Main Linac I:

Longitudinal Dynamics

Peter Tenenbaum, SLAC

Outline

- The basics:
  - Maxwell's equations in SI
  - The wave equation in free space
  - The wave equation in regular waveguide
- Single-celled accelerating cavities
- Multi-celled accelerating cavities
- Cavities with finite conductivity
- Introducing the RF power
- Introducing the Beam
- The ILC linear accelerating system

NB: there are class notes available on the ILC School's ilcagenda website. They contain a lot of the fussy details which are not covered in the lectures.

Maxwell's Equations in SI

$$\vec{\nabla} \bullet \vec{D} = \rho$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Where:

$$\vec{B} = \mu \vec{H}$$

$$\vec{D} = \varepsilon \vec{E}$$

Note that, most of the time in these lectures,

$$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

$$\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$$

The Wave Equation

Apply the vector calculus identity,

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}\right)-\nabla^2\vec{A}$$

To the electric curl Maxwell's equation:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{E}\right)-\nabla^2\vec{E}=-\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}$$

$\rightarrow$ 0 in absence of charges

Reverse order of derivatives (well-behaved functions)

$$\nabla^2\vec{E}=\mu\frac{\partial}{\partial t}\vec{\nabla}\times\vec{H}$$

Replace via magnetic curl equation

$$\nabla^2\vec{E}=\mu\left(\frac{\partial\vec{J}}{\partial t}+\frac{\partial^2\vec{D}}{\partial t^2}\right)$$

Zero current region

Replace *D* with *E*

$$\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=0$$

Similarly,

$$\nabla^2\vec{H}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}=0$$

Solution in Free Space

Consider a solution which is a superposition of plane waves which propagates in the *z* direction:

$$\vec{E} = \vec{E}_0 \exp\left[i\left(\omega t - kz\right)\right], \vec{H} = \vec{H}_0 \exp\left[i\left(\omega t - kz\right)\right]$$

In free space, due to symmetry, E_0 and H_0 must be constant over all time and space. Thus, derivatives in the wave equation operate on the exponential only:

$$\nabla^2 \vec{E} \rightarrow \frac{\partial^2 \vec{E}}{\partial z^2} = -k^2 \vec{E}_0 \exp\left[i\left(\omega t - kz\right)\right], \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = -\mu\varepsilon\omega^2 \vec{E}_0 \exp\left[i\left(\omega t - kz\right)\right]$$

$$k^2 = \mu\varepsilon\omega^2$$

Thus, solution is a wave with phase velocity (ω/k) and group velocity ($\partial\omega/\partial k$) = $1/\sqrt{\mu\varepsilon}$ which propagates in the *z* direction.

Solution in Free Space (2)

Now consider what happens when the electric divergence equation is applied:

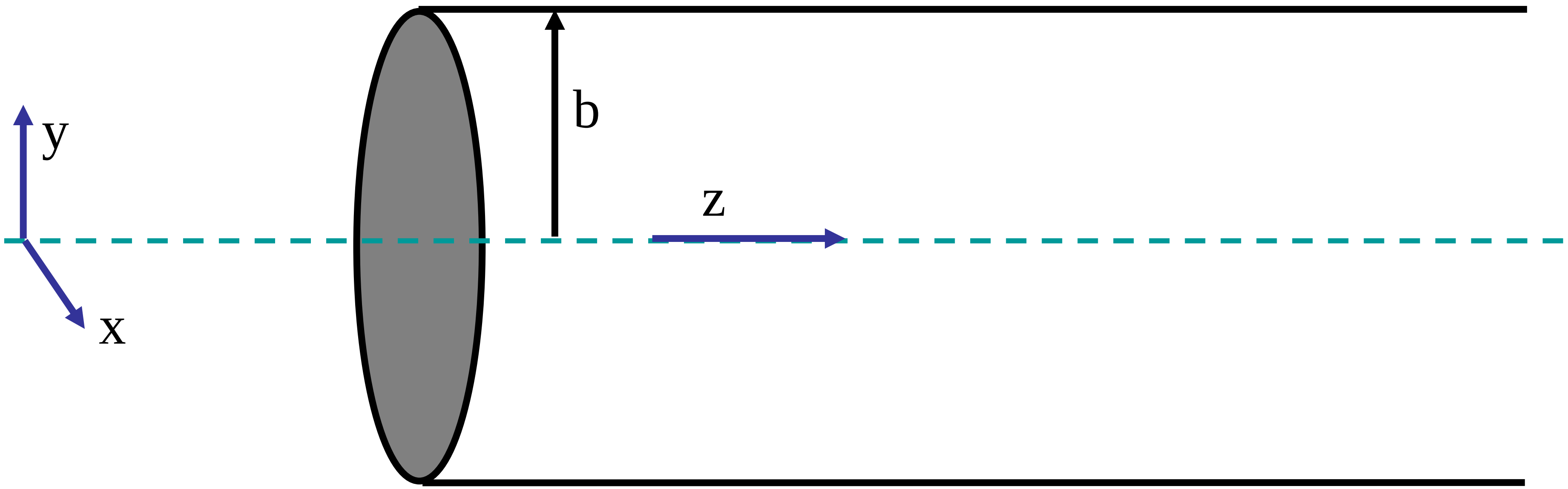
$$\nabla \bullet E = 0 = \frac{\partial}{\partial z}\left\{ E_{0,z} \exp\left[i\left(\omega t - kz \right) \right] \right\} = -ikE_{0,z} \exp\left[i\left(\omega t - kz \right) \right]$$

So if $k \neq 0$, then $E_{0,z}$ must be zero – no component of the field parallel to the direction of propagation (ie, light is a transverse wave, not exactly a shocking revelation)

This solution results in particles receiving acceleration $\perp$ z which alternates in direction over time – no net acceleration!

How to resolve this? We can have a nonzero $\partial E/\partial z$ if we allow $\partial E/\partial x$ and/or $\partial E/\partial y$ to be nonzero as well, then we can have a net zero divergence.

Need to break symmetry in x,y directions with some sort of boundary – consider a cylindrical boundary, a conducting pipe of radius b which extends infinitely in z…


y
b
z
x

Cylindrical Waveguide

Now,

$$\vec{E} = \vec{E}_0(r,\theta)\exp\left[i\left(\omega t - kz\right)\right], \vec{H} = \vec{H}_0(r,\theta)\exp\left[i\left(\omega t - kz\right)\right]$$

Boundary conditions: At r=b,

$$E_\theta = 0$$

$$E_z = 0$$

$$H_r = 0$$

$$\frac{\partial H_z}{\partial r} = 0$$

Define a new operator: $\nabla_\perp^2 \equiv \nabla^2 - \frac{\partial^2}{\partial z^2} = \nabla^2 + k^2$

Cylindrical Waveguide (2)

Considering just the z component of E: $$\nabla_\perp^2 E_z - k^2 E_z + \mu\varepsilon\omega^2 E_z = 0$$

Define a new variable: $$k_c^2 = \mu\varepsilon\omega^2 - k^2$$

New, improved equation: $$\nabla_\perp^2 E_{0,z} + k_c^2 E_{0,z} = 0$$

Solution is a Fourier series with sinusoids and Bessel functions (ugh!):

$$E_{0,z} = \sum_{n=0}^{\infty} a_n J_n\left(k_c r\right)\cos\left(n\theta + \theta_n\right)$$

Require that $E_z \rightarrow 0$ at r=b (boundary condition) – implies that $J_n(k_c r) \rightarrow 0$ at r=b. Can only be accomplished if $k_c r = z_{np}$, ie, p^{th} zero of J_n. We can write the field as a double sum instead of a single sum (terrific):

$$E_{0,z} = \sum_{p=1}^{\infty}\sum_{n=0}^{\infty} a_{np} J_n\left(k_{c,np} r\right)\cos\left(n\theta + \theta_n\right)$$ where $$k_{c,np} = \frac{z_{np}}{b} = \sqrt{\mu\varepsilon\omega^2 - k^2}$$

Cylindrical Waveguide (3)

$$k_{c,np} = \frac{z_{np}}{b} = \sqrt{\mu\varepsilon\omega^2 - k^2}$$

Note that for k $\rightarrow$ 0 (infinite wavelength), ω > 0.

Define the cutoff frequency, $\omega_{c,np:}$

$$\omega_{c,np} = \frac{1}{\sqrt{\mu\varepsilon}} \frac{z_{np}}{b}, \text{ thus } k^2 = \mu\varepsilon\left(\omega^2 - \omega_{c,np}^2\right)$$

Note the behavior for frequencies around $\omega_{c,np}$:

$\omega > \omega_{c,np}$: k real, sinusoidal wave

$\omega = \omega_{c,np}$: k = 0, no wave

ω < ωc,np: k imaginary, evanescent wave

We can calculate the phase and group velocity of these waves:

$$v_{\text{gr}} = \frac{1}{\sqrt{\mu\varepsilon}} \frac{\sqrt{\omega^2 - \omega_{c,np}^2}}{\omega} < c; v_{\text{ph}} = \sqrt{\frac{1}{\mu\varepsilon} + \frac{\omega_{c,np}^2}{k^2}} > c$$

Still no good for accelerating real particles – phase velocity of wave is greater than the speed of light!

TE and TM modes

Longitudinal electric field:

$$E_{0,z} = \sum_{p=1}^{\infty} \sum_{n=0}^{\infty} a_{np} J_n\left(k_{c,np} r\right) \cos\left(n\theta + \theta_n\right)$$

BC: $E_{0,z} \rightarrow 0$ at r = b, so $J_n(k_{c,np}b) = 0$

Longitudinal magnetic field:

$$H_{0,z} = \sum_{v=1}^{\infty} \sum_{u=0}^{\infty} f_{uv} J_u\left(k_{c,uv} r\right) \cos\left(u\theta + \theta_n\right)$$

BC: $\partial H_{0,z}/\partial r \rightarrow 0$ at r = b, so $J_n'(k_{c,uv}b) = 0$

In general, $k_{c,np} \neq k_{c,uv}$ for most values of n,p,u,v

Implies that a wave in a waveguide with a given frequency and wavelength can have a longitudinal *electric* field, or a longitudinal *magnetic* field, but not both!

