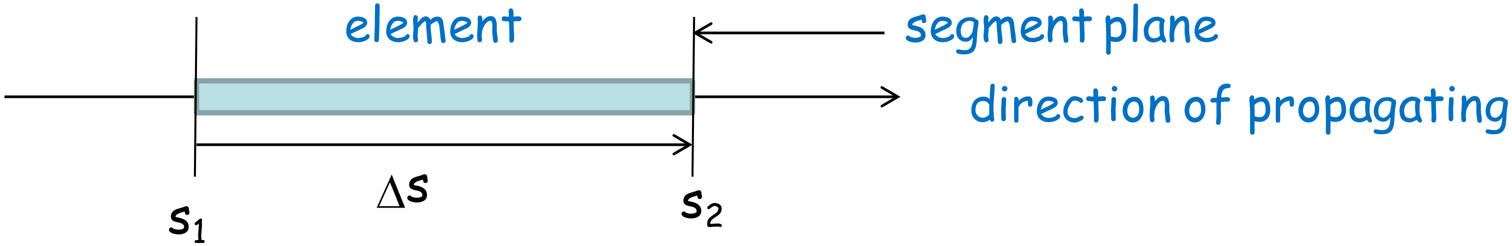
Lecture 2: Lie Algebra Method and Nonlinear Dynamics

Yunhai Cai
FEL & Beam Physics Department
SLAC National Accelerator Laboratory

November 6-17, 2011

6th International Accelerator School for Linear Collider, Pacific Grove, California, USA

Concept of Transfer Map



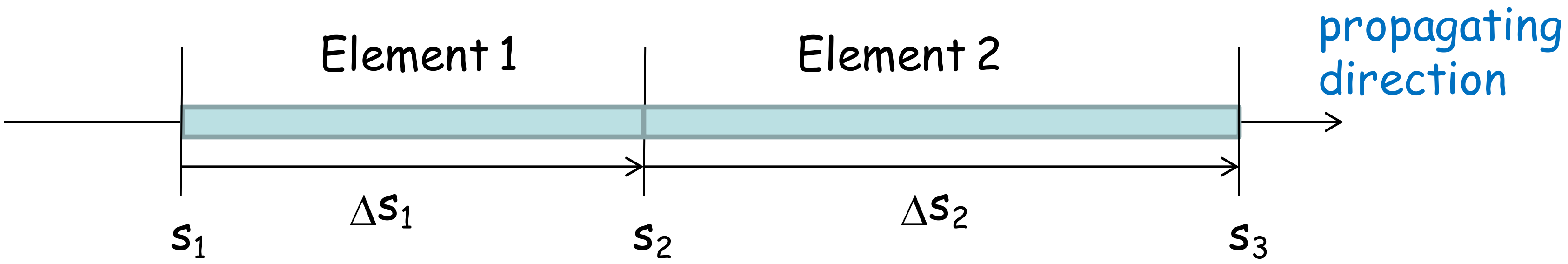
$$z(s_1) = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ \ell \end{pmatrix}_{|s_1} \quad \mathcal{M}_{1\to 2} \quad \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ \ell \end{pmatrix}_{|s_2} = z(s_2)$$

$$z(s_2) = \mathcal{M}_{1\to 2}(z(s_1)).$$

abbreviated map notation

A set (six) of functions of canonical coordinates. It's called symplectic if its Jacob is symplectic.

Concatenation of Maps



If we have the transfer map for each individual elements:

$$z(s_2) = \mathcal{M}_{1\to 2}(z(s_1)),$$

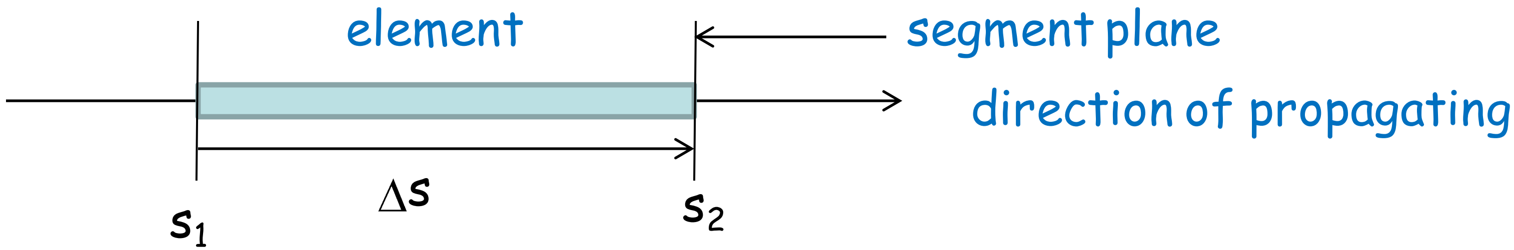
$$z(s_3) = \mathcal{M}_{2\to 3}(z(s_2)).$$

Then the transfer map for the combined elements is given by

$$z(s_3) = \underbrace{\boxed{\mathcal{M}_{1\to 2} \circ \mathcal{M}_{2\to 3}}}_{\mathcal{M}_{1\to 3}}(z(s_1)) \equiv \mathcal{M}_{2\to 3}(\mathcal{M}_{1\to 2} z(s_1)),$$

nested functions

Taylor Series and Exponential Lie Operator



For any function f(s), we have the Taylor expansion

$$f(s_2) = \sum_{n=0}^{\infty} \frac{\Delta s^n}{n!} \frac{d^n f}{ds^n}\bigg|_{s_1} \equiv e^{\Delta s \frac{d}{dx}} f(s)\big|_{s_1} \longleftarrow$$ a symbolic notation

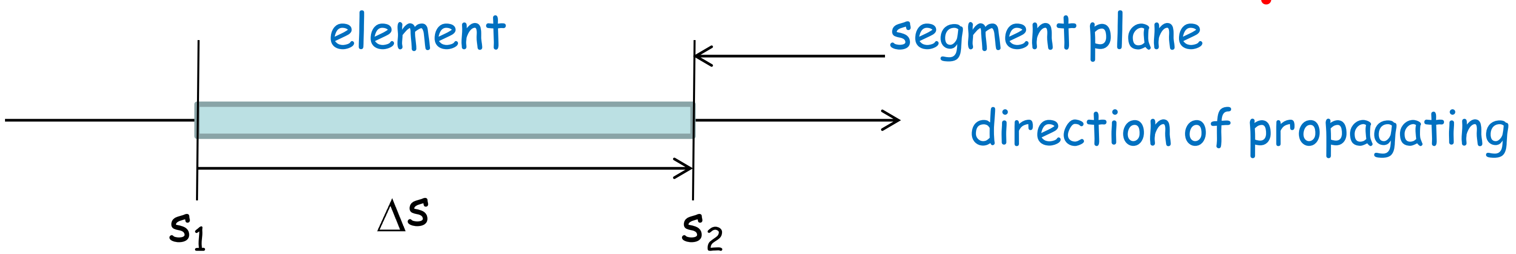
In particular, if there is no explicit dependent of s in the function f(s), namely f(s) = f(x(s),p_x(s),...), we have

$$\frac{df}{ds} = -[H, f] \equiv -:H:f, \longleftarrow$$ another symbolic notation

