Errors in 2D beam emittance measurement in FODO channel

<u>Purpose</u>: minimization of errors in beam emittance measurements through beam profile measurements

1. Intro

XX International Linac Conference, Monterey, California

MEASUREMENT AND CORRECTION OF CROSS-PLANE COUPLING IN TRANSPORT LINES*

M. Woodley, P. Emma, *SLAC*, Stanford, CA 94309, USA

Abstract

In future linear colliders the luminosity will depend on maintaining the small emittance aspect ratio delivered by damping rings. Correction of cross-plane coupling can be important in preventing dilution of the beam emittance. In order to minimize the vertical emittance, especially for a flat beam, it is necessary to remove all cross-plane (x-y) correlations. This paper studies emittance measurement and correction for coupled beams in the presence of realistic measurement errors. The results of simulations show that reconstruction of the full 4×4 beam matrix can be misleading in the presence of errors. We suggest more robust tuning procedures for minimizing linear coupling.

1 INTRINSIC EMITTANCE

A four-dimensional (4D) symmetric beam matrix, σ, contains ten unique elements, four of which describe coupling. The *projected* (2D) beam emittances, ε_x and ε_y, are defined as the square roots of the determinants of the *on*-diagonal 2×2 submatrices. If one or more of the elements of the *off*-diagonal submatrix is non-zero, the beam is x-y coupled. Diagonalization of the beam matrix yields the *intrinsic* beam emittances, ε_1 and ε_2.

$$\sigma=\begin{pmatrix} <x^2> & <xx'> & <xy> & <xy'> \\ <xx'> & <x'^2> & <x'y> & <x'y'> \\ <xy> & <x'y> & <y^2> & <yy'> \\ <xy'> & <x'y'> & <yy'> & <y'^2> \end{pmatrix}, \quad \overline{\sigma}=\overline{R}\sigma\overline{R}^T=\begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{pmatrix}$$

diamond symbols) are used to correct the <xy>, <x′y′>, <x′y>, and <xy′> beam correlations, respectively, at location 4. The horizontal and vertical betatron phase advances between the skew quadrupoles are also indicated on the figure. This scheme allows total correction of any arbitrary linearly coupled beam with correction range limited only by the available skew quadrupole strength.

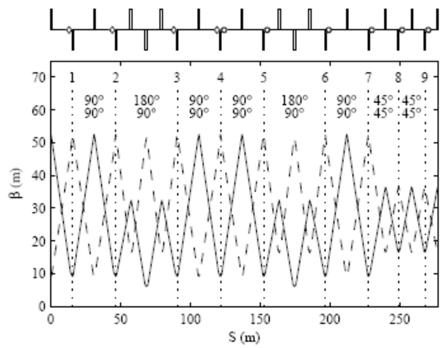


Figure 1: SCS (S=0-120 m) plus 4D emittance measurement section (S=120-270 m): β_x (solid), β_y (dash). Diamond symbols indicate skew quadrupoles; circles indicate wire scanners. The betatron phase advances between devices are shown in 2 rows above the plotted β–functions (x on top and y below).

3 4D EMITTANCE MEASUREMENT

The ideal 4D emittance measurement section contains the nominal NLC beam at 250 GeV ($\gamma\varepsilon_1=3\times10^{-[illegible]}$ m, $\gamma\varepsilon_2=3\times10^{-8}$ m). For these emittances, the ideal rms beam sizes at the wires range from 1.5-10 μm. In each simulation, the real beam size on each wire is given a gaussian distributed multiplicative random error of rms f_{err}

$$\sigma_{sim}=\left(1+f_{err}\right)\sigma_{ideal}$$

and the ensemble of simulated measurements is analyzed.

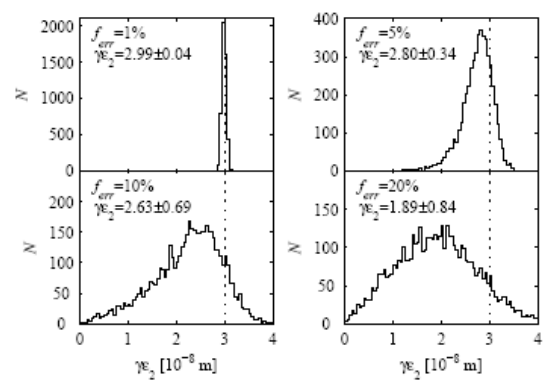


Figure 2: Results of simulations of 4D emittance measurement and reconstruction of $\gamma\varepsilon_2$ (coupled input beam). Vertical dotted lines show the actual value $\gamma\varepsilon_{20}$ used in the simulations.

$\vec{\sigma}$ - matrix of 4D beam:

$$\vec{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{vmatrix} \qquad (1\text{-}1)$$

Matrix (1-1) is symmetric, $\sigma_{ij} = \sigma_{ji}$, so only 10 elements of matrix are independent. Measurements are provided for beam sizes

$$\langle x^2 \rangle = \sigma_{11}, \ \langle y^2 \rangle = \sigma_{33}, \ \langle xy \rangle = \sigma_{13} \qquad (1\text{–}2)$$

at different locations . Explicit transformation for σ_{11}, σ_{33}, σ_{13} is:

$$\begin{vmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \end{vmatrix} = \begin{vmatrix} R_{11}^2 & 2R_{11}R_{12} & 2R_{11}R_{13} & 2R_{11}R_{14} & R_{12}^2 & 2R_{12}R_{13} & 2R_{12}R_{14} & R_{13}^2 & 2R_{13}R_{14} & R_{14}^2 \\ R_{31}^2 & 2R_{31}R_{32} & 2R_{31}R_{33} & 2R_{31}R_{34} & R_{32}^2 & 2R_{32}R_{33} & 2R_{34}R_{32} & R_{33}^2 & 2R_{33}R_{34} & R_{34}^2 \\ R_{11}R_{31} & R_{12}R_{31}+R_{11}R_{32} & R_{13}R_{31}+R_{11}R_{33} & R_{31}R_{14}+R_{11}R_{34} & R_{12}R_{32} & R_{13}R_{32}+R_{12}R_{33} & R_{32}R_{14}+R_{12}R_{34} & R_{13}R_{33} & R_{14}R_{33}+R_{13}R_{34} & R_{14}R_{34} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{24} \\ \sigma_{33} \\ \sigma_{34} \\ \sigma_{44} \end{vmatrix}_0 \qquad (1\text{-}3)$$

To determine 10 independent values of σ-matrix, we need 3 x 3 + 1 =10 equations from at least 4 independent measurement stations. To be physically correct, the following conditions for matrix elements have to be fulfilled (for details see BDS meeting note of YB, 09/12/2006):

$$\boxed{\sigma_{ii} > 0}\ , \qquad \boxed{\begin{vmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ij} & \sigma_{jj} \end{vmatrix} > 0}\ , \qquad i = 1,2,3,4; \quad j > i \tag{1-4}$$

because each projection of 4D beam onto 2D plane is an ellipse:

x-x' projection:

$$\boxed{\sigma_{22}x^2 + \sigma_{11}x'^2 - 2\sigma_{12}xx' = \sigma_{11}\sigma_{22} - \sigma_{12}^2} \tag{1-5}$$

x-y projection:

$$\boxed{\sigma_{33}x^2 + \sigma_{11}y^2 - 2\sigma_{13}xy = \sigma_{11}\sigma_{33} - \sigma_{13}^2}\text{, etc.} \tag{1-6}$$

Complete analysis of errors in 4D beam emittance measurements requires solution of 10x10 linear system with variable parameters. In this note we consider errors in 2D beam emittance measurements (uncoupled beam), where beam at each plane (x-x'), (y-y') is determined by 3 parameters α, β, $\ni$. Special case of 2D beam with $\alpha = 0$ was considered in BDS meeting note of YB, 10/24/2006.