Waves with a longitudinal electric field are called TM (Transverse Magnetic) modes

Waves with a longitudinal magnetic field are called TE (Transverse Electric) modes

Usually called TM_{np} or TE_{uv} waves

Fields of the TM_{np} Mode

$$E_{0,z} = J_n\left(k_{c,np} r\right)\cos\left(n\theta + \theta_{np}\right)$$

$$H_{0,z} = 0$$

$$E_{0,r} = \frac{-ik}{k_{c,np}} J'_n\left(k_{c,np} r\right)\cos\left(n\theta + \theta_{np}\right)$$

$$H_{0,r} = \frac{-i\omega\varepsilon n}{k_{c,np}^2 r} J_n\left(k_{c,np} r\right)\sin\left(n\theta + \theta_{np}\right)$$

$$E_{0,\theta} = \frac{ikn}{k_{c,np}^2 r} J_n\left(k_{c,np} r\right)\sin\left(n\theta + \theta_{np}\right)$$

$$H_{0,\theta} = \frac{-i\omega\varepsilon}{k_{c,np}} J'_n\left(k_{c,np} r\right)\cos\left(n\theta + \theta_{np}\right)$$

Note: for $n \neq 0$ there are 2 polarizations of every mode.

Get the fields for the 2nd polarization by replacing sine with cosine and cosine with –sine.

Single-celled Accelerating Cavity

Electromagnetic waves in free space: ***NO GOOD*** for acceleration

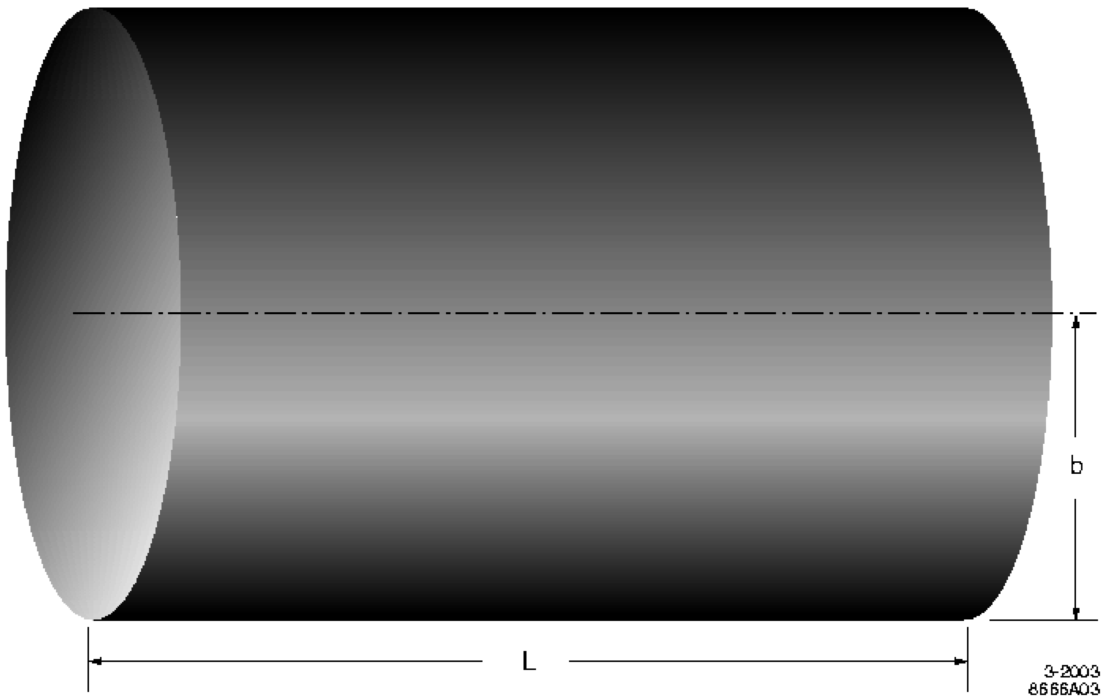
No longitudinal component, so particles get + and – acceleration in direction perpendicular to wave.

Electromagnetic wave in cylindrical waveguide: ***NO GOOD*** for acceleration

Wave have phase velocity > c, particles have velocity ≤ c, so waves “overtake” particles – acceleration oscillates between + and – in direction of wave propagation

How about if we accelerate the particles in a waveguide, and then separate them from the electromagnetic field before it can switch signs? (Equivalently: how about if we impose boundary conditions in z, such as perfectly-conducting walls?)

Single-Celled Cavity (2)



Additional Boundary Conditions:

At z=0 and z=L, the radial electric field must go to zero (electric field is always normal to conducting boundaries)

At z=0 and z=L, the longitudinal magnetic field must go to zero (magnetic field is always tangential to conducting boundaries)

At z=0 and z=L, $dH_\theta/dz = dH_r/dz = 0$ ($E_\perp \to 0$ on endcaps, and magnetic curl equation applies)

Single-Celled Cavity (3)

Consider a superposition of a leftward-propagating TM_{01} mode and a rightward-propagating TM_{01} mode with the same amplitude and frequency. The nonzero field vectors become:

$$E_z = J_0\left(k_{c,01} r\right)\cos(kz)\exp(i\omega t)$$

$$E_r = -\frac{k}{k_{c,01}} J'_0\left(k_{c,01} r\right)\sin(kz)\exp(i\omega t)$$

$$H_\theta = \frac{-i\omega\varepsilon}{k_{c,01}} J'_0\left(k_{c,01} r\right)\cos(kz)\exp(i\omega t)$$

Note that this is a *standing wave* solution!

TM_{npj} Modes

The boundary conditions are automatically satisfied if kL = jπ, j integer. Only certain frequencies are permitted in this case:

$$\omega_{npj} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{z_{np}}{b}\right)^2 + \left(\frac{j\pi}{L}\right)^2}$$

For j=0, the solution corresponds to a wave with a frequency equal to $\omega_{c,np}$ and a wave number of zero.

Picking a Mode

$J_n(0) \neq 0$ for n=0 only. If we want to accelerate the beam along the cavity axis, we have to pick n=0.

Recall that the standing wave is a superposition of leftward- and rightward- traveling waves. The rightward-traveling wave (which moves in the same direction as the beam) can always be made to accelerate the beam by injecting the beam when the fields in the cavity have the correct phase. The leftward-traveling wave can be shown to usually decelerate the beam, except when j=0.

Because the phase of the standing wave changes during the passage of the beam, for a TM_{0p0} wave with a field given by $E_z = E_0 \exp(i\omega t)$, the maximum acceleration achievable is $E_0 LT$, where $T = \sin(\psi/2)/(\psi/2)$ and $\psi = L\omega/c$ (ie, the amount that the phase in the cavitgy changes during the beam's passage).

R/Q

What is the relationship between the stored energy in a cavity and the achievable acceleration? Start with the equation for stored electromagnetic energy

$$U = \frac{1}{2}\left(\overrightarrow{E}\cdot\overrightarrow{D}^{*} + \overrightarrow{B}\cdot\overrightarrow{H}^{*}\right)$$

For TM_{0p0} mode, we can select a time when B and H are zero, and only have to evaluate the electric field in the cavity to compute the stored energy!

R/Q (2)

After lots of math, we find that: $$U = \frac{\pi}{2}\varepsilon E_0^2 b^2 L J_1^2\left(z_{0p}\right)$$

We also know that the max acceleration achievable is: $$V = E_0 L T$$

So we can compute V^2/U, which drops out the field E_0. The result is a big ugly mess, which we can simplify:

$$\frac{V^2}{\omega_{0p0} U} \approx Z \frac{T^2 L}{b}$$

Impedance of medium (~377 Ω for vacuum)

The quantity R/Q ("R over Q" or "R Upon Q") is defined as $V^2/\omega U$, and is purely a geometric quantity of the cavity! It has dimensions of impedance (ohms).

R/Q (3)

With additional substitutions,

$$\frac{R}{Q} = Z \frac{2}{z_{0p}} \frac{\sin^2(\psi/2)}{\psi/2} c\sqrt{\mu\varepsilon}$$

R/Q has a weak maximum at ψ ~ 134°. Since z_{0p} monotonically increases with p, p=1 is the most energy-efficient accelerating mode! So typical RF cavities accelerate in the TM_{010} mode.

NOTE: in a real cavity, with a more complicated shape and including the hole that lets the beam pass through, R/Q will not be exactly as given in these formulas! However, the general conclusions – that R/Q (or $V^2/\omega U$) is a function solely of the cavity geometry, and is a constant for any given cavity – remain true.