Used Hamiltonian equation and the definition of the Poisson bracket.
Combining these symbolic notations, we have the exponential Lie operator

$$f(s_2) = e^{-\Delta s:H:} f(s)\big|_{s_1}$$

Exponential Lie Operator as a Transfer Map



In the previous slide, we have shown that

$$f(s_2) = e^{-\Delta s:H:} f(s)\Big|_{s_1}.$$

If we apply this formula to a particular function: z=x, or p_x, or y, or p_y, or δ or ℓ, and then we have

$$z(s_2) = e^{-\Delta s:H:} z(s_1).$$

Therefore, this exponential Lie operator is a transfer map. We have

$$\mathcal{M}_{1\to 2} = e^{-\Delta s:H:}$$

An Example of a Drift

Hamiltonian in the paraxial approximation is given by

$$H_D = \frac{1}{2(1+\delta)}(p_x^2 + p_y^2).$$

It is easy to show that the exponential Lie operator indeed generates the transfer map we have found by solving the Hamiltonian equation. Namely, we have

$$x_f = e^{-\Delta s:H_D:} x_{|i} = x_i + \frac{p_{xi}}{1+\delta} \Delta s,$$

$$p_{xf} = e^{-\Delta s:H_D:} p_{x|i} = p_{xi},$$

$$y_f = e^{-\Delta s:H_D:} y_{|i} = y_i + \frac{p_{yi}}{1+\delta} \Delta s,$$

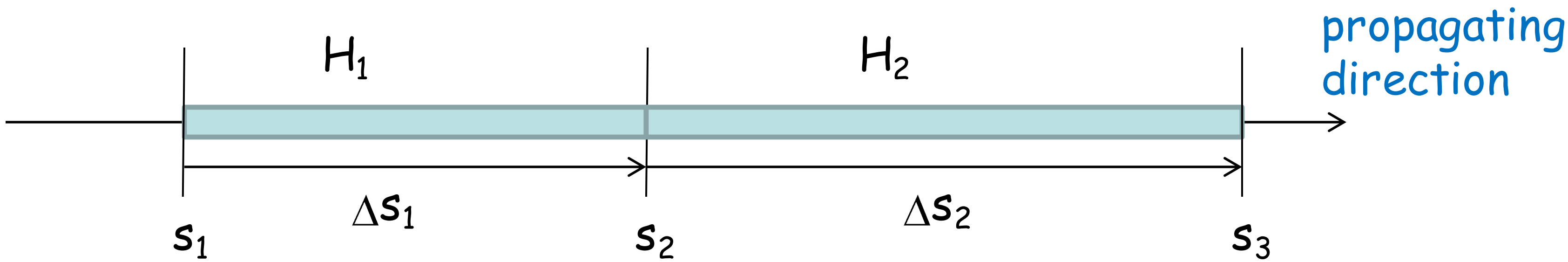
$$p_{yf} = e^{-\Delta s:H_D:} p_{y|i} = p_{yi},$$

$$\delta_f = e^{-\Delta s:H_D:} \delta_{|i} = \delta_i,$$

$$\ell_f = e^{-\Delta s:H_D:} \ell_{|i} = \ell_i + \frac{\Delta s}{2(1+\delta_i)^2}(p_{xi}^2 + p_{yi}^2).$$

However, most time, it is easier to obtain the transfer map by solving the Hamiltonian equation.

Lie Operators and Map Concatenation



just shown

It is obvious that

$$f(s+\Delta s)=e^{-\Delta s:H:}f(x,p_x,\ldots)=f(e^{-\Delta s:H:}x,e^{-\Delta s:H:}p_x,\ldots)=f(x(s+\Delta s),p_x(s+\Delta s),\ldots)$$

just shown

obviously true

The Lie operator acts only on the arguments of function . This precisely the definition of the map concatenation we introduced early. So we have

$$\mathcal{M}_{1\to 3}=\mathcal{M}_{1\to 2}\circ\mathcal{M}_{2\to 3}=e^{-\Delta s_1:H_1:}e^{-\Delta s_2:H_2:}.$$

The dot is removed because Lie operator automatically has the property.

The Cambell-Baker-Hausdorf (CBH) Theorem

To combine two exponential Lie operators, we have

$$e^{:A:}e^{:B:} = e^{:A+B+\frac{1}{2}[A,B]+\ldots:}$$

The bracket notes the Poisson bracket. This theorem can be shown easily using the definition of the exponential Lie operator and the Jacob identity for the Poisson brackets:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

In general, it should be considered as a part of perturbation theory. It is good when A and B are small.

1) If [A,B] = 0, then $e^{:A:}e^{:B:} = e^{:A+B:}$ (actually, this is an exact result)

This a necessary condition for the exponential Lie operator being a transfer map of the element that can be described by a Hamiltonian.

Similarity Transformation

$$e^{:A:}e^{:B:}e^{-:A:} = e^{:e^{:A:}B:}$$

Here is a proof. Set $f=e^{-:A:}g$, so we have

$$\begin{aligned}
e^{:A:}e^{:B:}f &= e^{:A:}\sum\frac{1}{n!}[B,[B,...[B,f]...]]] \\
&= \sum\frac{1}{n!}e^{:A:}[B,[B,...[B,f]...]] \\
&= \sum\frac{1}{n!}[e^{:A:}B,e^{:A:}[B,...[B,f]...]] \\
&= \sum\frac{1}{n!}[e^{:A:}B,[e^{:A:}B,...[e^{:A:}B,e^{:A:}f]...]] \\
&= \sum\frac{1}{n!}[e^{:A:}B,[e^{:A:}B,...[e^{:A:}B,g]...]] \\
&= e^{:e^{:A:}B:}g
\end{aligned}$$

We used $e^{:A:}[f_1,f_2]=[e^{:A:}f_1,e^{:A:}f_2]$ ($e^{:A:}[x,p_x]=[e^{:A:}x,e^{:A:}p_x]$)

Linear Similarity Transformation

A specially useful transformation is given by

$$\mathcal{M} e^{:B(z):} \mathcal{M}^{-1} = e^{:B(\mathcal{M} z):}.$$

Here, $\mathcal{M}$ is a symplectic linear map. As an example, let us to consider a pair of identical thin lens sextupoles with integrated strength S_2,separated by –I transformation. For simplicity, we limit to the transverse dimensions. The transfer map is given by

$$
\begin{aligned}
& e^{-:\frac{S_2}{6}(x^3-3xy^3):}(-\mathcal{I})e^{-:\frac{S_2}{6}(x^3-3xy^3):} \\
&= (-\mathcal{I})(-\mathcal{I})e^{-:\frac{S_2}{6}(x^3-3xy^3):}(-\mathcal{I})e^{-:\frac{S_2}{6}(x^3-3xy^3):} && \leftarrow \text{Similarity transformation} \\
&= (-\mathcal{I})e^{-:\frac{S_2}{6}((-x)^3-3(-x)(-y)^3):}e^{-:\frac{S_2}{6}(x^3-3xy^3):} && \leftarrow \text{CBH theorem} \\
&= (-\mathcal{I})
\end{aligned}
$$

We obtain the well-known result by Karl Brown.

Calculate Effective Hamiltonian

Consider a set of multipoles separated by linear maps. We can represent the nonlinear transfer map by

$$\mathcal{M}_{0,1}e^{-:V_1(z)}\mathcal{M}_{1,2}e^{-:V_2(z)}\ldots\mathcal{M}_{n-1,n}e^{-:V_n(z)}\mathcal{M}_{n,n+1}$$

$$=\mathcal{M}_{0,1}e^{-:V_1(z):}\mathcal{M}_{1,2}e^{-:V_2(z):}\ldots\mathcal{M}_{n-1,n}\mathcal{M}_{n,n+1}\mathcal{M}^{-1}_{n,n+1}e^{-:V_n(z):}\mathcal{M}_{n,n+1}$$

$$=\mathcal{M}_{0,1}e^{-:V_1(z):}\mathcal{M}_{1,2}e^{-:V_2(z):}\ldots\mathcal{M}_{n-1,n}\mathcal{M}_{n,n+1}e^{-:V_n(\mathcal{M}^{-1}_{n,n+1}z):}$$

$$=\mathcal{M}_{0,n+1}e^{-:V_1(\mathcal{M}^{-1}_{1,n+1}z):}e^{-:V_2(\mathcal{M}^{-1}_{2,n+1}z):}\ldots e^{-:V_n(\mathcal{M}^{-1}_{n,n+1}z):}$$

similarity transformation

$$=\mathcal{M}_{0,n+1}e^{-:H_{eff}:}$$

CBH theorem

z is the coordinates at the end.