2. Errors in 2D beam emittance measurement

Consider 2D beam emittance measurement problem for the beam propagating in FODO channel. Single particle transformation matrix

$$\begin{vmatrix} x \\ x' \end{vmatrix} = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \begin{vmatrix} x_o \\ x'_o \end{vmatrix} \qquad (2\text{-}1)$$

Beam ellipse transformation:

$$\begin{vmatrix} \beta \\ \alpha \\ \gamma \end{vmatrix} = \begin{vmatrix} C^2 & -2CS & S^2 \\ -CC' & CS'+SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{vmatrix} \begin{vmatrix} \beta_o \\ \alpha_o \\ \gamma_o \end{vmatrix} \qquad (2\text{-}2)$$

As far as $\boxed{\beta Э = R^2}$, where R is the beam size and Э is the beam emittance, equation for unknown beam parameters α_o, β_o, γ_o, is

$$\begin{vmatrix} R_1^2 \\ R_2^2 \\ R_3^2 \end{vmatrix} = \begin{vmatrix} C_1^2 & -2C_1S_1 & S_1^2 \\ C_2^2 & -2C_2S_2 & S_2^2 \\ C_3^2 & -2C_3S_3 & S_3^2 \end{vmatrix} \begin{vmatrix} \beta_o Э \\ \alpha_o Э \\ \gamma_o Э \end{vmatrix} \qquad (2\text{-}3)$$

Solution of Eq. (2-3) is

$$\boxed{\alpha_o \text{э} = \frac{C_3^2(R_2^2S_1^2 - R_1^2S_2^2) + C_1^2(R_3^2S_2^2 - R_2^2S_3^2) + C_2^2(R_1^2S_3^2 - R_3^2S_1^2)}{2(C_2S_1 - C_1S_2)(C_3S_1 - C_1S_3)(C_2S_3 - C_3S_2)}} \qquad (2\text{-}4)$$

$$\boxed{\beta_o \text{э} = \frac{-R_3^2S_1S_2(C_2S_1 - C_1S_2) + C_3S_3(R_2^2S_1^2 - R_1^2S_2^2) - S_3^2(C_1S_1R_2^2 - C_2S_2R_1^2)}{(C_2S_1 - C_1S_2)(C_3S_1 - C_1S_3)(C_2S_3 - C_3S_2)}} \qquad (2\text{-}5)$$

$$\boxed{\gamma_o \text{э} = \frac{R_1^2C_2S_3(C_2S_3 - C_3S_2) + C_1S_1(R_2^2C_3^2 - R_3^2C_2^2) + C_1^2(C_2S_2R_3^2 - C_3S_3R_2^2)}{(C_2S_1 - C_1S_2)(C_3S_1 - C_1S_3)(C_2S_3 - C_3S_2)}} \qquad (2\text{-}6)$$

Beam emittance

$$\boxed{\text{э} = \sqrt{(\beta_o \text{э})(\gamma_o \text{э}) - (\alpha_o \text{э})^2}} \qquad (2\text{-}7)$$

Error in beam emittance

$$
\begin{aligned}
\frac{d\text{э}}{\text{э}} = -\{ & 2(C_2S_1 - C_1S_2)^2(C_3S_1 - C_1S_3)^2(C_3S_2 - C_2S_3)^2 \\
& [- \frac{(C_1^2R_2^2 - C_2^2R_1^2)(C_1^2R_2^2\,\frac{dR_2}{R_2} - C_2^2R_1^2\frac{dR_1}{R_1})}{C_1^2C_2^2(C_2S_1 - C_1S_2)^2} - \\
& - \frac{(C_1^2R_3^2 - C_3^2R_1^2)(C_1^2R_3^2\,\frac{dR_3}{R_3} - C_3^2R_1^2\frac{dR_1}{R_1})}{C_1^2C_3^2(C_3S_1 - C_1S_3)^2} + \\
& + \frac{(C_2^2R_3^2 - C_3^2R_2^2)(C_2^2R_3^2\,\frac{dR_3}{R_3} - C_3^2R_2^2\frac{dR_2}{R_2})}{C_2^2C_3^2(C_3S_2 - C_2S_3)^2} + \\
& + \frac{2C_2(C_3^4R_1^4\,\frac{dR_1}{R_1} - C_1^4R_3^4\frac{dR_3}{R_3})}{C_1^2C_3^3(C_2S_1 - C_1S_2)(C_3S_1 - C_1S_3)} + \\
& \frac{2C_1(C_3^4R_2^4\,\frac{dR_2}{R_2} - C_2^4R_3^4\frac{dR_3}{R_3})}{C_2^2C_3^3(C_2S_1 - C_1S_2)(C_3S_3 - C_3S_2)} \;]\} \Big/ D
\end{aligned}
\tag{2-8}
$$

Denominator:

$$\boxed{D = G^4[(\frac{A+B}{G})^2 - 1]\ [(\frac{-A+B}{G})^2 - 1]} \qquad (2\text{-}9)$$