Multi-Celled Cavities

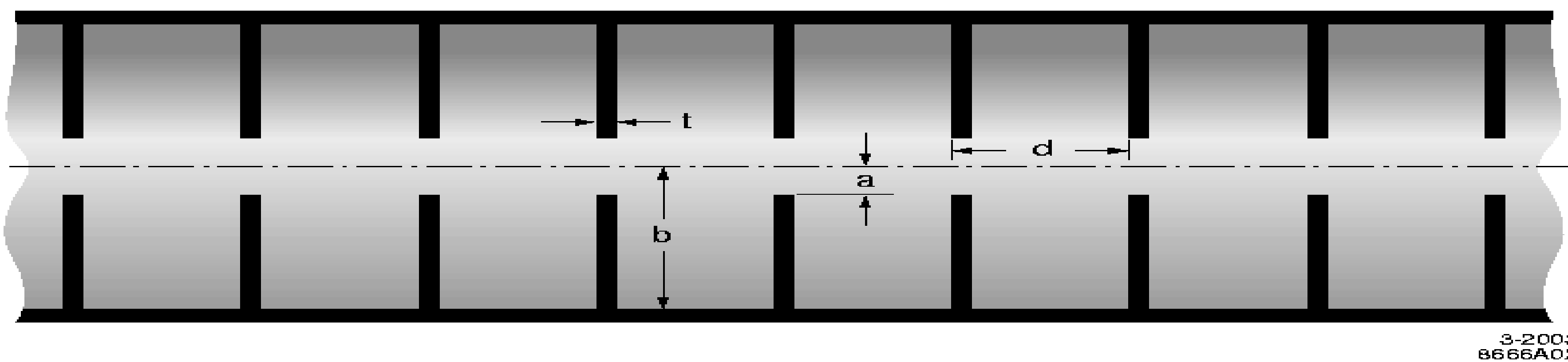
Why use more than one cavity?

- Remember that $V^2/\omega U$ is a constant for any given cavity
  - The stored energy goes as the square of the voltage
  - Take any realistic cavity, plug in V=250 GV, and the stored energy is absurdly huge
- For N cavities, the stored energy is reduced by 1/N compared to 1 cavity

Multi-Celled Cavities (2)

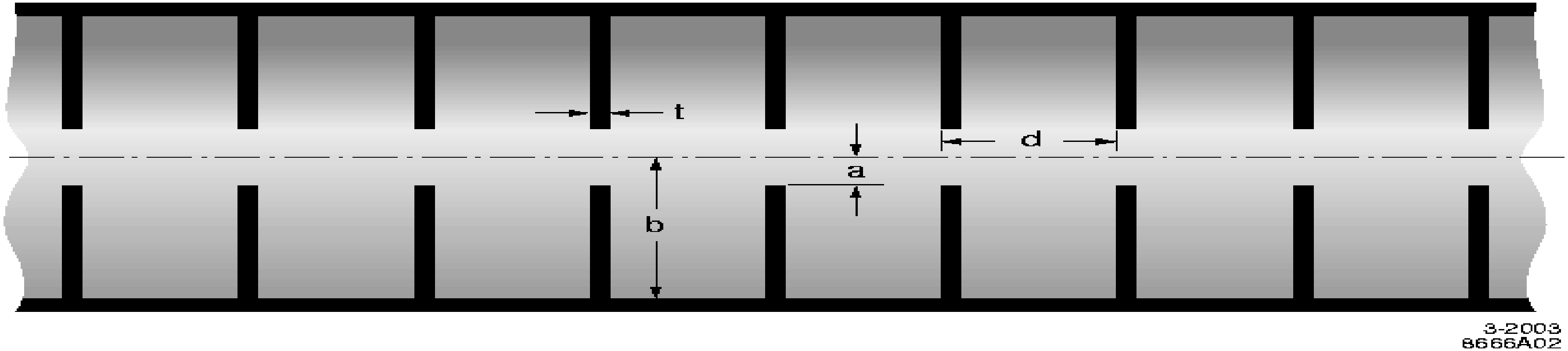
Why use multi-celled cavities rather than single-celled cavities?

- Every cavity needs some sort of connection to a microwave power source
- Engineering-wise, it's more efficient to use multi-celled cavities to reduce the number of connections



Properties of Multi-Celled Cavities

Consider an infinite periodic array of cells with beam holes, as shown below:



From Floquet's theorem, we expect that the solution has to have a certain symmetry:

$$\vec{E}(r,\theta,z+d,t)=\vec{E}(r,\theta,z,t)\exp\left[d\left(-\alpha+ik_z\right)\right]$$

We also expect that, as a $\rightarrow$ 0, the solution reduces to the single-cavity, no-hole solution for a TM_{010} field.

Define the fields for the unperturbed (no-holes) solution: $\vec{E}_1, \vec{H}_1$

Define the fields for the perturbed (periodic) solution: $\vec{E}_2, \vec{H}_2$

Consider the expression:

$$\oint d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^{*}-\vec{E}_2\times\vec{H}_1^{*}\right)$$

(Why?)

That Awful Integral

We can use the Divergence theorem: $\oint d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^* - \vec{E}_2\times\vec{H}_1^*\right) = \int dV\vec{\nabla}\bullet\left(\vec{E}_1\times\vec{H}_2^* - \vec{E}_2\times\vec{H}_1^*\right)$

And the vector calculus identity: $\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$

And Maxwell's equations (see slide 1) to find that:

$$\oint d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^* - \vec{E}_2\times\vec{H}_1^*\right) = i\left(\omega_2-\omega_1\right)\int dV\left(\mu\vec{H}_1^*\cdot\vec{H}_1 + \varepsilon\vec{E}_1^*\cdot\vec{E}_1\right)$$

The RHS integrand is 2x the stored energy in the unperturbed cavity. Thus,

$$\left(\omega_2-\omega_1\right) = \frac{-i\oint d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^* - \vec{E}_2\times\vec{H}_1^*\right)}{2U_1}$$

That awful integral (2)

In the unperturbed (no-hole) cavity, E is normal to the surface everywhere. Therefore, (E_1 x H_2) must be perpendicular to the surface everywhere (or zero), so dA.(E1 x H2) is zero everywhere. Half the problem is solved!

Moreover, outside of the area near the hole, E_2 must also be normal to the surface everywhere. So the problem of evaluating the surface integral reduces to the problem of evaluating (E_2 x H_1) in the area of the hole between cavities.

After a lot of calculus and some dodgy approximations, we find the following approximate relation for the frequency of the multi-celled cavity:

$$\omega = \frac{z_{01}}{b\sqrt{\mu\varepsilon}}\left\{1 + \frac{2}{3\pi J_1^2\left(z_{01}\right)}\frac{a^3}{b^2 d}\left[1 - \exp\left(-z_{01}h/a\right)\cos\left(k_z d\right)\right]\right\}$$

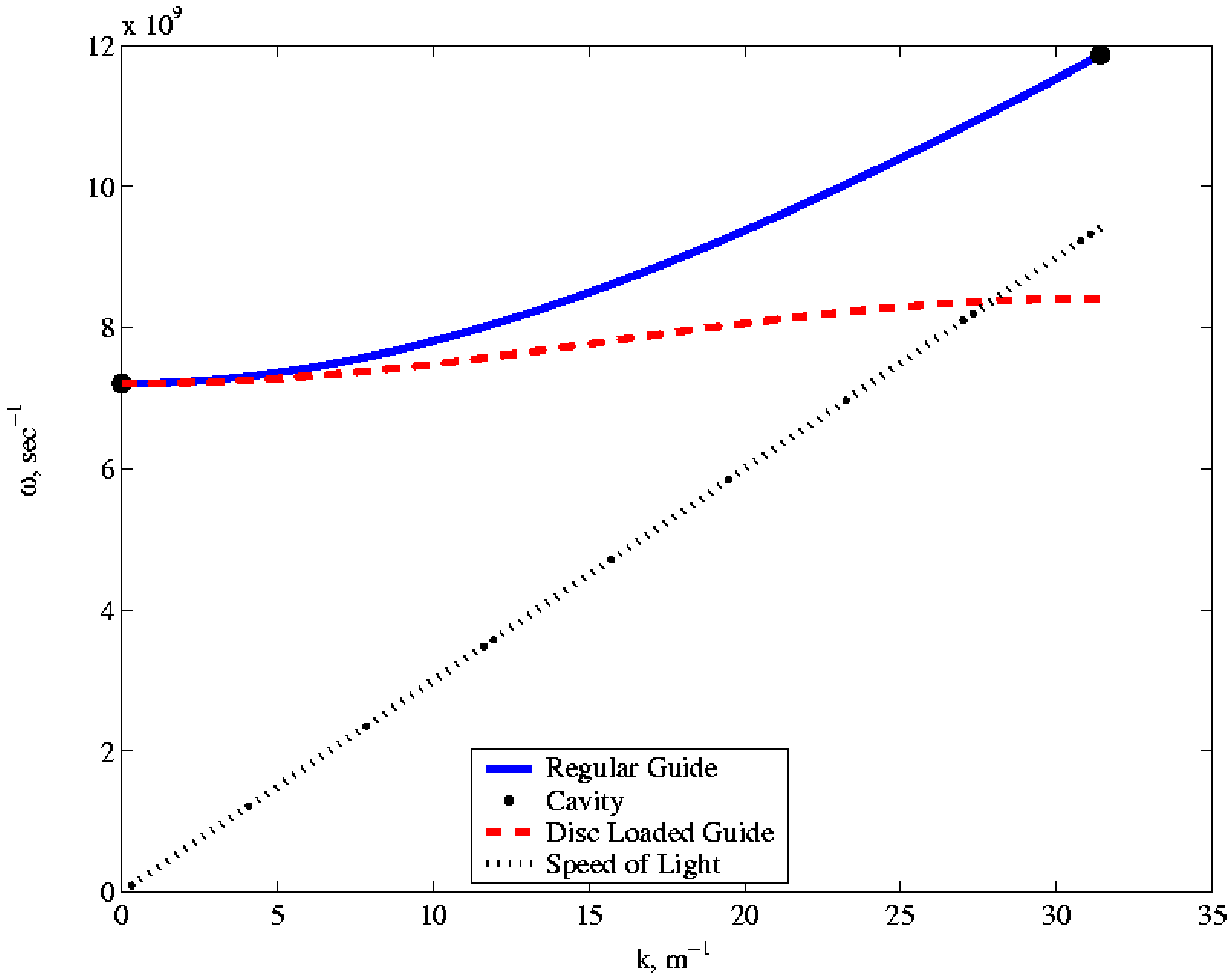
Note that, for a $\rightarrow$ 0, this reduces to the formula for the TM_{010} frequency, as required.