We can use the similarity transformation and CBH theorem to obtain an effective Hamiltonian so that the transfer map consists of a linear map followed by an exponential Lie operator for the nonlinearity. It is very useful for understanding of the nonlinear effects and their compensation. Clearly, it is an approximation to a real accelerator.

Perturbation of Sextupoles

For a sextupole magnet, we have the potential

$$V_S(x,y) = \frac{S_2}{6}(x^3 - 3xy^2).$$

So

$$V_S(\mathcal{M}^{-1}_{i,n+1} z) = \frac{S_2}{6}(x_i^3 - 3x_i y_i^2),$$

with

$$x_i = \sqrt{\beta_x(s_i)}(\cos\Delta\psi_{i,n+1} x - \sin\Delta\psi_{i,n+1} p_x),$$

$$y_i = \sqrt{\beta_y(s_i)}(\cos\Delta\varphi_{i,n+1} y - \sin\Delta\varphi_{i,n+1} p_y),$$

where x, p_x, y, p_y are the normalized coordinates at n+1 position. The effective Hamiltonian based on the CBH theorem is

$$H_{eff} = \sum_{i=1}^{n} \frac{S_2^i}{6}(x_i^3 - 3x_i y_i^2) - \frac{1}{2}\sum_{i<j}^{n} \frac{S_2^i S_2^j}{36}[(x_i^3 - 3x_i y_i^2),(x_j^3 - 3x_j y_j^2)]$$

The first term is the same as the first-order canonical perturbation theory. Except the reference point is the end rather then the beginning.

Effective Hamiltonian

The Poisson bracket can be evaluated, we have

$$
\begin{aligned}
H_{eff} &= \sum_{i=1}^{n} \frac{S_2^i}{6}(x_i^3 - 3x_i y_i^2) \\
&+ \frac{1}{2}\sum_{i<j}^{n} S_2^i S_2^j \sqrt{\beta_{x,i}\beta_{x,j}} \{\sin(\Delta\varphi_{i,n+1} - \Delta\varphi_{j,n+1})\beta_{y,i}\beta_{y,j} x_i x_j y_i y_j \\
&+ \sin(\Delta\psi_{i,n+1} - \Delta\psi_{j,n+1})(\beta_{x,i}x_i^2 - \beta_{y,i}y_i^2)(\beta_{x,j}x_j^2 - \beta_{y,j}y_j^2)/4\}
\end{aligned}
$$

We see that two sextupoles generate octopole like terms.

Second-Order Symplectic Integrators

Separate Hamiltonian into two exactly solvable parts:

$$H = H_0 + H_1$$

Approximation with symplectic integrators:

propagator

$$e^{-:H:L} = \prod_{i=1}^{n} e^{-:H:\Delta s} \approx \prod_{i=1}^{n} [e^{-\frac{:H_0:}{2}\Delta s} e^{-:H_1:\Delta s} e^{-\frac{:H_0:}{2}\Delta s} + O(\Delta s)^3]$$

exact kick error

1. The residual term can be easily see from the CBH theorem
2. It becomes the exact solution at the limit of infinite number of segments
3. Preserves symplectic condition during the integration

$$e^{-\frac{1}{2}\Delta s:H_0:} e^{-\Delta s:H_1:} e^{-\frac{1}{2}\Delta s:H_0:} = e^{-\frac{1}{2}\Delta s:H_0:} e^{-\Delta s:(H_1 - \frac{1}{2}H_0) + \frac{\Delta s^2}{4}[H_1, H_0] : + O(\Delta s)^3 :}$$

$$= e^{-\Delta s:(H_1 + H_0): + O(\Delta s)^3}$$

Presentations for Magnetic Elements

Lie factors $\quad \mathcal{M}^1 e^{:f_3:} e^{:f_4:} \cdots$

- engine in MARYLIE (A. Dragt)
- violates symplecticity when evaluates

Dragt-Finn

Taylor map $\quad \mathcal{M}^n(z)$

- engine in TRANSPORT, MAD, COSY (K. Brown and M. Berz), simple R-matrix
- but high-order one violates

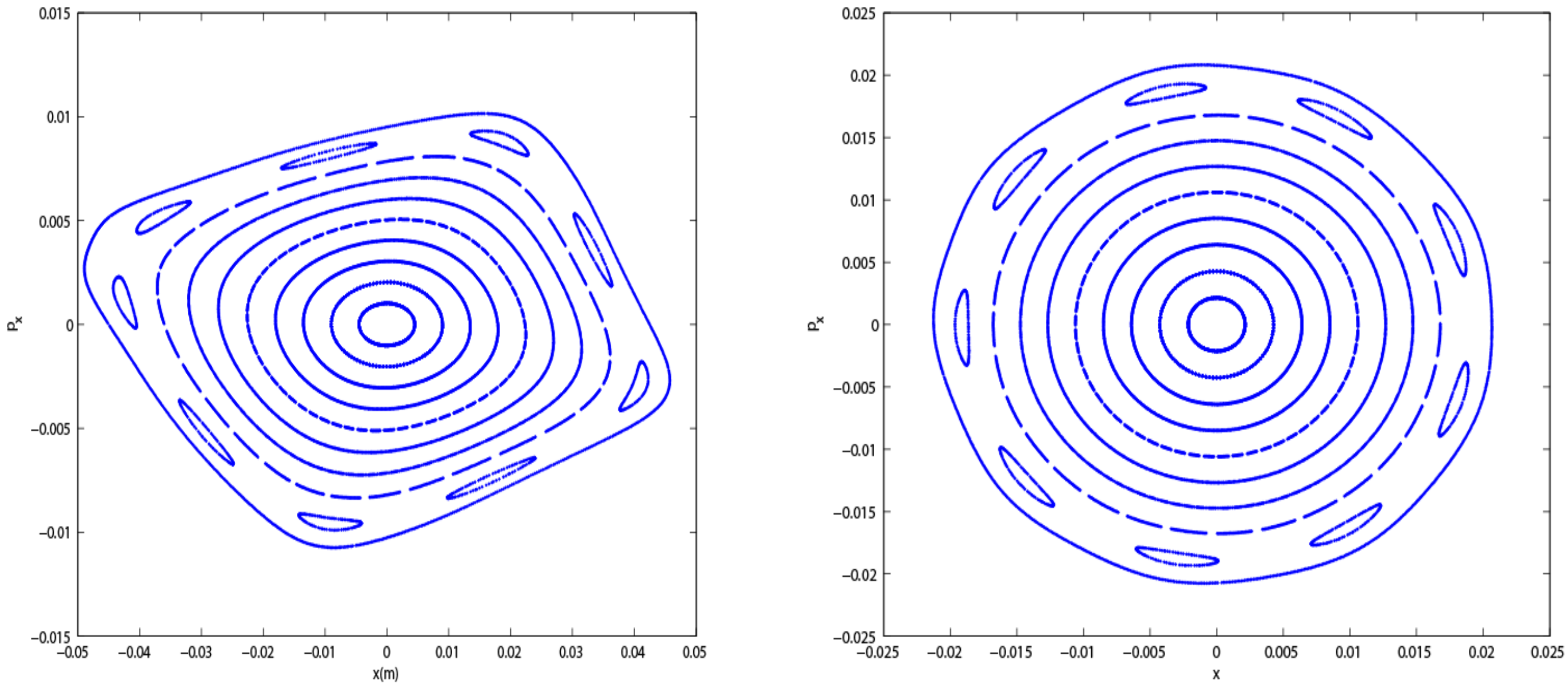
TPSA

Symplectic Integrator

$$\prod_{i=1}^{n} e^{-\frac{:H_0:}{2}\Delta s} e^{-:H_1:\Delta s} e^{-\frac{:H_0:}{2}\Delta s}$$