$$\boxed{A = R_1(C_2S_3 - C_3S_2)} \qquad (2\text{-}10)$$

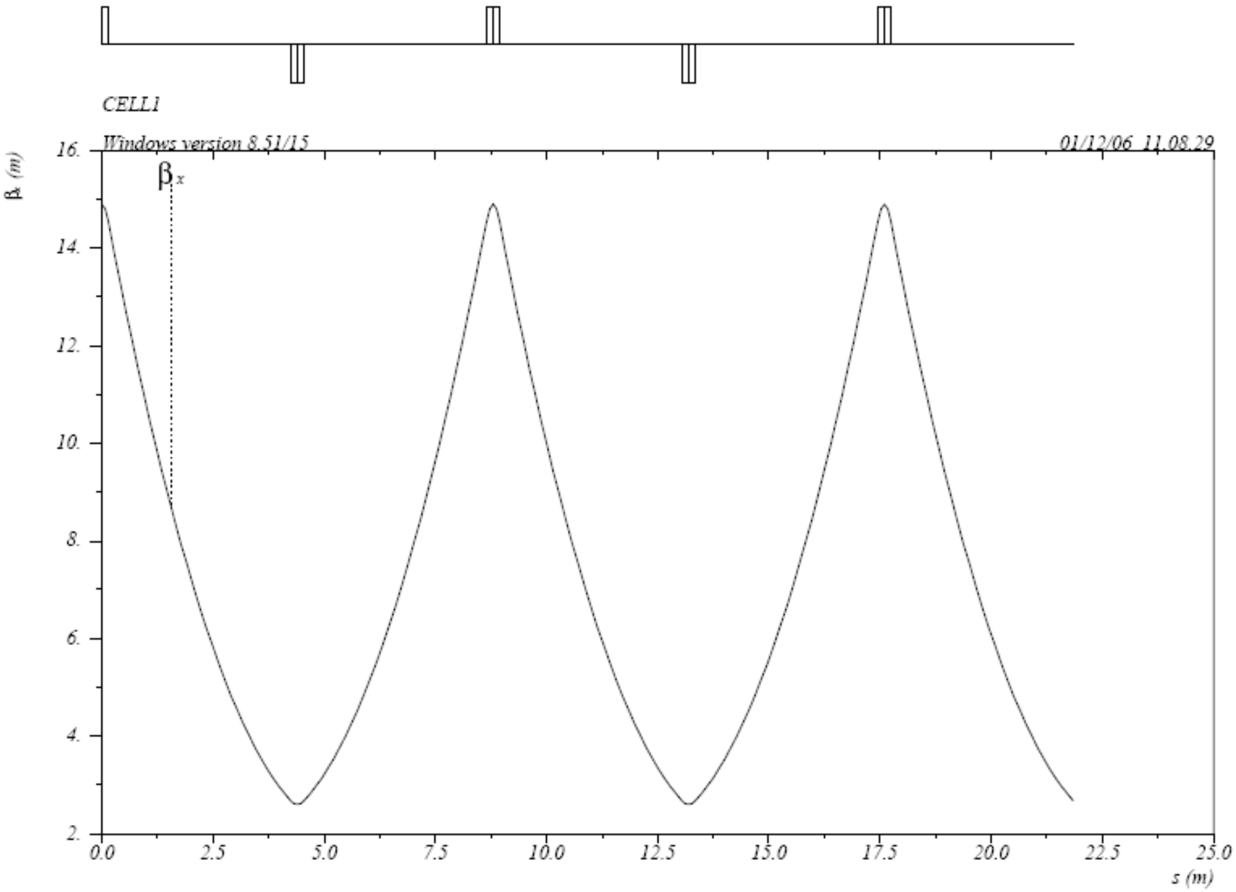
$$\boxed{B = R_2(C_3S_1 - C_1S_3)} \qquad (2\text{-}11)$$

$$\boxed{G = R_3(C_2S_1 - C_1S_2)} \qquad (2\text{-}12)$$

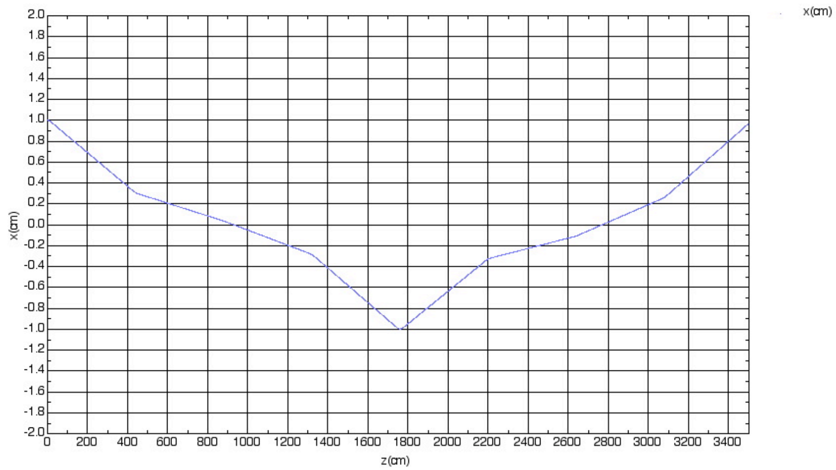
Large error in beam emittance is expected if

$$\boxed{a = \frac{A+B}{G} \approx 1} \quad \text{or} \quad \boxed{b = \frac{B - A}{G} \approx 1} \qquad (2\text{-}13)$$

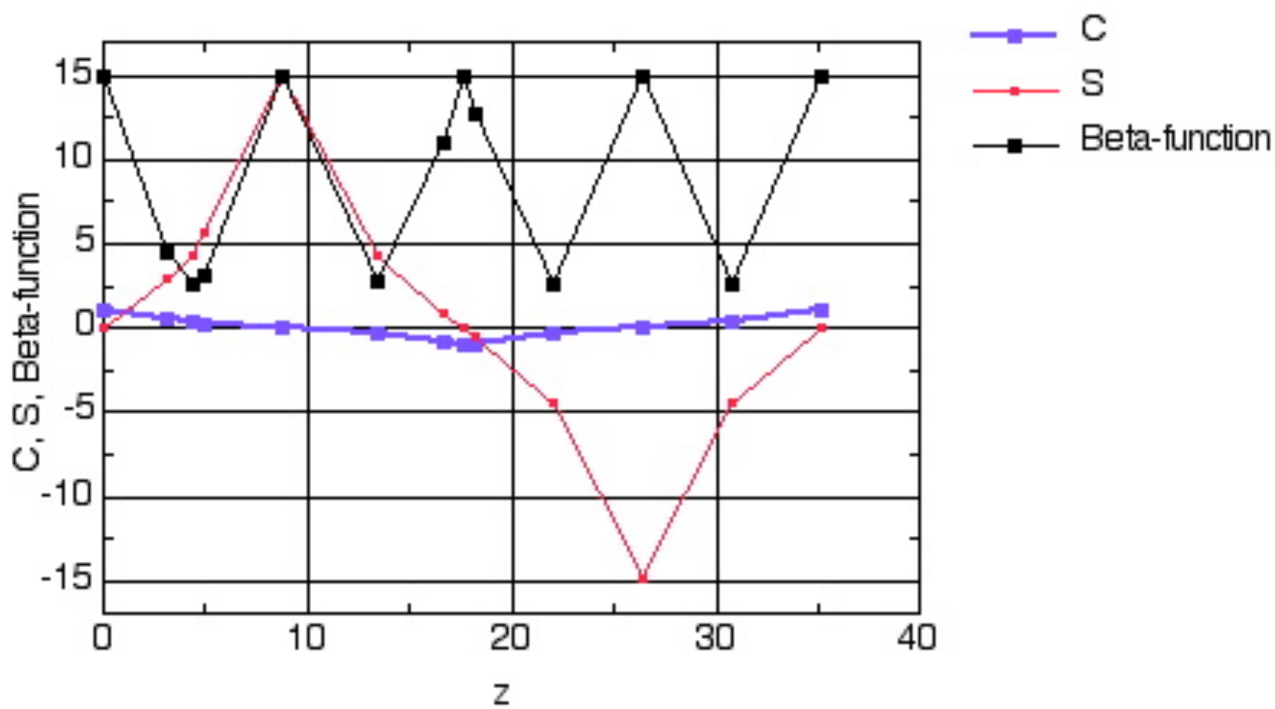
Consider 90^o phase advance FODO channel:



Beam energy	5 GeV
Lens Gradient	17.8 T/m
Quad length	0.3 m
FODO period	8.8 m
Distance between quads	4.4 m

