Bhabha Scattering at NNLO

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LCWS08 - Chicago, November 18, '08

Outline

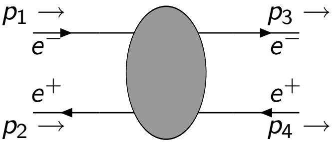
1. Why NNLO QED Corrections to Bhabha Scattering?
2. NNLO Electron Loop Corrections
3. NNLO Photonic Corrections
4. NNLO Heavy Flavor and Hadronic Loop Corrections
5. Conclusions

. . . in the Last Twelve Months

- R. Bonciani, AF, and A. Penin,
  Heavy-flavor contribution to Bhabha scattering,
  Phys. Rev. Lett. **100**, 131601 (2008) [arXiv:0710.4775 [hep-ph]]
- S. Actis, M. Czakon, J. Gluza and T. Riemann,
  Virtual Hadronic and Leptonic Contributions to Bhabha Scattering,
  Phys. Rev. Lett. **100**, 131602 (2008) [arXiv:0711.3847 [hep-ph]]
- R. Bonciani, AF, and A. Penin,
  Calculation of the Two-Loop Heavy-Flavor Contribution to Bhabha Scattering,
  JHEP **0802**, 080 (2008) [arXiv:0802.2215 [hep-ph]]
- J. Kühn and S. Uccirati,
  Two-loop QED hadronic corrections to Bhabha scattering,
  Nucl. Phys. B **806**, 300 (2008) arXiv:0807.1284 [hep-ph]
- S. Actis, M. Czakon, J. Gluza and T. Riemann,
  Virtual Hadronic and Heavy-Fermion $O(\alpha^2)$ Corrections to Bhabha Scattering,
  arXiv:0807.4691 [hep-ph].

Bhabha Scattering and Luminosity-I

$$e^+e^- \longrightarrow e^+e^-$$



$$s \equiv -P^2 = -(p_1+p_2)^2 = 4E^2 > 4m^2 \quad t \equiv -Q^2 = -(p_1-p_3)^2 = -4(E^2-m^2)\sin^2\frac{\theta}{2} < 0$$

- Effective tool for the Luminosity measurement @ e^+e^- colliders

$$\sigma_{\text{exp}} \equiv \frac{N}{L} \qquad L = \frac{N}{\sigma_{\text{bh-th}}}$$

Bhabha Scattering and Luminosity-II

- In the region employed for L measurements the Bhabha scattering cross section is large, QED dominated, and measured with very high precision (1 permille at KLOE/DAFNE)
- SABH is employed at LEP and ILC, while LABH is employed at colliders operating at $\sqrt{s} = 1 - 10$GeV
- Due to beam-beam interactions, at ILC the colliding energy $\sqrt{s}$ shows a continuous spectrum: the LABH can also be used to determine the luminosity spectrum
- Realistic simulation of Bhabha events are performed by sophisticated MC generators which take into account the detector geometry, experimental cuts, theoretical input (fixed order calculations, resummation)

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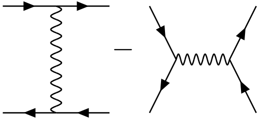
$\Longrightarrow$ study of radiative corrections to Bhabha scattering

Warning

$\mathcal{M} =$ 

- In this talk we consider the QED process only
- We consider differential cross-sections summed over the spins of the final state particles and averaged over the spin of the initial ones

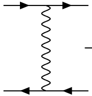
$$
\begin{aligned}
\frac{d\sigma_0(s,t)}{d\Omega} &= \frac{\alpha^2}{s}\left\{ \frac{1}{s^2}\left[st + \frac{s^2}{2} + (t-2m^2)^2\right] + \frac{1}{t^2}\left[st + \frac{t^2}{2} + (s-2m^2)^2\right]\right. \\
&\quad \left. + \frac{1}{st}\left[(s+t)^2 - 4m^4\right] \right\}
\end{aligned}
$$

Virtual Corrections to the Cross Section -I

$$\frac{d\sigma(s,t)}{d\Omega} = \frac{d\sigma_0(s,t)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t)}{d\Omega} + \mathcal{O}\left((\alpha/\pi)^3\right)$$

The $\mathcal{O}(\alpha^3)$ virtual corrections (one-loop $\times$ tree-level) are well known (in the full SM), no problem in keeping $m_e \neq 0$