- engine in TEAPOT, SAD, TRACY, **LEGO**, PTC (E. Forest, R. Ruth, and K. Hirata)
- preserves symplecticity
- simple and based on several known solutions
- emphasis on numerical process

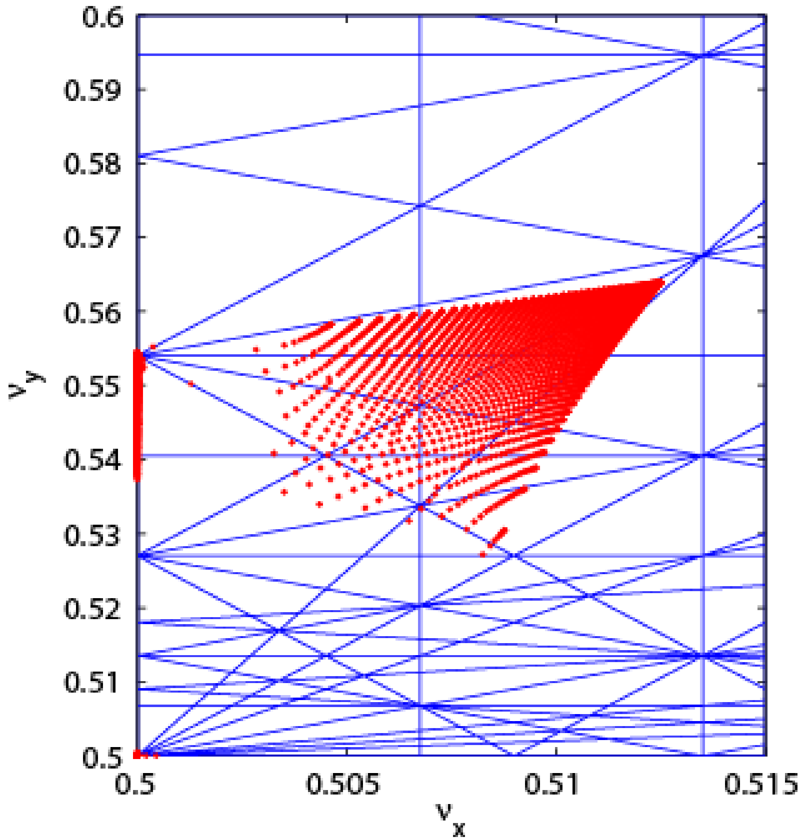
Nonlinear Normal Form



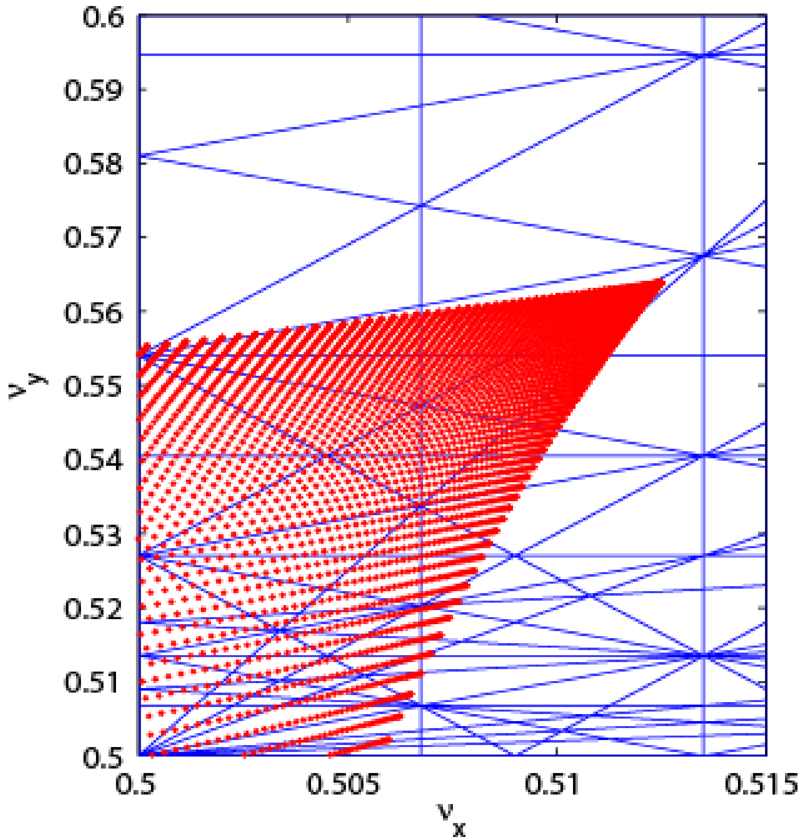
Physical coordinates $\longrightarrow$ Normalized coordinates