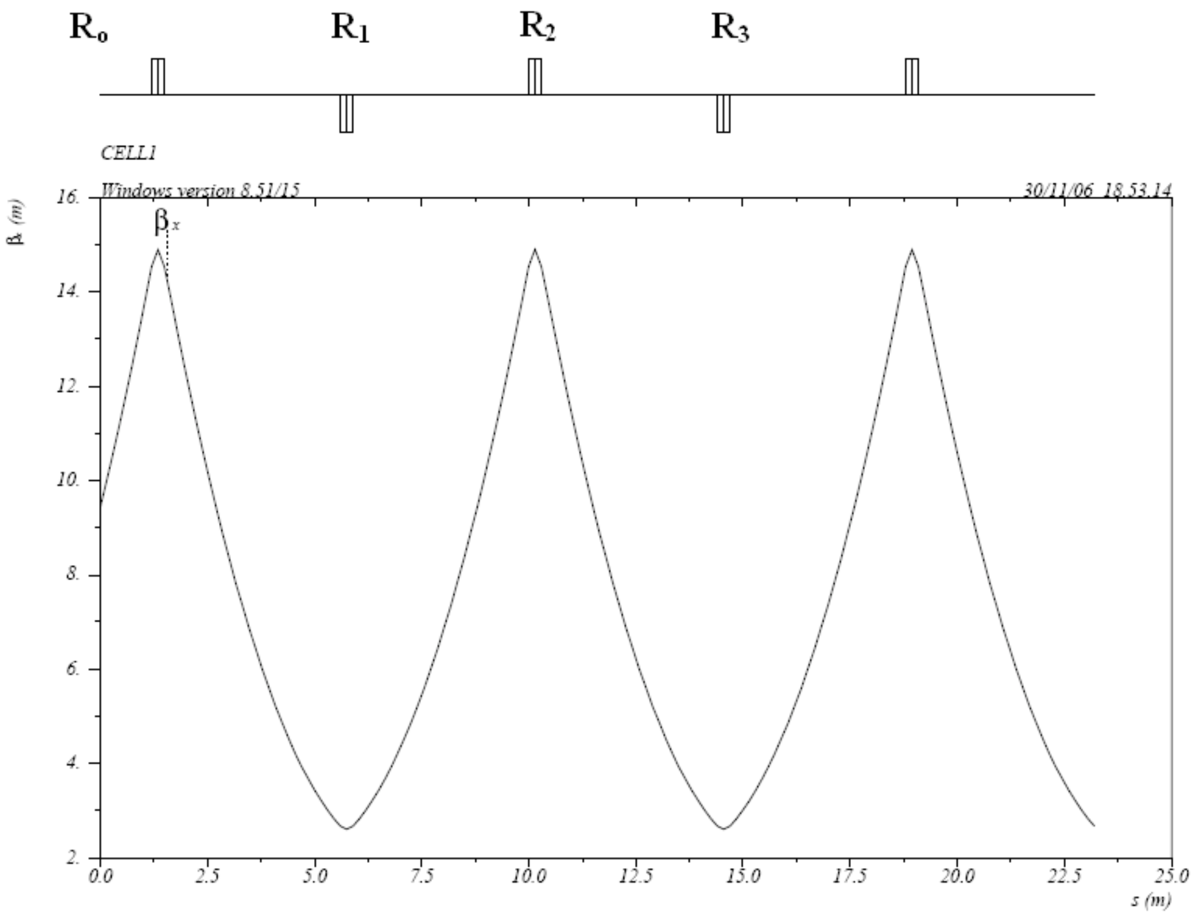


Single particle trajectory



Matrix parameters and β-function of the structure.

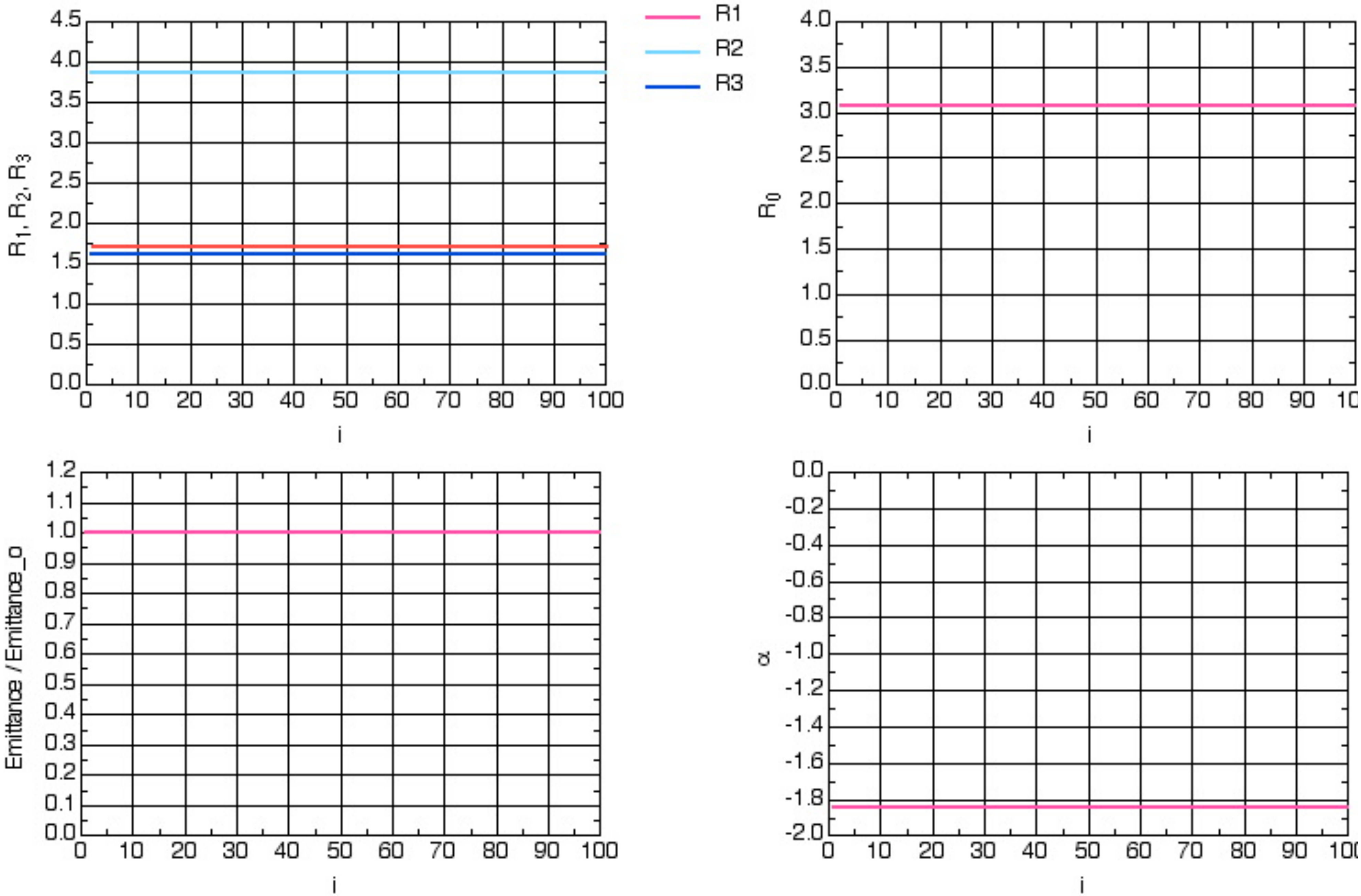
Example1: Stable beam emittance measurement



C1 = -0.428950
S1 = 3.87470
C2 = -2.44000
S2 = 11.7880
R1 = 1.61200
R2 = 3.86000
C3 = -1.01256
S3 = 3.08830
R3 = 1.61200

a = 0.17
b = 5.78

Ro = 3.07
э = 1
α = -1.8



Determination of beam parameters э, R_o, α for error in beam sizes R_1, R_2, R = 0.