M. Consoli (1979),
M. Böhm, A. Denner, and W. Hollik (1988),
M. Greco (1988),...

$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Bigg\{\Bigg($$

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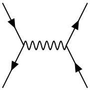
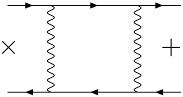
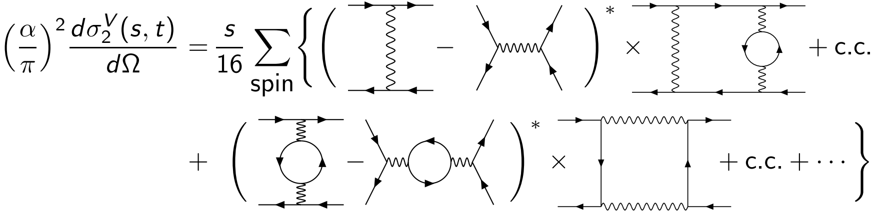

$\Bigg)^* \times$

$$+\,\text{c.c.} + \cdots\Bigg\}$$

Order α^4 *QED* corrections:

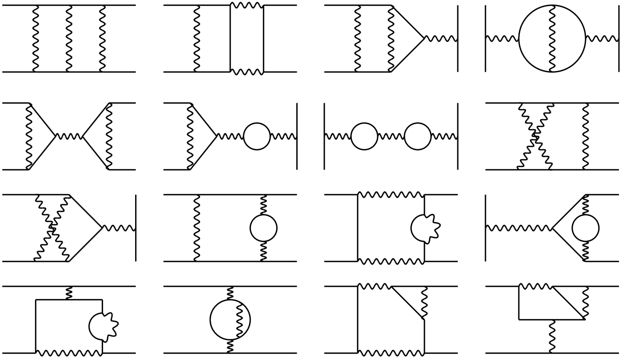
- Contributions from two-loop $\times$ tree-level and one-loop $\times$ one-loop
- Can be divided in three sets,
  i) with a closed electron loop,
  ii) closed heavy(er) flavor loop, and
  iii) photonic (without fermion loops)



RADIATIVE CORRECTIONS IN $m_e = 0$ APPROXIMATION

The virtual corrections where first obtained in the massless electron approximation

Z. Bern, L Dixon, and A. Ghinculov ('00)



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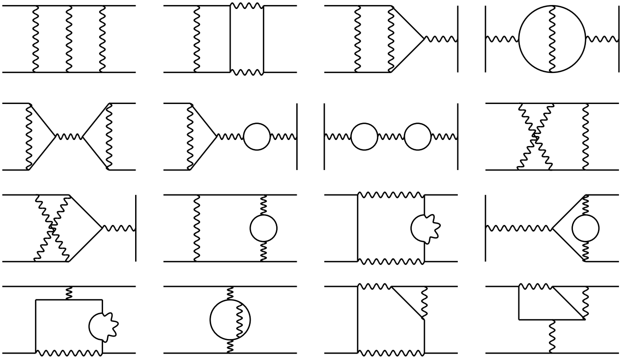
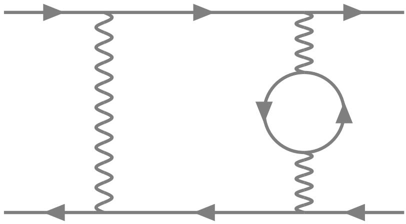
However, in order to interface the fixed order calculation with the existing MC, it is necessary to keep the electron mass as a collinear regulator

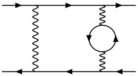


Electron Loop Corrections

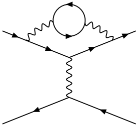


α^4 Electron Loop Corrections

All the two-loop graphs including a closed electron loop can be calculated also keeping $m_e \neq 0$ and without relying on any approximation or expansion



$\otimes$ Born Amplitude



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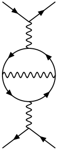

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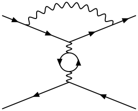

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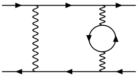
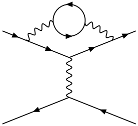
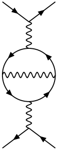
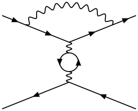
 $\otimes$ Born Amplitude

 $\otimes$ Born Amplitude

 $\otimes$ Born Amplitude

 $\otimes$ Born Amplitude

- The relevant integrals can be reduced to combination of a relatively small set of Master Integrals employing the Laporta algorithm
- The MIs (including the ones for the box) can be evaluated employing the differential equation method