Transformation approximated by a 10th order Taylor map

Footprint in Tune Space

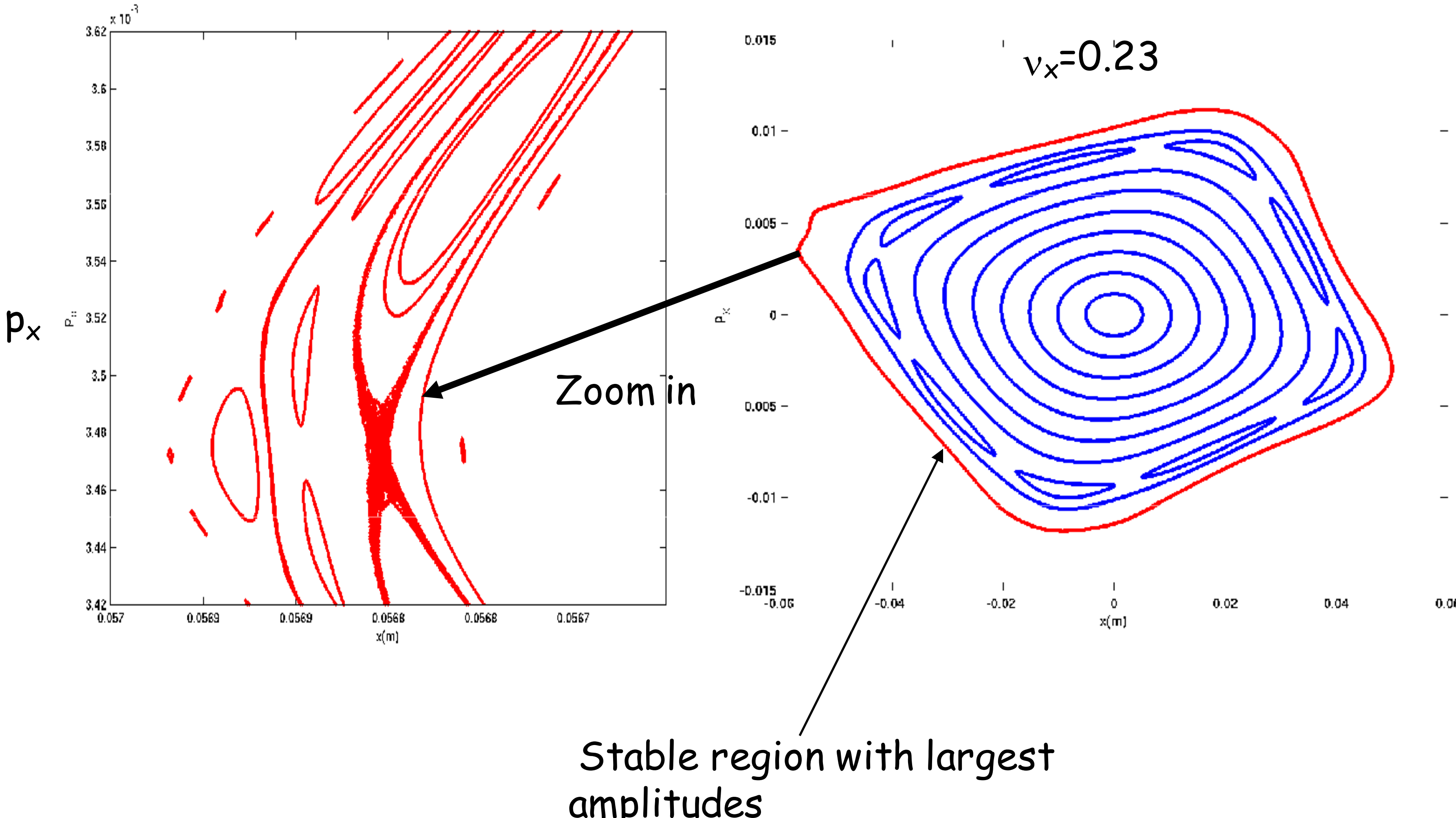


Frequency analysis
Tracking & FFT

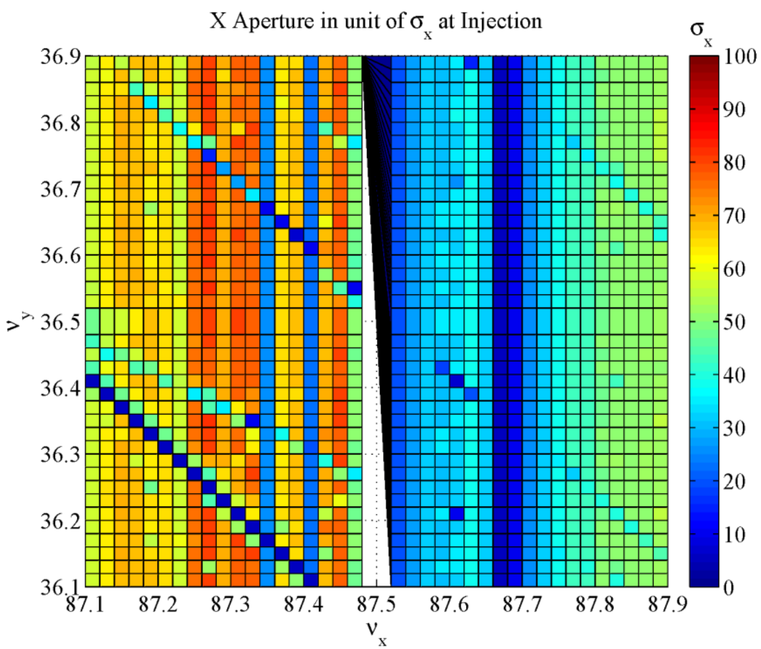


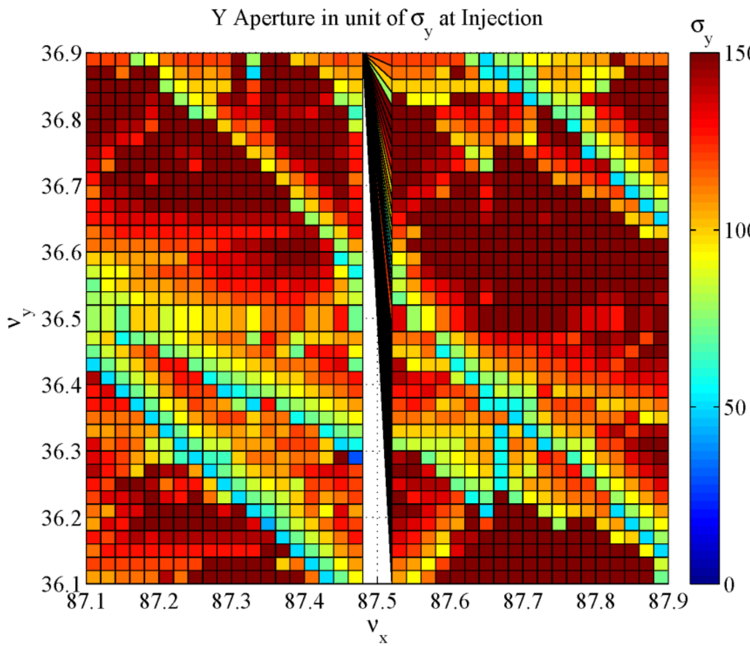
Normal form analysis
Taylor map & Lie form

Characteristics of Phase Space in Electron Storage Rings



Resonance in Storage Rings





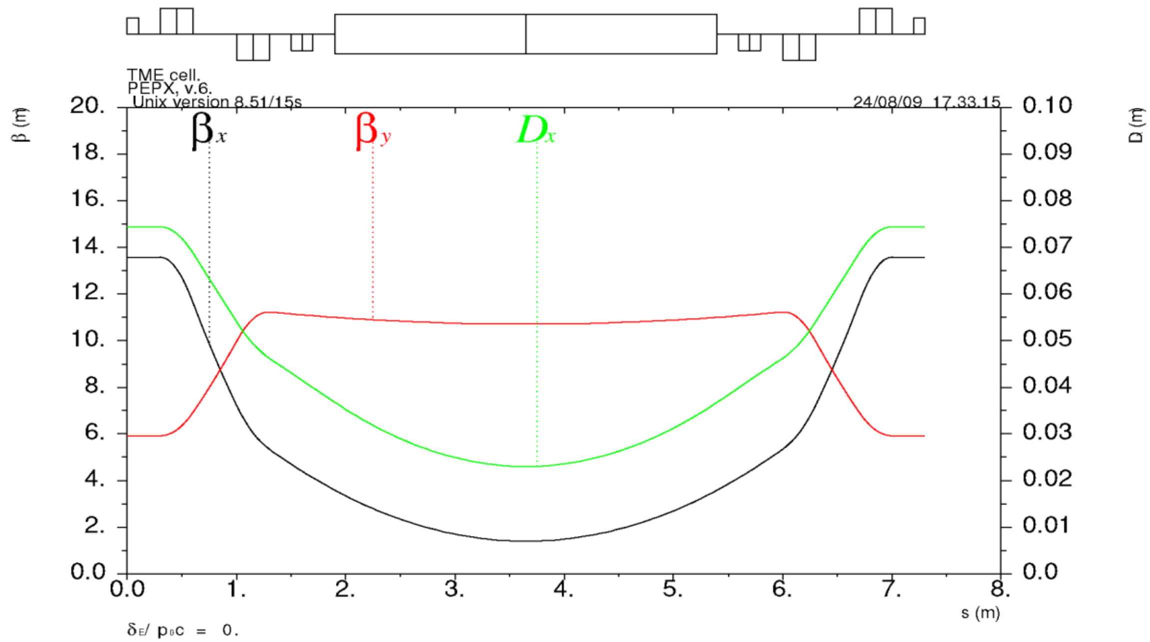
Dynamic aperture in a two-dimensional tune scan for the baseline design of PEP-X.

Where these resonances come from?