Variation of measured beam sizes:

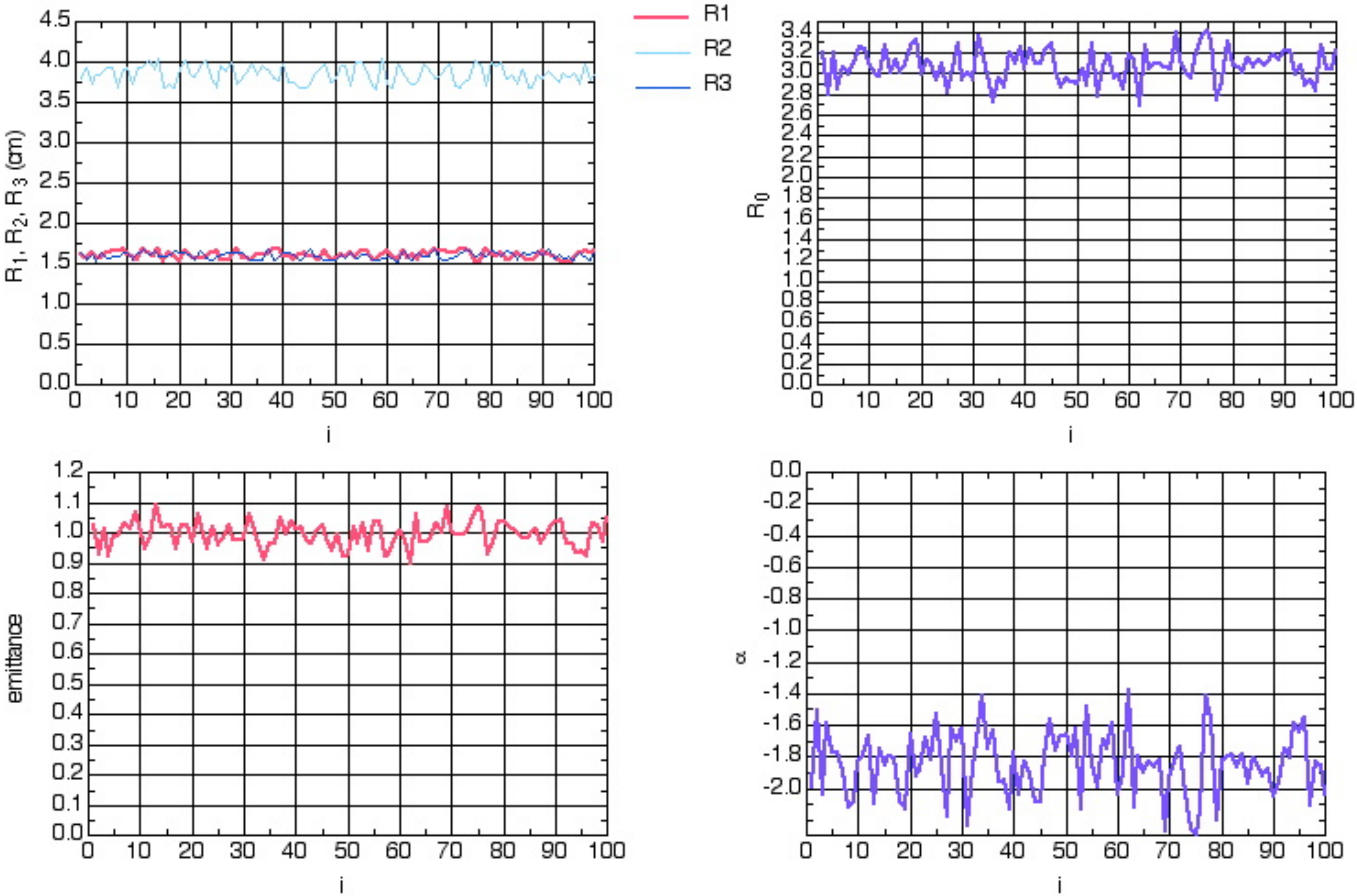
$$\boxed{R_1 = R_1^{(0)}(1 + f)}$$

$$\boxed{R_2 = R_2^{(0)}(1 + g)}$$

$$\boxed{R_3 = R_3^{(0)}(1 + h)}$$

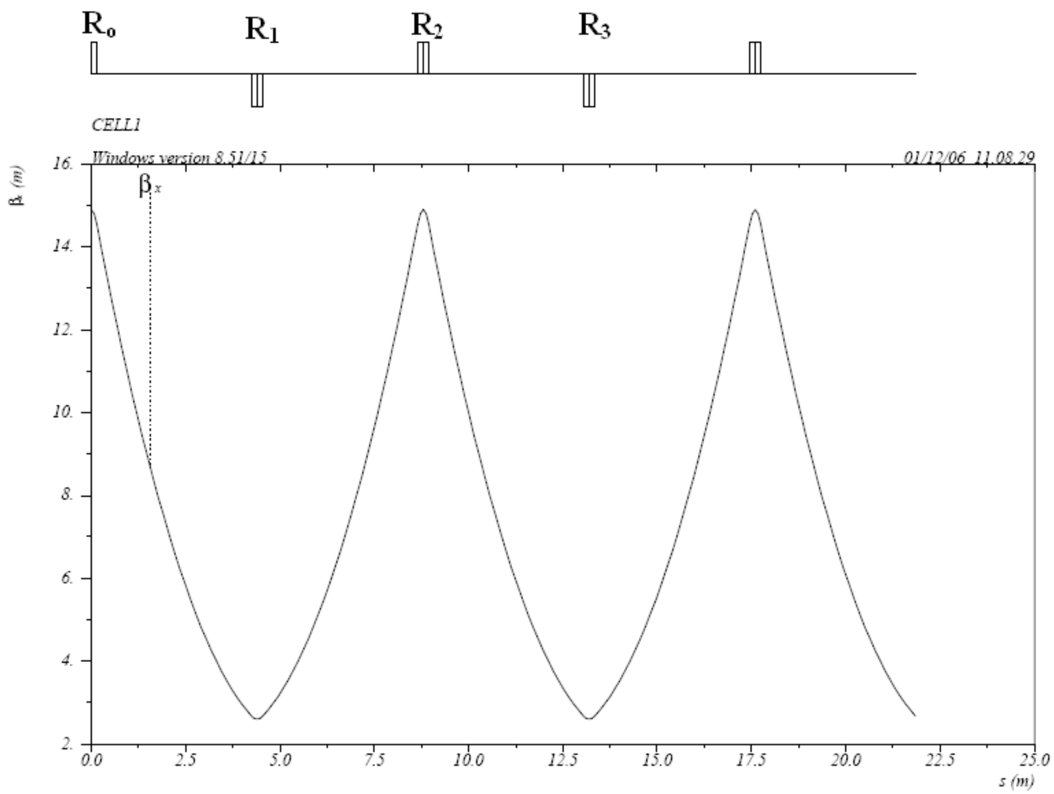
where $R_1^{(0)}$, $R_2^{(0)}$, $R_3^{(0)}$ - unperturbed values of measured beam sizes,
f, g, h – generators of random numbers uniformly distributed within interval [-a, a]
Values of R_1, R_2, R_3 are distributed with