Pass Band and Dispersion Relation

The equation for ω shows that the multi-celled cavity has a pass band, since $\cos(k_z d)$ varies between +1 and -1 – a small band of frequencies will propagate, others will not.

Consider 3 cases – regular waveguide with b = 10 cm, a single-celled cavity with b = 10 cm and d = 10 cm, an infinitely-long multi-celled cavity with b = 10 cm, d = 10 cm, and a = 5 cm (for now let h << d).

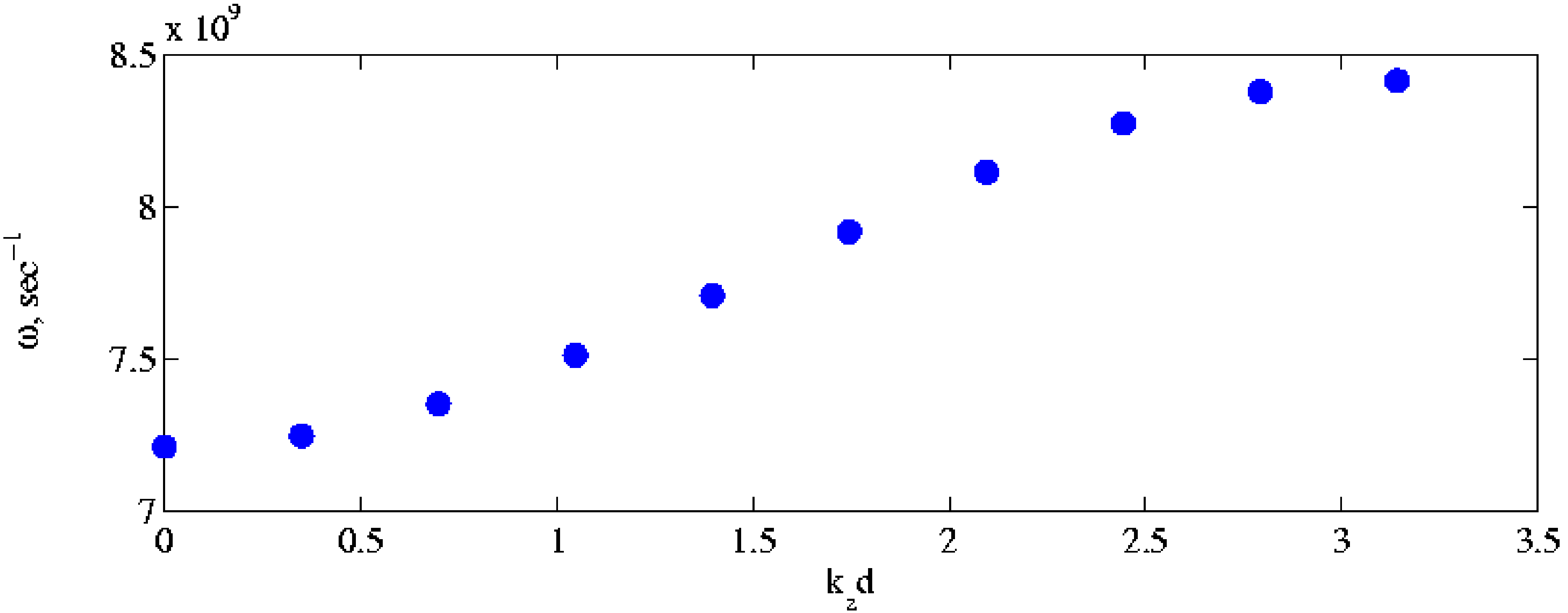
The periodic “loading” from the discs dramatically changes the dispersion relation of the waveguide!



Finite Number of Cells

When the number of cells is infinite, the pass band of the multi-cell cavity is continuous.

When the number of cells is finite (N), the cavity acts like N coupled oscillators, with N modes which are uniformly spaced in k_z, from $k_z d = 0$ to $k_z d = \pi$.



You want to tune the dimensions such that one of the modes has a phase velocity of c.

Selection of Operating Mode

Continuing with our example cavity (10 cells, 10 cm radius, 10 cm length, 5 cm iris radius), the two modes which are closest to v_{ph}=c are 8π/9 and π.

Parameter	9^{th} Mode	10^{th} Mode
$K_z d$	8π/9	π
f	1.366 GHz	1.373 GHz
v_{ph}	1.03 c	0.92 c
Energy gain	100%	~0

If we tune the 8π/9 mode to have v_{ph} = c, the next mode (0.5% different in frequency) has v_{ph} ~ 0.9 c, and so v=c particles get no net acceleration in the cavity!

Too many cells → mode frequencies get close together → RF power source will excite both useful and useless modes

Finite Conductivity

The DC conductivity of Niobium in the superconducting state is, for all practical purposes, infinite. This is not true of its AC conductivity!

Finite conductivity permits the EM fields in the cavity to penetrate a short distance into the walls of the material. Similarly, the currents which are at the surface in a perfect conductor extend some distance into the material. The presence of fields and currents results in dissipation of power in the walls.

We can relate the power density (power/area) to the surface magnetic field. Skipping the tedious math, we find:

$$\frac{dP}{dA} = R_S \left| \vec{H}_S \right|^2$$

R_S, the "surface resistance" (dimensions of resistance, unit=ohms) is the surface resistivity divided by the skin depth.

Finite Conductivity (2)

The surface resistance for elemental Niobium is approximately given by:

$$R_S\left[\Omega\right] \approx 9\times10^{-5}\frac{f^2\left[\mathrm{GHz}^2\right]}{T\left[\mathrm{K}\right]}\exp\left(-\alpha_{SC}\frac{T_C}{T}\right)$$

Where T_C is the critical temperature of the material (for Nb, 9.2 K), and α_{SC} ~ 1.76. Surface resistance is minimized by selecting an operating temperature and a frequency which are both as low as possible.

Note that impurities and defects in the material will produce a residual surface resistance even in the limit of T $\rightarrow$ 0. The practical limit for residual surface resistance appears to be ~ 3 nΩ.

In any event, we can evaluate the total power loss in the walls of a cavity with the surface resistance and the H of a TM_{010} mode:

$$P = \frac{\pi E_0^2 R_S}{Z^2} J_1^2\left(z_{01}\right) b\left(L+b\right)$$

Q of a Cavity

Both P (=dU/dt) and U are proportional to E_0^2, thus they are proportional to one another. Define the "Wall Q" (for "Quality Factor") of the cavity:

$$Q_W \equiv \frac{\omega U}{P} = \frac{z_{01} Z L}{2 R_S (L + b)}$$

As expected, we want a low value of R_S to limit the power lost into the walls. The definition of Q_w implies that, in the absence of external power:

$$U(t) = U(t=0)\exp(-\omega t / Q_w)$$

Sample calculation: for our 1.37 GHz cavity, with L = 10 cm and b = 10 cm, if we operate at 2 K the surface resistance is 23.1 nΩ and $Q_w \sim 1.9 \times 10^{10}$ – it takes seconds for the stored energy to dissipate into the walls of this cavity!

Choice of Operating Mode

In our sample SC cavity, the time for loss of energy into the walls is on the order of seconds.

For the $8\pi/9$ mode, the group velocity is ~8.1% of c, so power will travel through the structure in about 42 nsec. What do we do with the power when it reaches the far end of the cavity? Either remove it, or leave it alone.

If we remove the power, then the cavity is operating very inefficiently – we have to replenish its stored energy every 42 nsec, even though the time needed for the power to dissipate in the walls is seconds.

If we allow the power to reflect off the far end of the cavity, it will propagate back to the input point and leave the cavity. Again, we need to replenish the stored energy much more often than we should.

We can avoid this problem if we tune the cavity to operate in the π mode. In this mode, the group velocity of the RF power is zero, so the only time constant involved in the entry and exit of stored energy is the time constant of the wall losses.

Shunt Impedance

So far we have defined R/Q: $\frac{R}{Q} \equiv \frac{V^2}{\omega U}$ And Q_W: $Q_W \equiv \frac{\omega U}{P}$

The product of these quantities relates the voltage to the power losses in the walls, and has dimensions of resistance and unit of ohms. This is the *shunt impedance*:

$$R_{\text{cav}} = \frac{V^2}{P} = \frac{R}{Q} Q_W$$

For a multi-cell cavity, the shunt impedance is approximately equal to the single-cell shunt impedance x the number of cells. There is also a correction for the hole beam hole:

$$R_{cell} \approx \frac{R_{cav}}{1 + 30.5\left(a/\lambda\right)^2}$$

Introducing the RF Power

RF power from a source is brought into the cavity via an RF coupler – basically a specialized hole in the cavity which lets power “leak” into it.

By symmetry, a hole which allows power to leak in also allows power which is trapped in the cavity to leak out. Since we want to take advantage of the low losses of the SC cavity, we want this “hole” to be small, and the time needed to fill the RF cavity with power becomes correspondingly long.

How does this work? Consider a cavity-coupler system with an incident electromagnetic wave with field amplitude E_{in}, reflected wave amplitude E_{ref}, and “escaping” wave amplitude E_e (“e” for “emitted from the cavity”). Let the coefficient of reflection, Γ, be close to -1 (ie, almost all the incident wave is reflected, with a 180 degree phase change). The net amplitude of the wave flowing away from the cavity, E_{out}, is given by $E_e+\Gamma E_{in}$, by superposition.