PEP-X: ALLTME CELLS

Parameter	Value
Energy, E [GeV)]	4.5
Horizontal emittance, ε_x [pm-rad]	94.6
Damping time, τ_x [ms]	202
Tunes, ν_x, ν_y, ν_s	89.66, 39.57, 0.003
Momentum compaction, α_c	7.58×10^{-5}
Bunch length, σ_z [mm]	3.00
Energy spread, σ_e/E	3.6×10^{-4}
Chromaticity, ξ_x, ξ_y	-149.62, -64.12
Energy loss per turn U_0 [MeV]	0.33
RF Voltage, V_{RF} [MV]	1.16

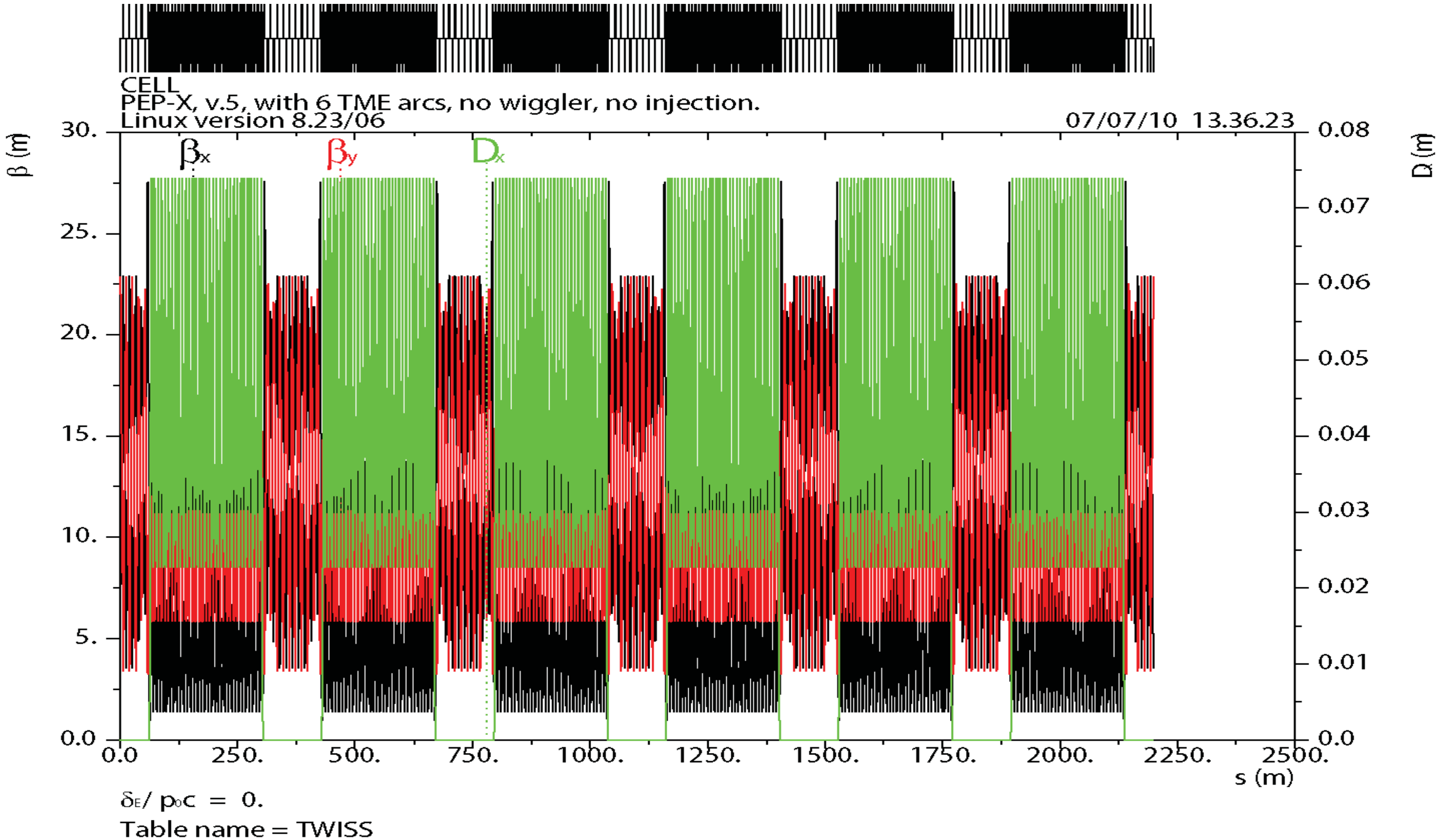
Theoretical Minimum Cell (TME)



When phase advance is 135^0 and 45^0 in x and y respectively, all 3^{rd} order driving terms generated by the sextuples in 32 cells are canceled out and the first-order chromatic beta beating canceled as well.

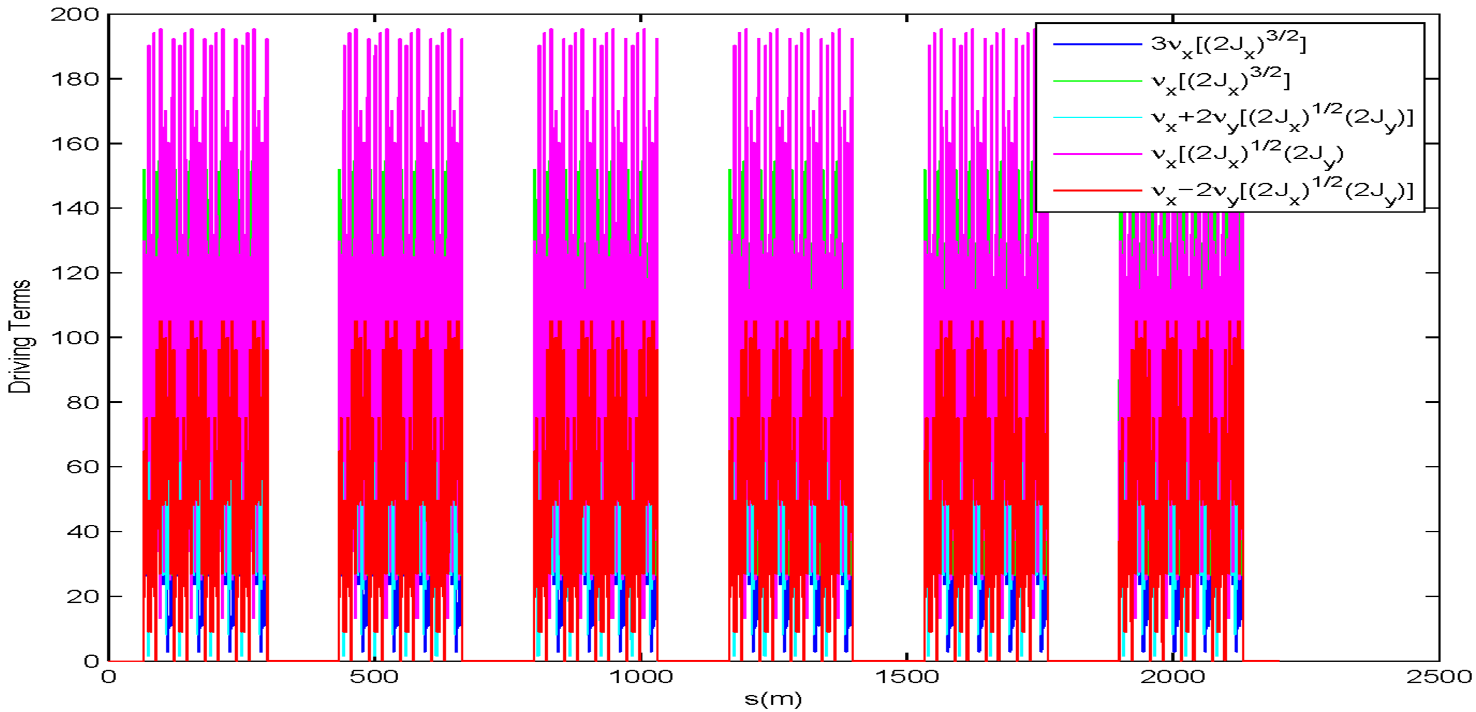
PEP-X: ALLTME CELLS ($135^0/45^0$)

One of the Earliest Designs



Third-Order Achromat (Karl Brown)

(linear in sextupole strengths)