$$\boxed{\sigma = \frac{a}{\sqrt{3}}}$$



Determination of beam parameters with errors in beam sizes R_1, R_2, R_3 = ± 5% (σ_R/R = 2.88%). Error in measured value of beam emittance is approximately ± 10% ($\sigma_э/э$= 4.3%).

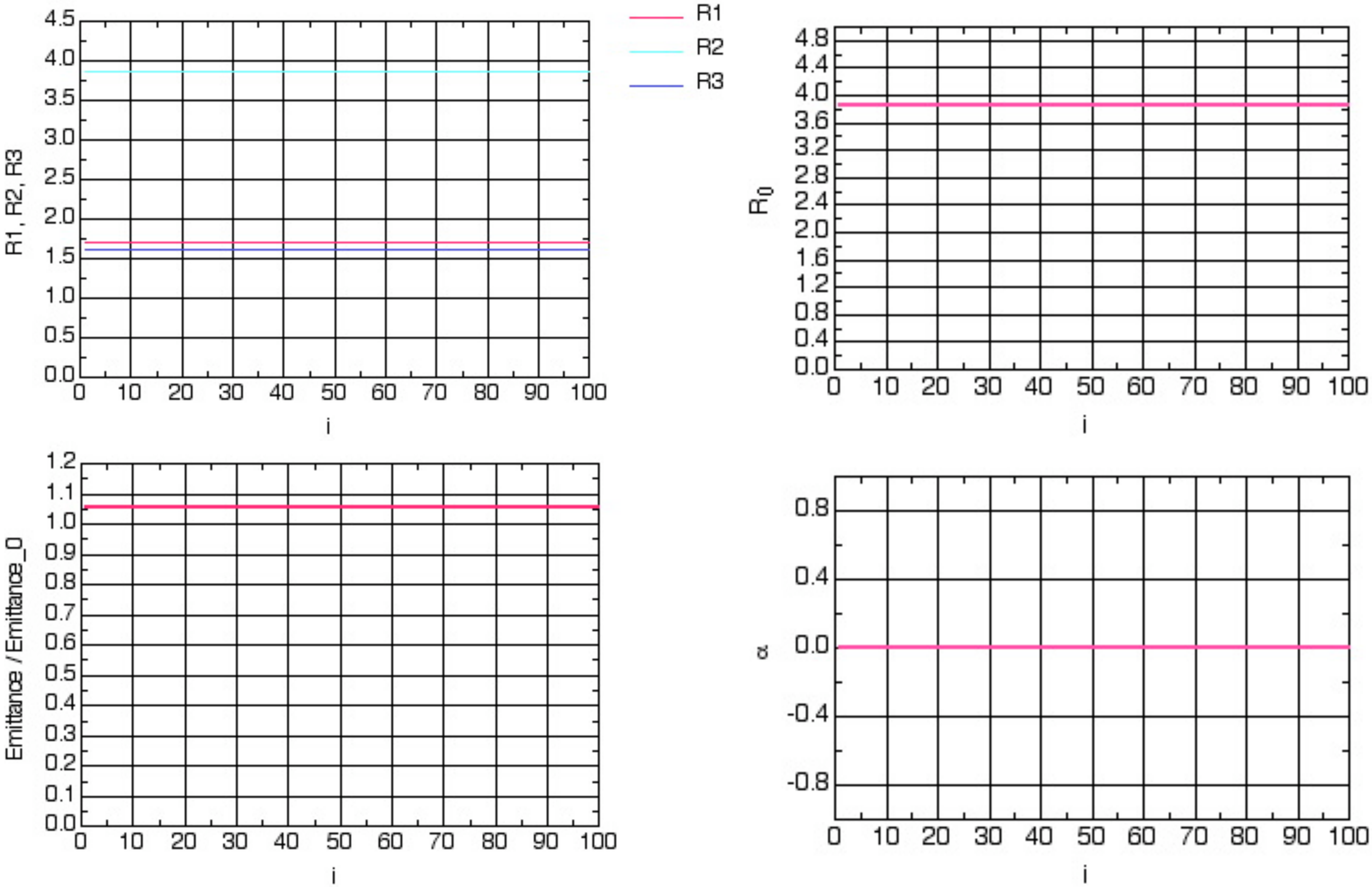
Example 2: Stable beam emittance measurements (phase shift 45^o, 90^o, 135^o)