RF Power (2)

Conservation of energy: the power incident on the cavity has to be equal to the sum of the power going away from the cavity, the power going to increase the cavity stored energy, and the energy lost in the walls:

$$P_{\text{in}} = P_{\text{out}} + P_{\text{c}} + \frac{dU_{\text{c}}}{dt}$$

By definition: $P_{\text{c}} = \dfrac{\omega U_{\text{c}}}{Q_w}$ Define coupler coefficient β_c: $\beta_{\text{c}} \equiv \dfrac{P_{\text{out}}}{P_{\text{c}}}, \text{ when } P_{\text{in}} \to 0$

Since power is proportional to E^2, we can rewrite the conservation of power expression in terms of EM field amplitude:

$$E_{\text{in}}^2 = \left(E_{\text{e}} + \Gamma E_{\text{in}}\right)^2 + \frac{1}{\beta_{\text{c}}} E_{\text{e}}^2 + \frac{2Q_w}{\omega\beta_{\text{c}}} E_{\text{e}} \frac{dE_{\text{e}}}{dt}$$

RF Power (3)

Define a few useful quantities: $Q_L \equiv \dfrac{Q_w}{1+\beta_c}$ Is the "loaded Q" – the Q of the cavity when the "leakage" through the input coupler is included

$$t_c \equiv \frac{2Q_L}{\omega}$$

Is the "e-folding" time for the cavity including "leakage" through the input coupler

Recalling that Γ ≈ -1, we can write the differential equation from the previous slide in a fairly compact form:

$$t_c \frac{dE_e}{dt} + E_e = \frac{2\beta_c}{1+\beta_c} E_{in}$$

If E_e and E_{in} are zero at t=0, and at that time E_{in} jumps to a nonzero value which is constant in time, the solution to this equation is straightforward:

$$E_e = \frac{2\beta_c}{1+\beta_c} E_{in} \left(1 - e^{-t/t_c}\right) \xrightarrow[\text{math}]{\text{Annoying}} U_c(t) = t_c P_{in} \frac{2\beta_c}{1+\beta_c} \left(1 - e^{-t/t_c}\right)^2$$

RF Power (4)

Similarly, and maybe more usefully, we can see the time-evolution of the voltage:

$$V(t) = \left(1 - e^{-t/t_c}\right)\sqrt{\frac{R}{Q}\omega t_c P_{in} \frac{2\beta_c}{1+\beta_c}}$$

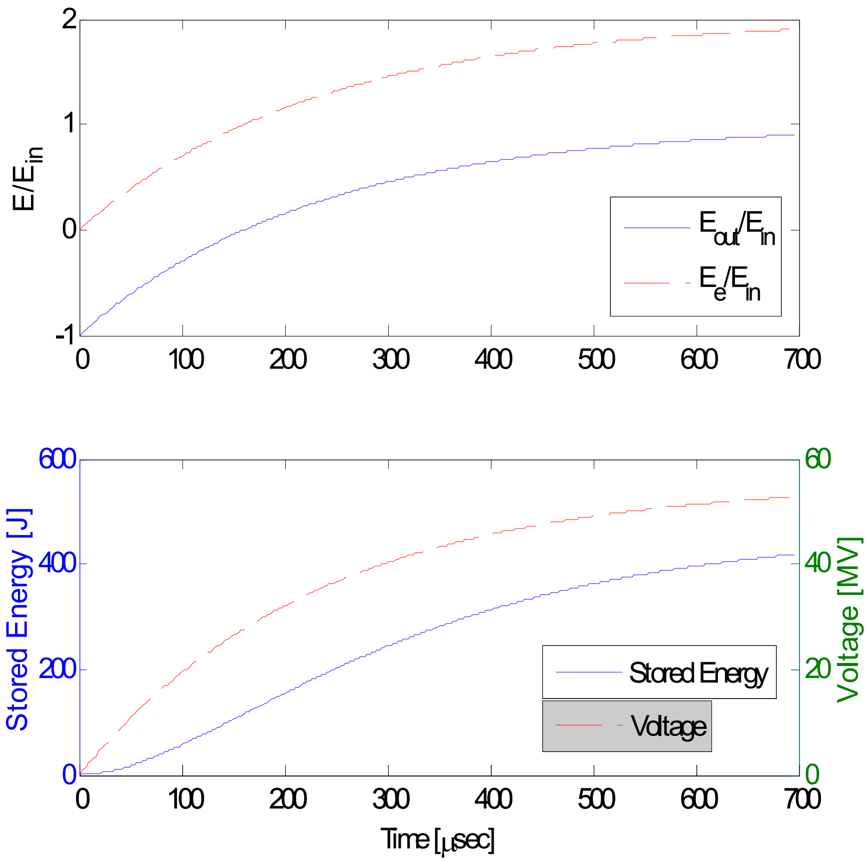
At $t \to \infty$, the stored energy reaches the steady-state, $dU_c/dt \to 0$. It can be shown that, for β_c=1, P_{in} and P_c are balanced, and $P_{out} \to 0$. This state yields the maximum stored energy and maximum achievable voltage.

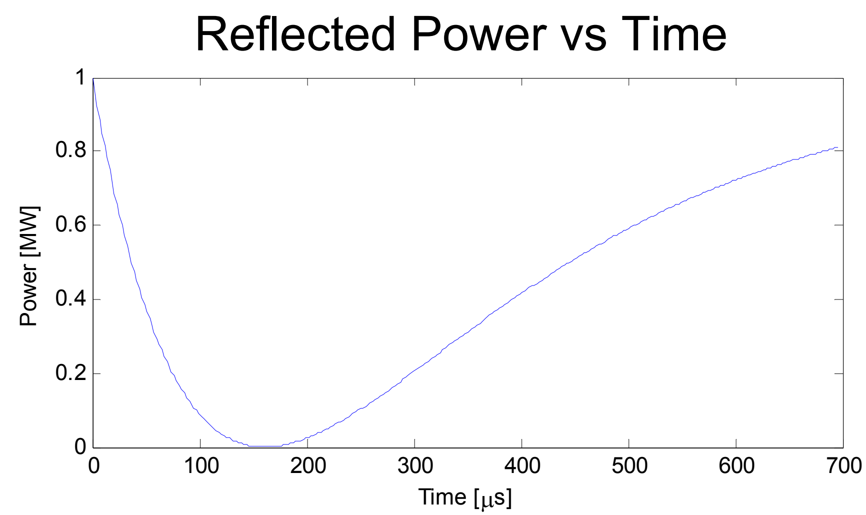
For β « 1 or for β» 1, P_{in} is mainly balanced by P_{out} and P_c is relatively small. When β « 1 the coupler does not let much RF power in, the stored energy in the cavity is small, and P_{out} is dominated by reflected power. Conversely, when β» 1, P_{out} is dominated by emitted power.

Note – when β» 1, P_{out} has to be dominated by reflected power at t « t_c since the cavity is empty, but it goes to emitted power later. This means that there is a phase change in the power coming back from the cavity, and that at some intermediate time P_{out} is zero.

A Concrete Example

Take as an example our 10-cell cavity at 1.37 GHz, and put on an input coupler such that $Q_L = 10^6$. Attach a power source with 1 MW power. The time t_c = 232 μsec, and the voltage at $t \to \infty$ is 55.5 MV.





Introducing the Beam

What we really want to do is accelerate a beam! What are the properties of our beam?

- Steady state current I_{beam}
- Charge per bunch q
- Inter-bunch interval $t_b = q / I_{beam}$
- RMS Bunch length $\sigma_z \ll \lambda$
- Duty factor $H \leq 1$
  - Average current = HI_{beam}
  - Beam is organized in *bunch trains*

The Beam (2)

We accelerate the beam in a cavity with some voltage, V. What are our requirements on this process?

- Every bunch in the bunch train should get the same acceleration
- The RMS energy spread in each bunch should be minimized
- Energy efficiency of the acceleration should be maximized.

The Beam (3)

First things first – we need to modify our original equation to take into account the fact that, when it's accelerated, the beam absorbs energy from the cavity:

$$P_{\text{in}} = P_{\text{out}} + P_{\text{c}} + \frac{dU_{\text{c}}}{dt} + P_{\text{beam}} = P_{\text{out}} + P_{\text{c}} + \frac{dU_{\text{c}}}{dt} + V(t) I_{\text{beam}}\left(t\right)$$

Recall that dU_c/dt is a function of time. If we leave the beam off while the cavity fills, and then at a time t when $V(t)I_b = dU_c/dt$ we turn the beam on, then the beam will take away all the power which would otherwise have gone to filling the cavity. This will lock the cavity into a steady-state, where V becomes a constant so all bunches get the same acceleration.

The efficiency is maximized by minimizing the sum of P_{out} and P_c at the moment that the beam is turned on. If we limit ourselves to $Q_L \ll Q_w$ (which is needed in any event to get an acceptably-fast fill time), then $P_c \ll P_{out}$ and we can solve the problem by turning the beam on when $P_{out} = 0$.

Selection of Parameters

We have 4 parameters: V, I_{beam}, P_{in}, Q_L.

We have 2 constraints: $V\, I_{beam} = P_{in}$, and V = the voltage at the time when $P_{out}=0$.