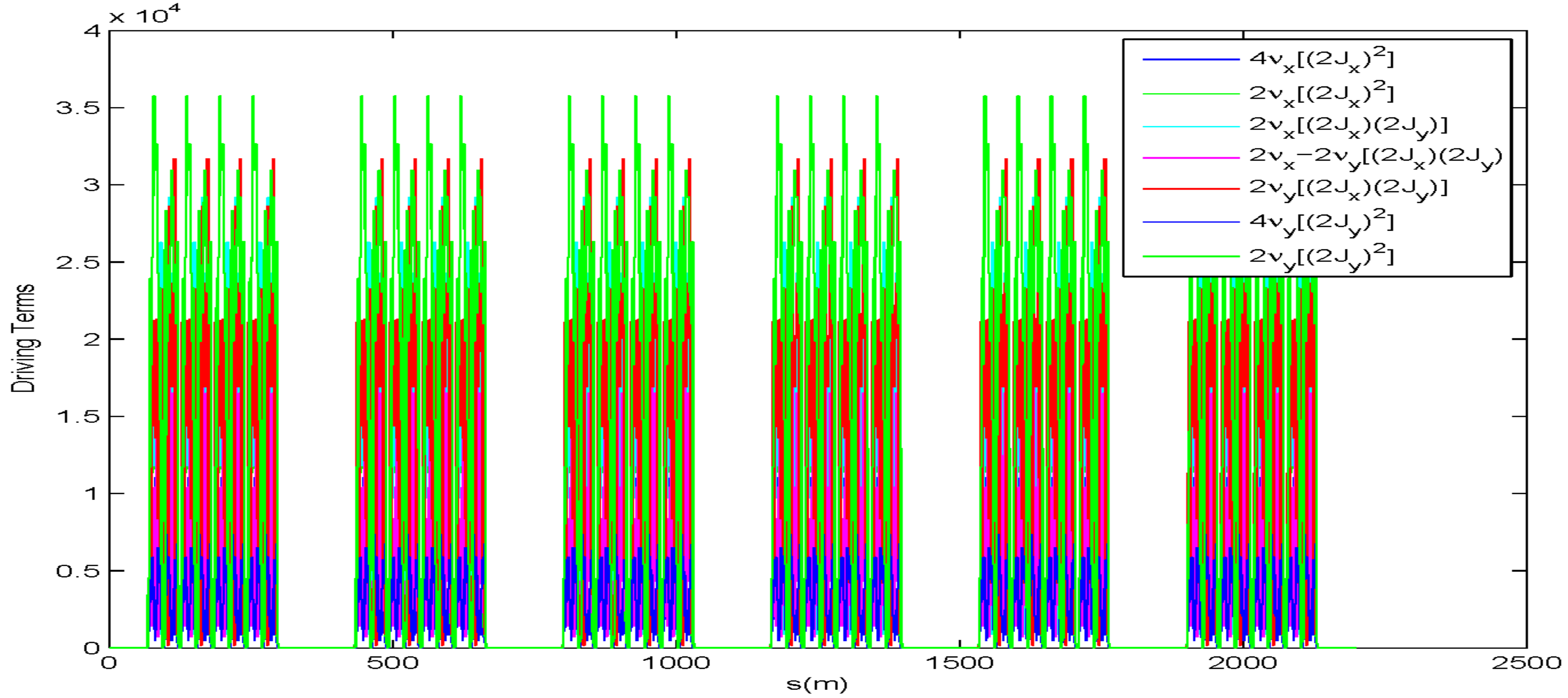


Nonlinear effects of sextupole magnets, calculated with LEGO. The results show that eight $135^0/45^0$ cells make a "third-order" achromat.

Yunhai Cai, Nucl. Instr. And Meth. A 645 (2011) 168-174.

Fourth-Order Achromat?

(quadratic in sextupole strengths)

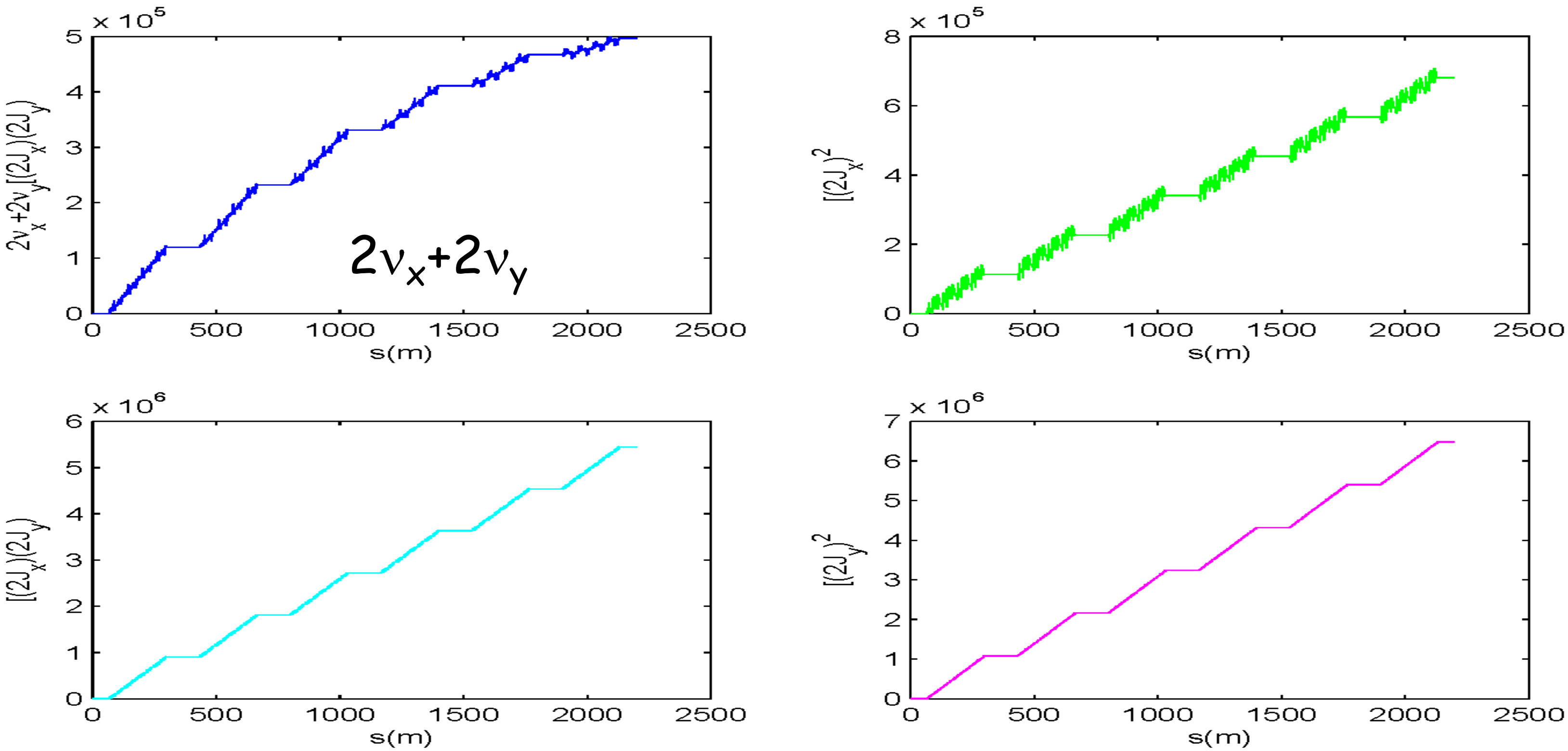


Interferences among sextupole magnets, calculated with LEGO. The results show that eight $135^0/45^0$ cells nearly make a "fourth-order" achromat. These cancelations came as a great surprise.

Yunhai Cai, Nucl. Instr. And Meth. A 645 (2011) 168-174.