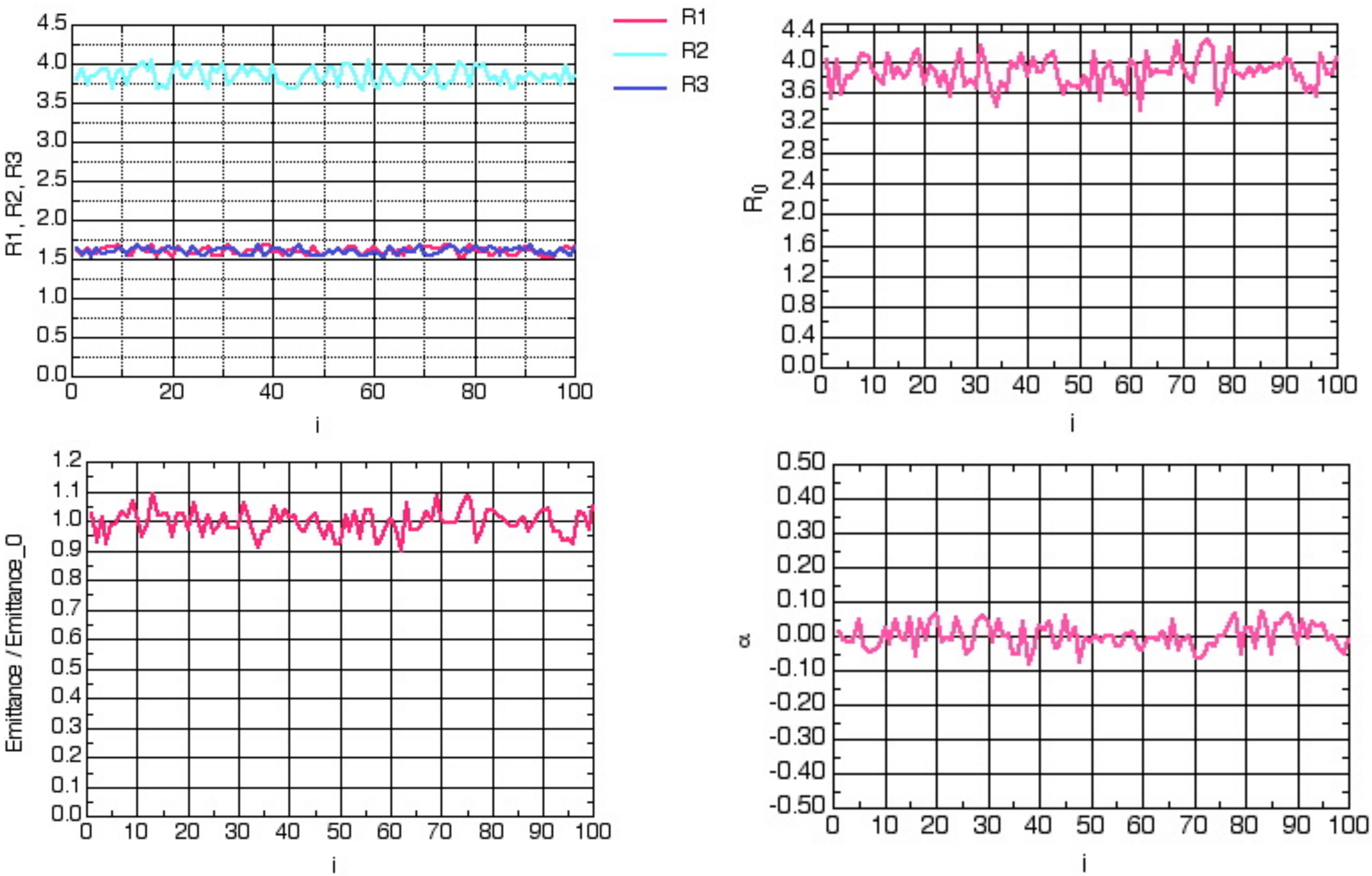
C1 = 0.295340
S1 = 4.40000
C2 = 0.000000
S2 = 14.9000
R1 = 1.70000
R2 = 3.86000
C3 = -0.295340
S3 = 4.40000
R3 = 1.61200

a = 0.17
b = 5.28

Ro = 3.86
э = 1
$\alpha = 0$

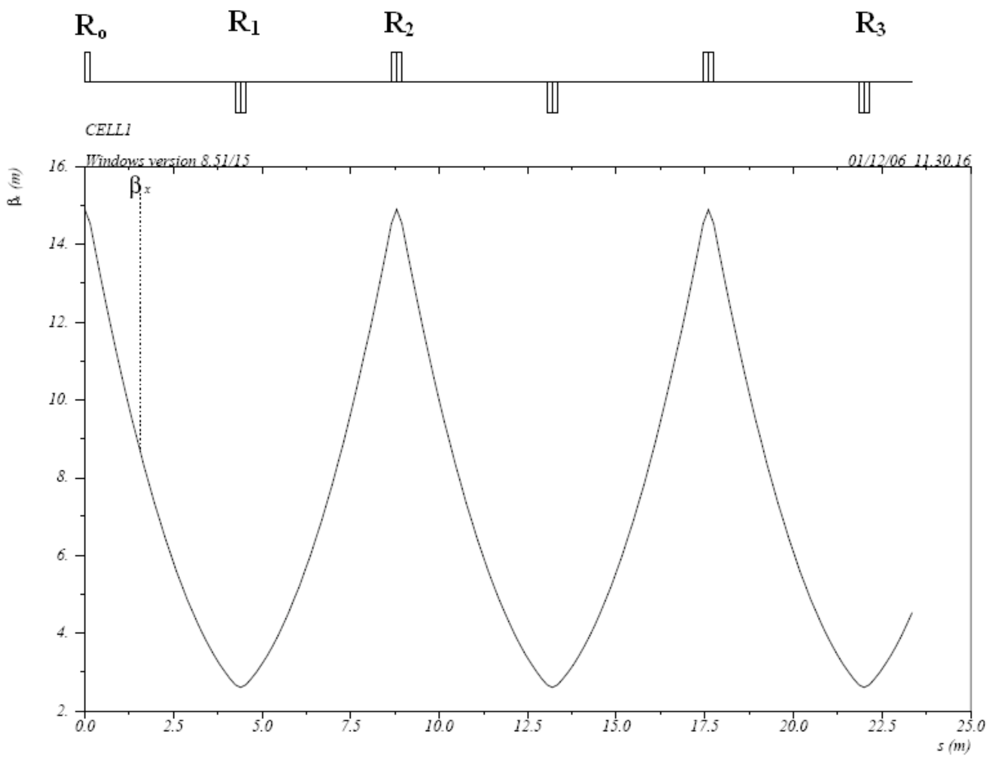


Determination of beam parameters э, R_o, α with error in beam sizes R_1, R_2, R_3 = 0.



Determination of beam parameters with errors in beam sizes R_1, R_2, R_3 = ± 5% (σ_R/R = 2.88%). Error in measured value of beam emittance ≈ ± 10% ($\sigma_э/э$= 4.29%).

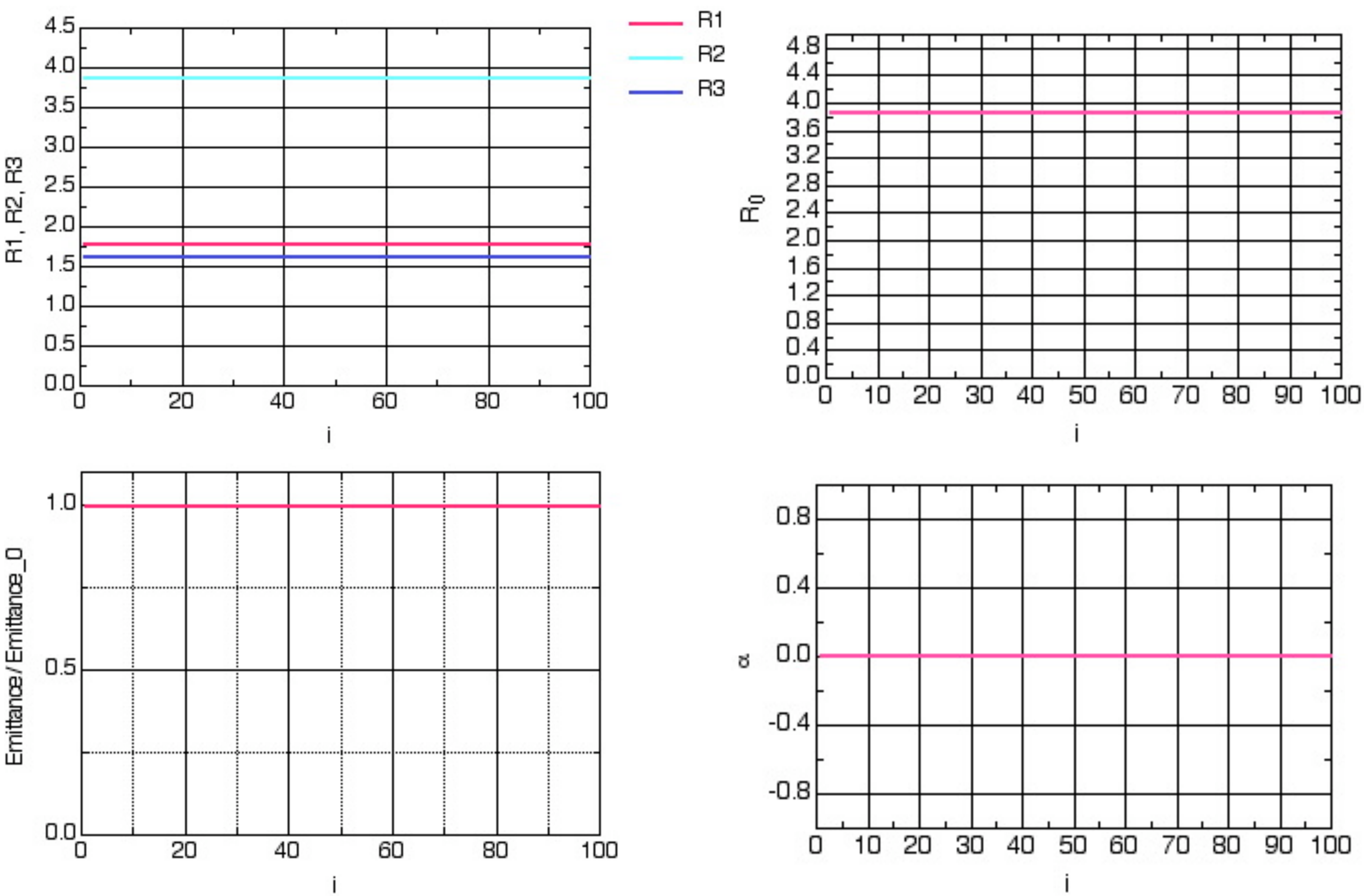
Example 3: Unstable beam emittance measurements (45^o, 90^o, 225^o)