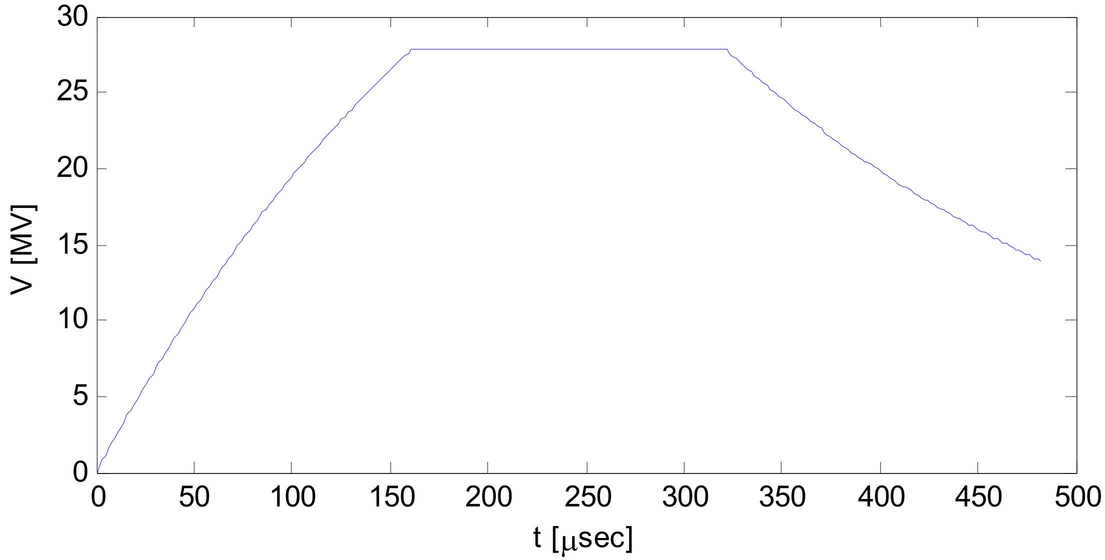
Recalling that $E_{\mathrm{e}} = \frac{2\beta_{\mathrm{c}}}{1+\beta_{\mathrm{c}}} E_{\mathrm{in}} \left(1-e^{-t/t_{\mathrm{c}}}\right)$ and that $E_e = E_{in}$ at the moment when $P_{out}=0$,

We find that we want $e^{-t/t_{\mathrm{c}}} = \frac{1}{2}$ or equivalently $t = t_{\mathrm{c}} \ln 2$

At which time $V = \sqrt{Q_L P_{\mathrm{in}} \frac{R}{Q}}$ Thus $Q_L = \dfrac{V}{I_{\mathrm{beam}} \frac{R}{Q}}$

If you consider our example cavity, with its 1 MW input power and 10^6 Q_L value, you find a steady-state voltage of 27.8 MV and a matched current of 36 mA.

Life-Cycle of an RF Pulse



- The cavity is filled until the steady-state voltage is achieved
  - Stored energy in the cavity increases
  - Voltage of the cavity increases
- When steady-state voltage is achieved, the beam is turned on
  - Beam loading absorbs all incident RF power
  - Cavity stored energy stops rising
  - Voltage becomes fixed in time
- After the last bunch exits the cavity the RF power is turned off
  - The stored energy in the cavity leaks out the input coupler until it's all gone
- Repeat over and over again!

More on Beam Loading

Wangler: "The beam is more than just a medium which absorbs energy from the cavity and acts as a resistive load, but it is in fact a source of fields in the cavity as well!"

Consider an infinitely-short bunch (or a bunch with $\sigma_z \ll \lambda$, which for our purposes is the same thing). It has an infinitely-short "pancake" of EM fields which accompanies it.

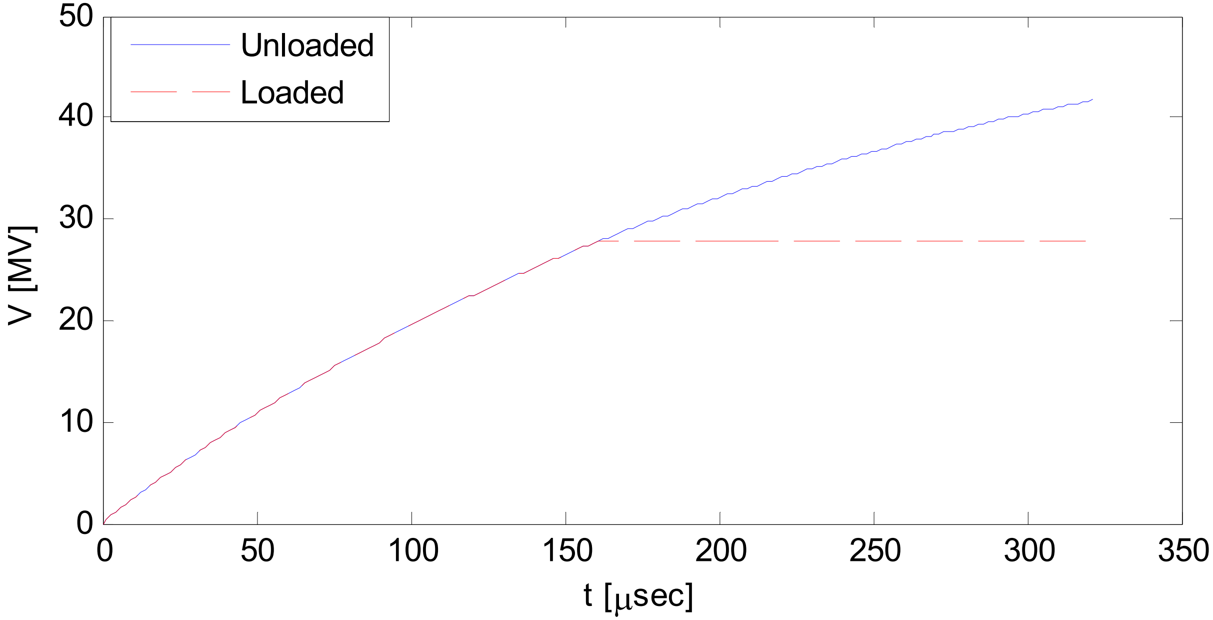
The Fourier transform of the fields has an infinitely-wide spectrum.

Thus, the bunch spectrum has some energy at the RF cavity's resonant frequency.

When the beam passes through an unpowered cavity, the stored energy in the "pancake" which is at the cavity frequency will excite the cavity. The beam's energy will be reduced, and the cavity's stored energy will be increased.

Beam Loading (2)

Let's go back to the powered cavity with the beam passing through. We can consider this case a superposition of a powered cavity with no beam, and a beam passing through an unpowered cavity (the fields have to superpose).



The fields induced by the beam in the unpowered cavity must be given by the difference between these two lines!

Beam Loading (3)

To achieve the exact behavior exhibited here, the voltage induced by a beam in a cavity where the beam is introduced at a time t_1 is given by:

$$V(t > t_1) = -\left[1 - e^{-(t-t_1)/t_c}\right] I_{\mathrm{beam}} Q_L \frac{R}{Q}$$

Note that if the RF power from the source has some phase offset φ relative to the bunches of the beam, the applied voltage always has that phase offset but the beam-induced voltage is always purely decelerating:

$$V = 2\left(1 - e^{-t/t_c}\right)\sqrt{Q_L P_{\mathrm{in}} \frac{R}{Q}} \cos\varphi - \left[1 - e^{(t-t_1)/t_c}\right] I_{\mathrm{beam}} Q_L \frac{R}{Q}$$

So the net phase and voltage of the cavity will vary (or "slew") during beam-time if there is a phase offset in the applied voltage. How do we correct this?

Beam Loading (4)

At the instant of beam arrival, the RF power and phase must change to compensate the loading. Conceptually, one can imagine that the change is a "jump" (step-transition to a new phase and power, which remain valid for the duration of the beam) or a "ramp" (continual variation in the phase and power during beam time). It can be shown that the "jump" is correct for standing-wave cavities. The new phase φ_1 and the new power P_1 are given by:

$$2\sqrt{Q_L P_1 \frac{R}{Q}} \cos\varphi_1 - I_{\mathrm{beam}} Q_L \frac{R}{Q} - V\cos\varphi = 0$$

$$\sqrt{Q_L P_1 \frac{R}{Q}} \sin\varphi_1 - V\sin\varphi = 0$$

Solving systems of equations full of trig functions is not very fun, but at least the system is well-determined, with 2 equations in 2 unknowns. If in doubt, just hand the whole mess to Matlab or some similar application and let it find the solution.

Single-Bunch Beam Loading

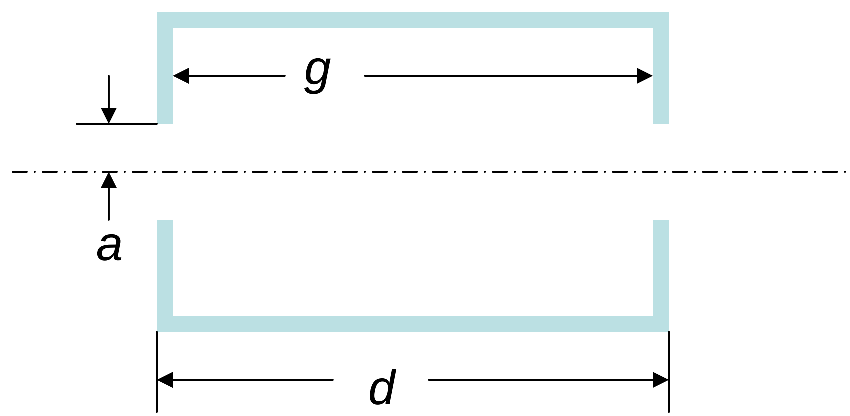
A single bunch will excite both the fundamental mode of the cavity and all the other modes with a longitudinal electric field. By the time bunch 2 arrives, most of the modes have decayed, but between head and tail of the bunch a near-infinite number of modes can be excited.

This results in a decelerating *wakefield* – the tail is decelerated by fields left by the head of the bunch.