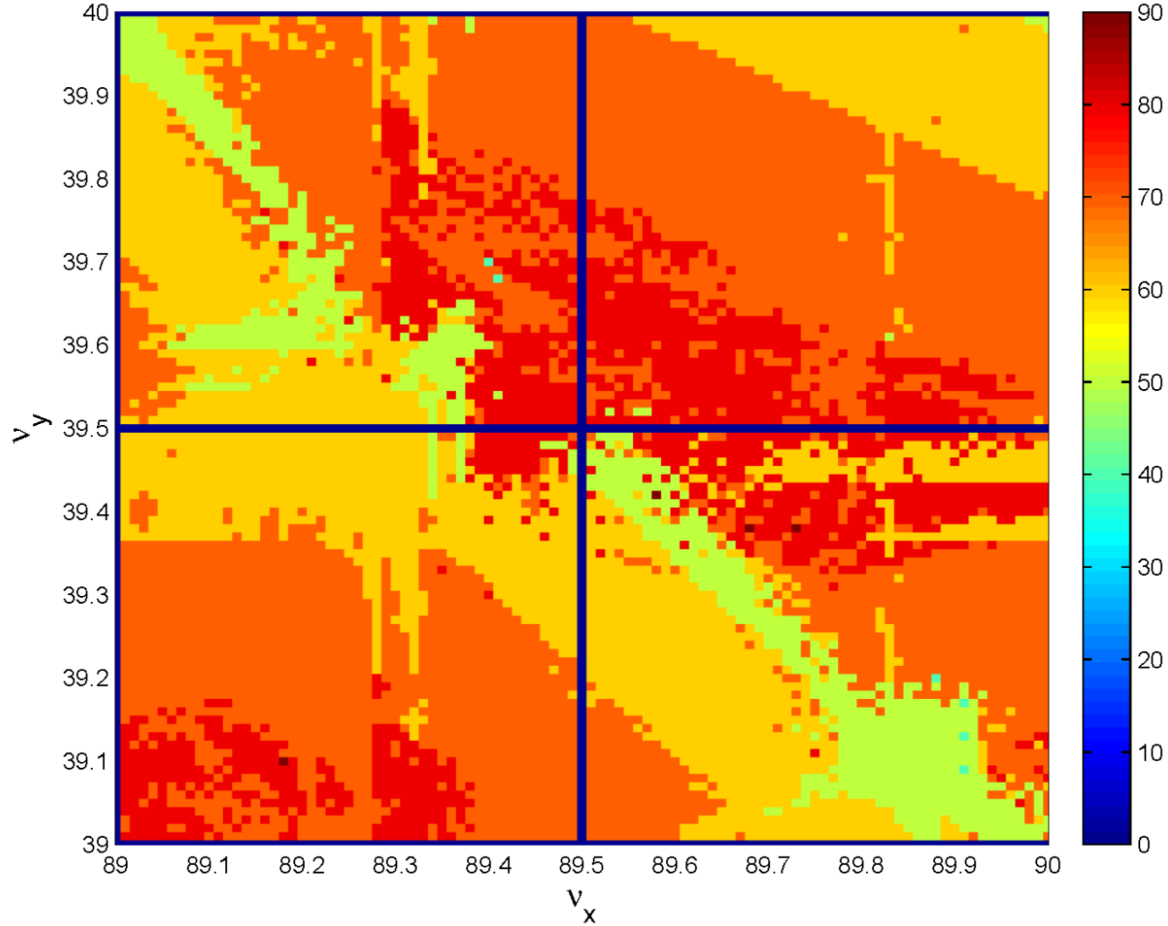
Fourth-Order Residual Terms

(quadratic in sextupole strengths)



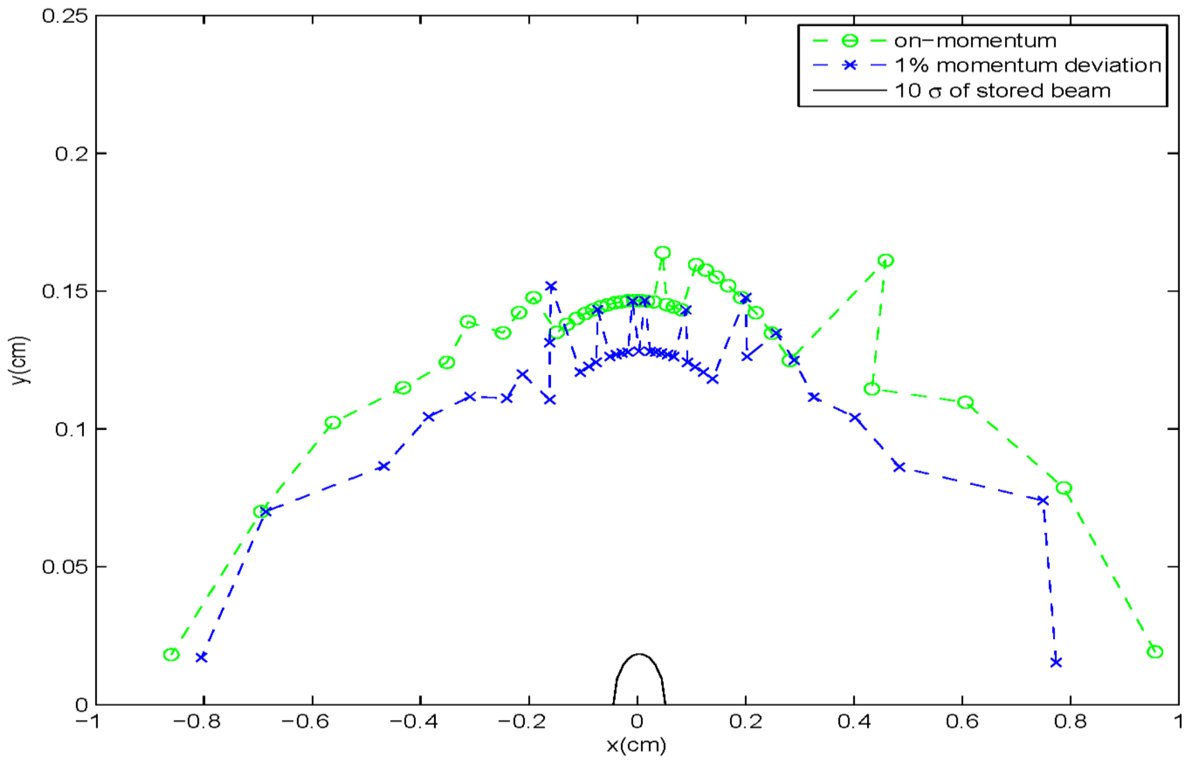
Interferences among sextupole magnets, calculated using LEGO. The results show that eight $135^0/45^0$ cells generate a systematic 4^{th} order resonance, namely $2\nu_x+2\nu_y=258$.

Tune Scan of Dynamic Aperture



Confirmation of the numerical analysis using the map.

Dynamic Aperture of PEP-X



Dynamic aperture is much larger then the size of stored beam and is enough also for an off-axis injection system and top-off operation.

Summary

- Lie algebra is a powerful method for nonlinear analysis. It is equivalent to the Hamiltonian perturbation
- Exponential Lie operator is a representation of the transfer map
  - **Used to derive symplectic integrator**
  - **Define transfer map of a beamline**
- Similarity transformation and the CBH theorem are two important tools in the Lie method. They can be used to derive the effective Hamiltonian

References

1. A.J. Dragt, J. Opt. Soc. An. 72, 372 (1982); A. Dragt et al. Ann. Rev. Nucl. Part. Sci. 38, 455 (1988).
2. John Irwin, "The application of Lie algebra techniques to beam transport design," SLAC-PUB-5315, Nucl. Instr. Meth. A 298, 460 (1990).
3. Yunhai Cai, "Single-particle dynamics in electron storage rings with extremely low emittance," Nucl. Instr. Meth. A 645, 168 (2011).