R. Bonciani *et al.*('03-'04)

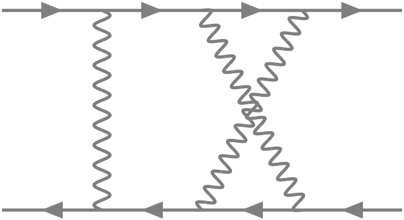
In the electron loop corrections to the CS

- both UV and IR divergences are regularized within the DIM REG scheme
- the UV renormalization is carried out in the on-shell scheme
- the graphs are at first calculated in the non physical region $s < 0$ and then analytically continued to the physical region $s > 4m_e^2$
- the cross section can be expressed in terms of HPLs and 2dHPLs with arguments

$$x = \frac{\sqrt{s} - \sqrt{s - 4m_e^2}}{\sqrt{s} + \sqrt{s - 4m_e^2}} \quad y = \frac{\sqrt{4m_e^2 - t} - \sqrt{-t}}{\sqrt{-t} + \sqrt{4m_e^2 - t}} \quad z = \frac{\sqrt{4m_e^2 - u} - \sqrt{-u}}{\sqrt{-u} + \sqrt{4m_e^2 - u}}$$

- The residual IR poles are eliminated by adding the contribution of the soft photon radiation

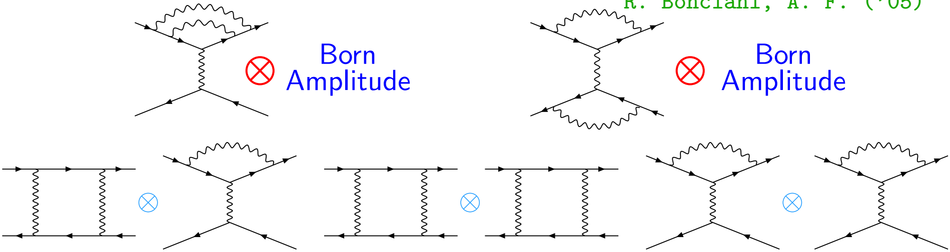
Photonic Corrections



$\mathcal{O}(\alpha^4)$ Photonic Corrections

With the same techniques employed in obtaining the $\mathcal{O}(\alpha^4(N_F = 1))$ non-approximated differential CS, it is possible to calculate the photonic virtual corrections (and related soft photon emission) to the CS at order $\mathcal{O}(\alpha^4)$, except for the ones arising from the the two loop photonic boxes

R. Bonciani, A. F. ('05)



The one- and two-loop Dirac form factors in the t-channel are sufficient to determine completely the small angle cross section

$$\frac{d\sigma_2}{d\sigma_0} = 6({F_1}^{(1l)}(t))^2 + 4{F_1}^{(2l)}(t)$$

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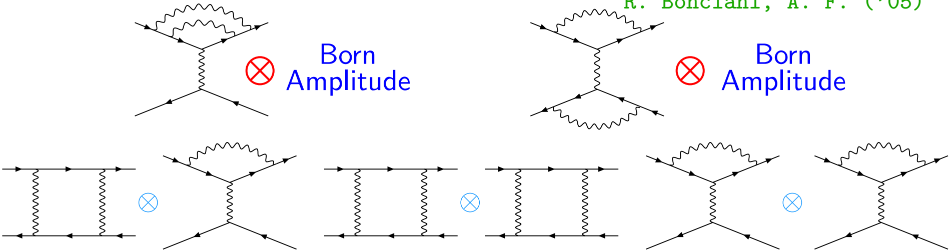

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α^4 Photonic Corrections-m_e^2/s Expansion

- Building on the BDG result and on works by A. B. Arbuzov *et al.*, B. Tausk, N. Glover, and J. J. van der Bij ('01) obtained the terms proportional to $L = \ln m_e^2/s$ of the full (virtual + soft) photonic CS (i. e. graphs including a closed electron loop have been neglected)
- A. Penin ('05) obtained also the constant terms of the photonic CS in the m_e^2/s expansion

Therefore, in the expansion

$$\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_2^{(1)} \ln\left(\frac{s}{m_e^2}\right) + \delta_2^{(0)} + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

$\delta_2^{(2)}$, $\delta_2^{(1)}$, and $\delta_2^{(0)}$ are known

- Several partial cross-checks of this results were possible by comparing it with the $m_e^2/s \to 0$ limit of the exact result for the photonic vertex and one-loop by one-loop corrections

Mass from Massless-I

As can be seen from Penin's result, when neglecting positive powers of the electron mass, the problem is equivalent to change the regularization scheme for the collinear singularities:

Is it possible to calculate graphs employing DIM REG to regulate both soft and collinear singularities and then translate a posteriori the collinear poles into collinear logs?