Calculating the wakefield is a complicated computational physics problem. A useful for the wake which is a distance *z* behind a driving charge is the following:

$$W_L(z) = \frac{Zc}{\pi a^2}\exp\left(-\sqrt{\frac{z}{s_z}}\right), \text{ where}$$

$$s_z \approx 0.41\frac{a^{1.8}g^{1.6}}{d^{2.4}}$$

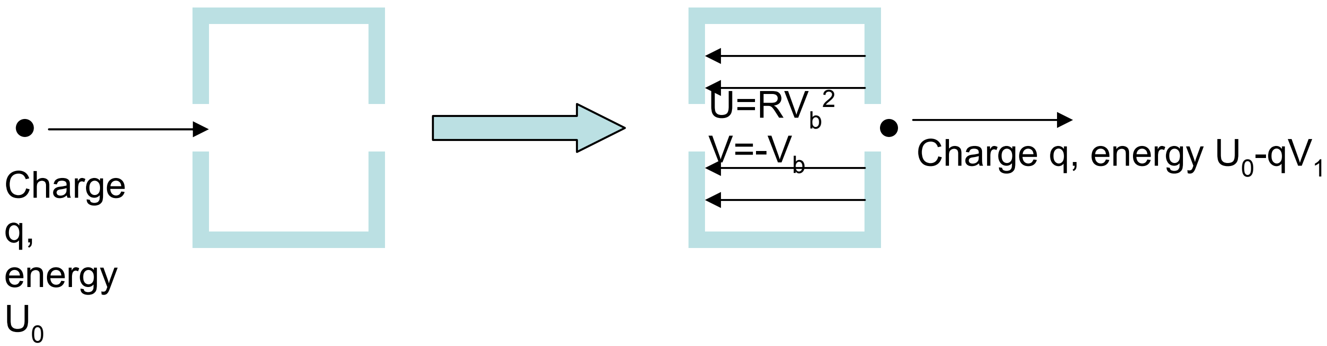


W has dimensions of V/C/m – multiply by charge of driving particle and length of cavity to get the voltage

Single-Bunch Loading (2)

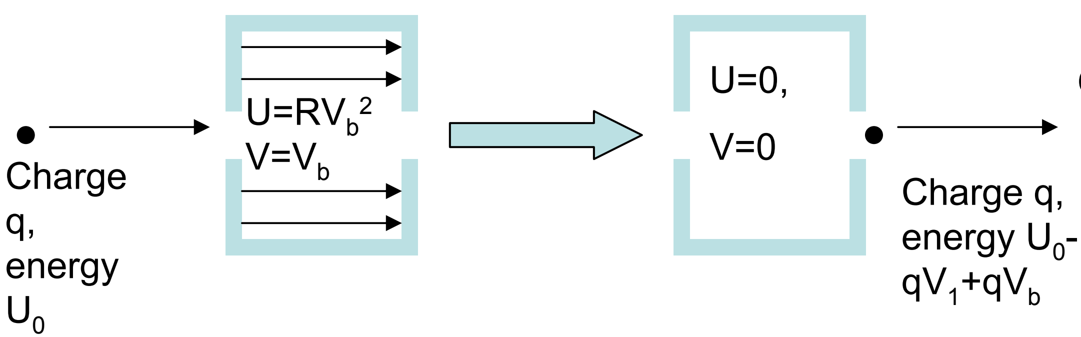
Note that $W_z(0) = Zc/(\pi a^2)$. A single electron experiences its own wakefield!

In fact, the self-loading is given by $W_z(0)qL/2$. To see this, consider the following thought experiment:



Conservation of energy:
$qV_1 = RV_b^2$.

Wait (n+1/2) RF cycles, where n is small, so field becomes accelerating, and introduce another bunch with charge q and energy U_0



Conservation of energy:

$qV_b – 2qV_1 = 0$, or $V_1 = V_b/2$

Fundamental Theorem of Beam Loading

Compensation of Single-Bunch Loading

Consider a 2-particle model, with the particles separated by $2\sigma_z$, so the RMS length of the bunch is σ_z, and a total charge q. The decelerating voltage seen by the two particles is given by:

$$V_1 = \frac{Lq}{4} W_L(0)$$

$$V_2 = \frac{Lq}{4} W_L(0) + \frac{Lq}{2} W_L(2\sigma_z) \approx \frac{Lq}{4} W_L(0)\left(1 + 2e^{-\Delta}\right), \text{ where}$$

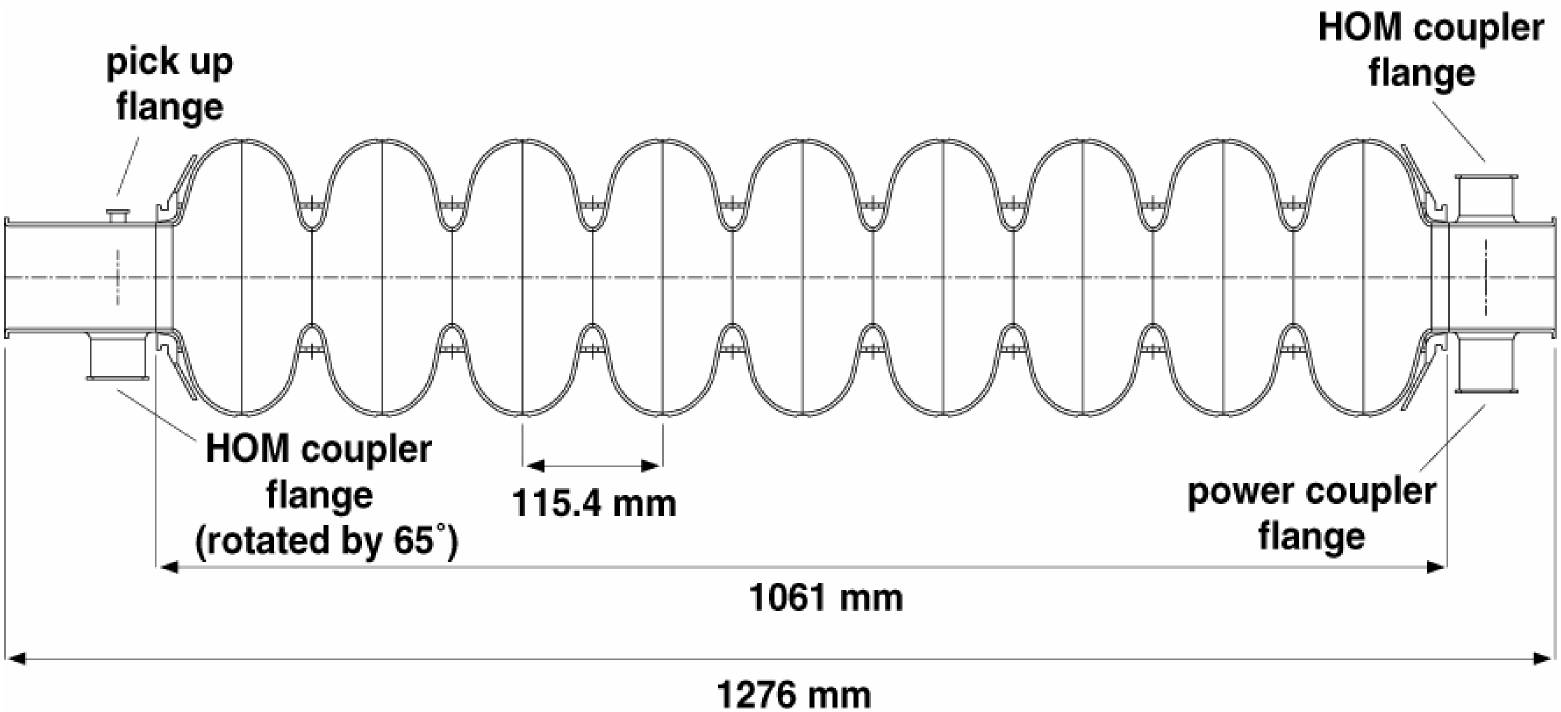
$$\Delta = \sqrt{2\sigma_z / s_z}$$

Note that, even for $\Delta \to 0$ there is a net head-tail energy spread induced by the wakefield.

We can compensate the resulting energy spread by running off-crest, such that the acceleration for the leading particle is reduced by exactly the amount that the trailing particle is decelerated by wakefields. After some trigonometry and taking some appropriate limits:

$$\varphi \approx \frac{LqW_L(0)}{8\pi V} \frac{\lambda}{\sigma_z}$$

The ILC Accelerating Cavity

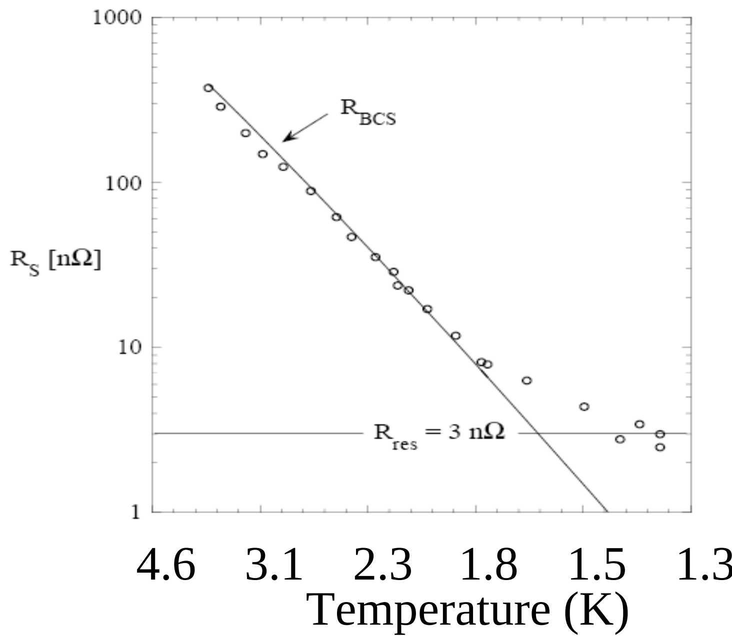


The ILC Cavity (2)

We want a cavity with a low surface resistance (favors low f), short filling time (favors large f). Compromise = 1.3 GHz.