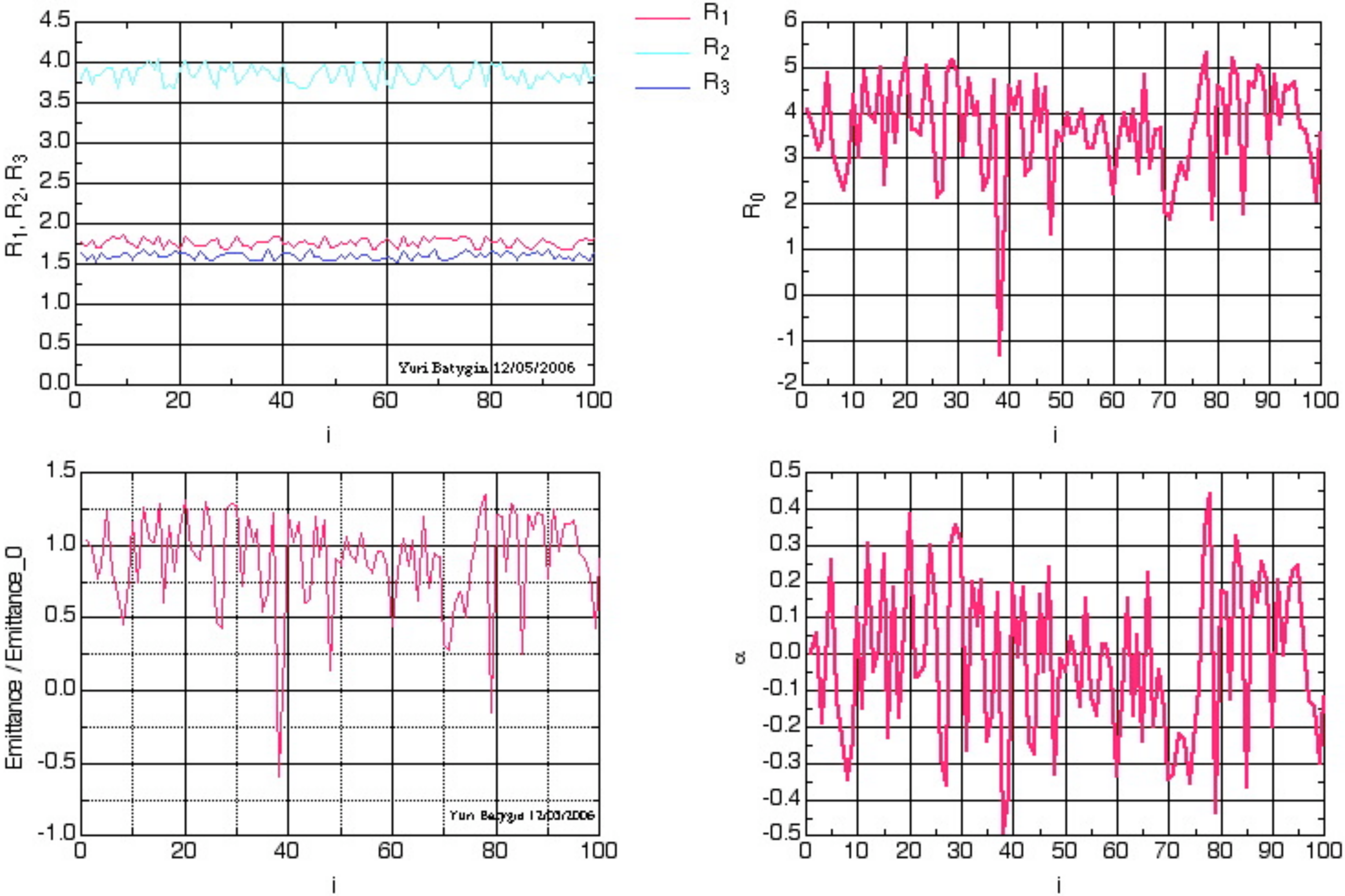
C1 = 0.255000
S1 = 5.68000
C2 = 0.000000
S2 = 14.9000
R1 = 1.77220
R2 = 3.86000
C3 = -0.295340
S3 = -4.40000
R3 = 1.61200

a = 0.85
b = 2.63

Ro = 3.86
э = 1
α = 0



Determination of beam parameters э, R_o, α with error in beam sizes R_1, R_2, R_3 = 0.



Unstable determination of beam parameters with errors in beam sizes R_1, R_2, R_3 = ± 5%.

3. Summary

1. Error in determination of beam emittance is larger than error in measured beam sizes.

2. Determination of emittance through beam size measurements at different locations

R_1, R_2 R_3 is performed with significant error if the following conditions are fulfilled:

$$\boxed{A = R_1(C_2S_3 - C_3S_2)}$$

$$\boxed{B = R_2(C_3S_1 - C_1S_3)}$$

$$\boxed{G = R_3(C_2S_1 - C_1S_2)}$$

$$\boxed{a = \frac{A+B}{G} \approx 1} \quad \text{or} \quad \boxed{b = \frac{B-A}{G} \approx 1}$$

where S_1, S_2, S_3, C_1, C_2, C_3 are single particle matrix transformation elements.

3. Next step: error analysis for 4D beam.