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For a generic QED/QCD process, with no closed fermion loops

$$\mathcal{M}^{(m\neq 0)} = \prod_{i \in \{\text{all legs}\}} Z_i^{\frac{1}{2}}(m,\varepsilon)\mathcal{M}^{(m=0)}$$

where Z is defined through the Dirac form factor

$$F^{(m\neq 0)}(Q^2) = Z(m,\varepsilon)\, F^{(m=0)}(Q^2) + \mathcal{O}(m^2/Q^2)$$

A. Mitov and S. Moch ('06)

Mass from Massless-II

with a similar technique applied to Bhabha scattering it was possible to calculate all the NNLO corrections in the limit $s, |t|, |u| \gg m_f^2 \gg m_e^2$

T. Becher and K. Melnikov ('07)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-r+r^2}{r}\right)\left[1 + \frac{\alpha}{\pi}\delta_1 + \left(\frac{\alpha}{\pi}\right)^2\delta_2\right]$$

$(r = 1/2(1-\cos\theta))$

$$\delta_2 = \delta_2^{\text{photonic}} + \delta_2^{\text{electron loop}} + \delta_2^{\text{heavy flavor loop}}$$

- photonic corrections in agreement with A. Penin ('05)
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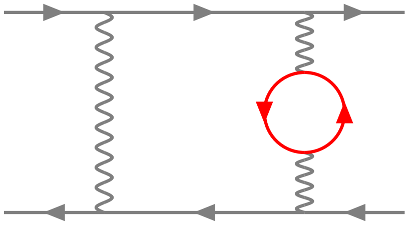
Heavy Fermion Corrections



Beyond $s \gg m_f^2$

In any realistic case the approximation $s, |t|, |u| \gg m_e^2$ is more than enough
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for example

- τ loop at KLOE, where $\sqrt{s} = 1\,\mathrm{GeV} < m_\tau$
- top quark loop at ILC, where $\sqrt{s} \approx 500\,\mathrm{GeV}$ and $m_t^2/t, m_t^2/u < 1$ just in the angular region $40^o < \theta < 140^o$

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It is necessary to calculate the NNLO corrections including an heavy fermion loop by retaining the exact dependence on m_f

$$s, |t|, |u|, m_f^2 \gg m_e^2$$

this is a non trivial problem involving four-scale two-loop boxes ...

$\Longrightarrow$

Structure of the Collinear Poles

What is the collinear structure of these corrections?

$$\delta_2 = \delta_2^C(s, t, m_f^2) \ln\left(\frac{s}{m_e^2}\right) + \delta_2^R(s, t, m_f^2) + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

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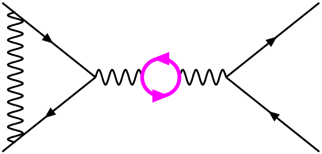
It is possible to show that the collinear logarithm arises from trivial reducible graphs only



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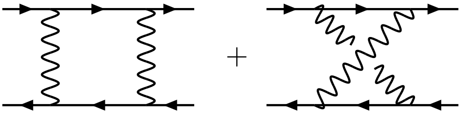

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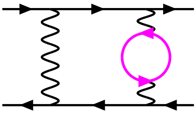
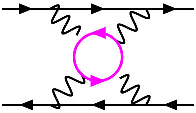
 $+$  $=$ Free of collinear poles

$$u = -s - t$$

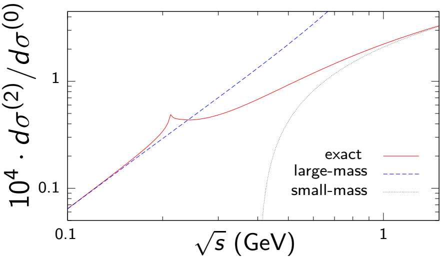
After UV renormalization, the only remaining poles are the IR (soft) ones

Technical Details

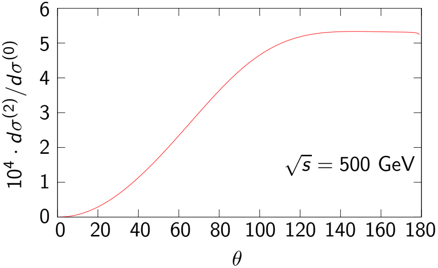
R. Bonciani, AF, and A. Penin ('07-'08)