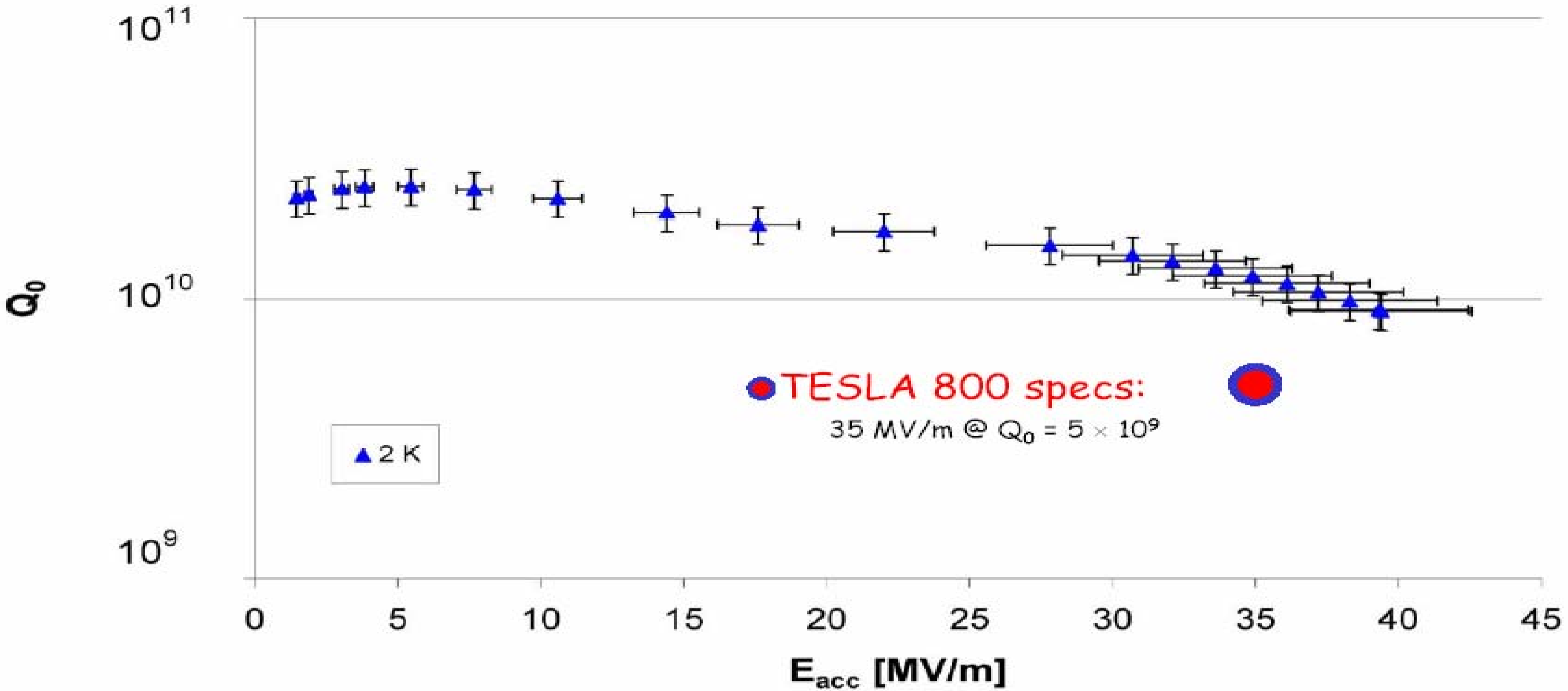
We want a cavity which has a small # of couplers per unit length (favors large # of cells), large frequency separation between π mode and next-nearest mode (favors small # of cells). Compromise = 9 cells, so 1.04 m cavity length (convenient).

R/Q = 1036 Ω.



Operate at 2 K – below this level, resistance starts to be dominated by residual, rather than natural SC AC resistivity!

The ILC Cavity (3)



Operating accelerating gradient is a compromise between real estate bill (favors high gradient) and cooling bill (favors low gradient). Compromise = 31.5 MV/m. At 43 MV/m, magnetic field on cavity iris is ~"critical field", causing cavity to suddenly leave SC state ("quench").

ILC Cavity (4) – Beam Parameters

- Average current of 45 μA * 250 GeV = 11 MW beam power
- Limiting refrigerator power to this limit → duty factor < 1.67%
  - Corresponds to beam current of 2.7 mA
- Damping ring sets upper limit of beam current to ~1-2 A
- Choose 9.0 mA beam current
  - Duty factor 0.5%

ILC Cavity (5) – Loaded Q etc

- With voltage (31.5 MV/m * 1.04 m) and beam current (9.0 mA), we can set:
  - $Q_L = 3.52 \times 10^6$
  - Power = 295 kW
  - Filling time before beam arrival = 597 μsec
- Bunch train length = 1 msec
- 1 msec train length and 0.5% duty cycle = 200 msec between trains
  - IE, 5 Hz repetition rate

ILC Cavity (6) – Practical Issues

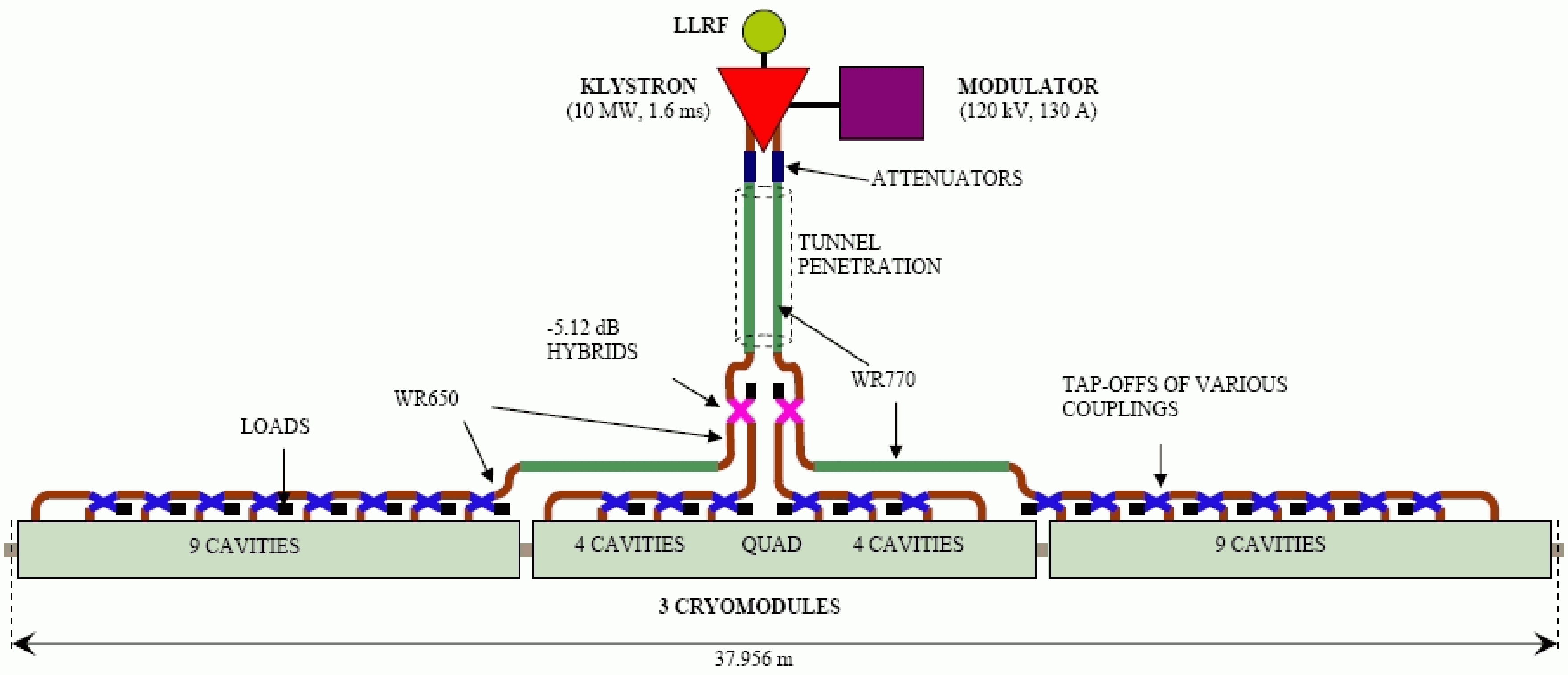
With $Q_L \sim 3.5 \times 10^6$ and f = 1.3 GHz, bandwidth of cavity ~ 370 Hz (!)
Cavity fabrication errors at ~ μm level can change resonant frequency this much – implies that cavity needs a mechanical tuner to set it onto the 1.3 GHz resonance and keep it there.

Cavities which are not powered are taken off the 1.3 GHz resonance with the tuner, otherwise they become "power brakes" for the beam.

Electromagnetic forces on the cavity cause a change in resonant frequency – amplitude ~ 1 Hz/(MV/m)^2 ("Lorentz force detuning"). At 31.5 MV/m, get ~ 1 kHz detuning. Main tuners too slow to keep cavity within BW of 1.3 GHz during 600 μsec build-up of gradient.

Solution: cavity has 2 tuners, a slow bulk tuner and a piezo tuner which compensates the Lorentz force detuning during the fill. Piezo tuner has small range but is really fast!

ILC Standard RF Unit



Positron linac uses 278 RF units to accelerate beam from 15 GeV to 250 GeV; electron linac uses 282 units (4 units make up energy loss in positron production process).