- It was possible to calculate the boxes for $m_e = 0$ and generic $s, |t|, |u|, m_f^2 \gg m_e^2$, therefore effectively eliminating one mass scale from the most challenging part of the calculation
- We employed IBPs and Differential Eq. Method
- The analytical result can be expressed in terms of HPL and a few GHPLs of a new class. The latter can be expressed in closed form in terms of polylogs
- by expanding the exact result it was possible to recover the result of Actis *et al* and Becher Melnikov
- It now is possible to study the τ loop effects at intermediate energies and top loop effects at ILC energies, where the $s, |t|, |u| \gg m_f^2$ approximation is not valid

Results



Two-loop corrections to the Bhabha scattering differential cross section at $\theta = 60^\circ$ due to a closed muon loop



Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 500$ GeV due to a closed loop of t-quark ($m_t = 170.9$ GeV).

The large logs depending on the IR cut-off ω are excluded from the numerical analysis

Numerical Analysis

The actual impact of the two-loop virtual corrections on the theoretical predictions can be determined only after the corrections are implemented into MC event generators

However, for ILC it is possible to conclude that

- The heavy flavor corrections are dominated by the collinear logs $\ln\left(s/m_e^2\right)$
- Top corrections (including $\mathcal{O}(\alpha\alpha_s)$) corrections reach ~ 0.5 permille for $\theta > 140^o$
- Very large terms proportional to $\ln^3\left(s/m_f^2\right)$ cancel against the contribution of (soft) real f-pair emission
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Dispersion Relation Approach

Alternative way to calculate diagrams with closed fermion loops

- replace the photon propagator on which one wants to insert a fermion loop as follows

$$\frac{\delta_{\mu\nu}}{q^2} \longrightarrow \frac{\delta_{\mu\nu}}{q^2}\left(q^2\delta_{\mu\nu} - q_\mu q_\nu\right)\Pi\left(q^2\right)\frac{\delta_{\mu\nu}}{q^2}$$

- apply the subtracted dispertion relation

$$\Pi\left(q^2\right) = -\frac{q^2}{\pi}\int_{4m^2}^{\infty} dz\, \frac{\mathrm{Im}\Pi\left(z\right)}{z}\frac{1}{q^2+z}$$

- relate the self energy imaginary part to the decay rate of an off-shell photon

$$\mathrm{Im}\Pi\left(z\right) = -\frac{\alpha}{3}R(z) \equiv -\frac{\alpha}{3}\frac{\sigma\left(e^+e^- \to \gamma^* \to f\overline{f}\right)}{4\pi\alpha^2/3}$$

- perform first the integration over the photon momentum q and then the integration over z

Dispersion Relations and Bhabha Scattering

S. Actis *et al* ('07-'08)
J. Kühn, S. Uccirati ('08)

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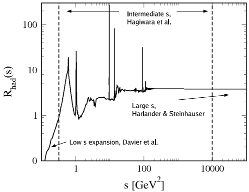
Hadronic Corrections

The corrections related to the hadronic vacuum polarization can be obtained by taking $R_{\text{had}}(z)$ from $e^+e^- \to$ hadrons data



At the ILC, hadronic corrections give the largest effect at large angles ($< 2\%$)

Summary & Conclusions

- A precise knowledge of the Bhabha scattering cross section (both at small and large angle) is crucial in order to determine the luminosity at e^+e^- colliders
- Photonic and electron loop corrections have been known for some time. Recently, we calculated also the heavy flavor NNLO corrections. This was done with a technique that effectively eliminates one mass scale from the most challenging part of the calculation and provides result in closed analytical form
- The hadronic corrections were evaluated by two groups with an approach based on dispersion relations
- The calculation of the virtual NNLO QED corrections is basically complete, but there is still work to be done: ex. one-loop hard photon emission
- These fixed order results must be included/interfaced with MC generators

Backup Slides

Penin's technique (in a Nutshell)

- Consider the amplitude of the two loop virtual corrections to the cross-section in which collinear and IR divergencies are regularized by m_e and λ: $\mathcal{A}^{(2)}(m_e, \lambda)$
- Build an auxiliary amplitude $\overline{\mathcal{A}}^{(2)}(m_e, \lambda)$ with the same IR singularities of the $\mathcal{A}^{(2)}(m_e, \lambda)$ but sufficiently simple to be evaluated in the small mass expansion
- The quantity $\delta\mathcal{A}^{(2)} = \mathcal{A}^{(2)} - \overline{\mathcal{A}}^{(2)}$ has a finite limit when m_e and λ tend to zero
- $\delta\mathcal{A}^{(2)}$ is regularization scheme independent and it can be reconstructed from the known results for the virtual corrections calculated by setting $m_e = \lambda = 0$ from the start

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$\Longrightarrow$ The method cannot be applied to the $\alpha^4(N_F = 1)$ corrections

Collinear Poles & Gauge Invariant Sets

- In a physical (Coulomb or axial) gauge, the collinear divergencies factorize and can be reabsorbed in the external field renormalization

J. Frenkel and J. C. Taylor ('76)

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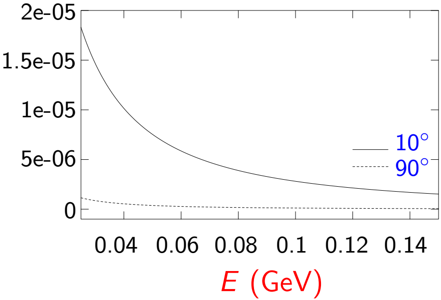
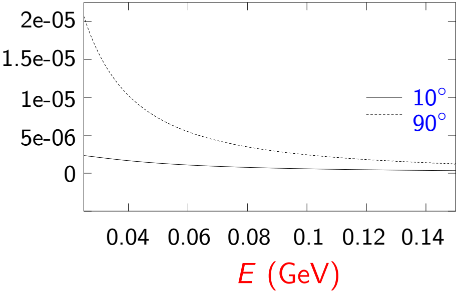
In the Feynman gauge we employ in the calculation, single box diagrams show collinear divergencies that cancel in the sum over all the box diagrams

(Ir)Relevance on the Terms $\propto m_e^2$





$$D_{\texttt{Vert}} = \left(\frac{\alpha}{\pi}\right)^2 \frac{\left| \left. \left| \frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega} - \frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega} \right| \right|_L \right|}{\frac{d\sigma_0}{d\Omega} + \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1}{d\Omega}}$$

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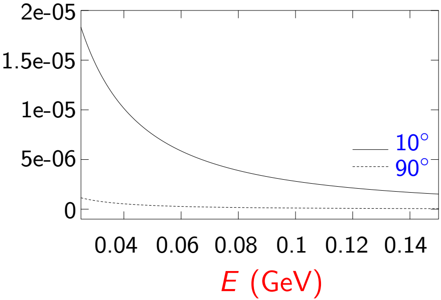

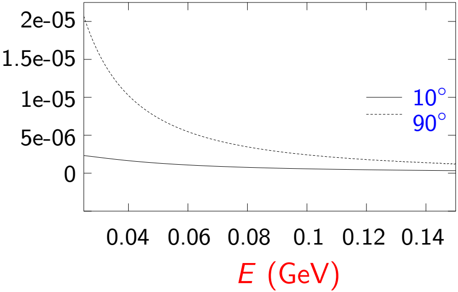

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the terms proportional to m_e become negligible for values of E that are very small with respect to the ones encountered in e^+e^- experiments

NNLO Heavy Flavor CS at $\sqrt{s} = 500$ GeV

$\sqrt{s} = 500$ GeV: θ	e (10^{-3})	μ (10^{-3})	τ (10^{-3})	t (10^{-3})
1°	3.4957072	0.9690710	0.1542329	0.0000575
2°	4.1203687	1.2491270	0.3573661	0.0002466
3°	4.5099086	1.4146106	0.5140242	0.0005763
50°	7.5740980	2.3185800	1.8411736	0.1707137
60°	7.7965875	2.3446744	1.9274750	0.2340996
70°	8.0081541	2.3708714	2.0072240	0.2998535
80°	8.2164081	2.3981523	2.0829886	0.3635031
90°	8.4172449	2.4207950	2.1521199	0.4202418
100°	8.5982864	2.4282953	2.2085332	0.4655025
110°	8.7451035	2.4090920	2.2456055	0.4979010
120°	8.8465287	2.3536259	2.2585305	0.5181602
130°	8.8954702	2.2543834	2.2446158	0.5287459

Table: *The second-order electron, μ, τ-lepton, and top-quark contributions to the differential cross section of Bhabha scattering at $\sqrt{s} = 500$ GeV in units of 10^{-3} of the Born cross section. The top-quark contribution also includes $\mathcal{O}(\alpha\alpha_s)$*