Update in beam emittance measurement study: minimal error in FODO channel

Purpose: minimization of errors in beam emittance measurements in FODO channel as a function of phase advance per cell

1. 2D Beam Emittance Measurement

Single particle transformation matrix

$$\begin{vmatrix} x \\ x' \end{vmatrix} = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \begin{vmatrix} x_o \\ x'_o \end{vmatrix} \tag{1-1}$$

Beam ellipse transformation:

$$\begin{vmatrix} \beta \\ \alpha \\ \gamma \end{vmatrix} = \begin{vmatrix} C^2 & -2CS & S^2 \\ -CC' & CS'+SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{vmatrix} \begin{vmatrix} \beta_o \\ \alpha_o \\ \gamma_o \end{vmatrix} \tag{1-2}$$

Beam emittance is determined from 3 independent measurements of beam size at different locations R_1, R_2, R_3 via solution of 3x3 matrix:

$$\begin{vmatrix} R_1^2 \\ R_2^2 \\ R_3^2 \end{vmatrix} = \begin{vmatrix} C_1^2 & -2C_1S_1 & S_1^2 \\ C_2^2 & -2C_2S_2 & S_2^2 \\ C_3^2 & -2C_3S_3 & S_3^2 \end{vmatrix} \begin{vmatrix} \beta_o \epsilon \\ \alpha_o \epsilon \\ \gamma_o \epsilon \end{vmatrix} \tag{1-3}$$

Beam emittance

$$\text{э} = \sqrt{(\beta_o \text{э})(\gamma_o \text{э}) - (\alpha_o \text{э})^2} \qquad (1\text{-}4)$$

Large error in determination of beam emittance is expected if (see BDS meeting note of YB, 12/05/2006)

$$a = \left(\frac{A + B}{G}\right)^2 \approx 1 \quad \text{or} \quad b = \left(\frac{B - A}{G}\right)^2 \approx 1 \qquad (1\text{-}5)$$

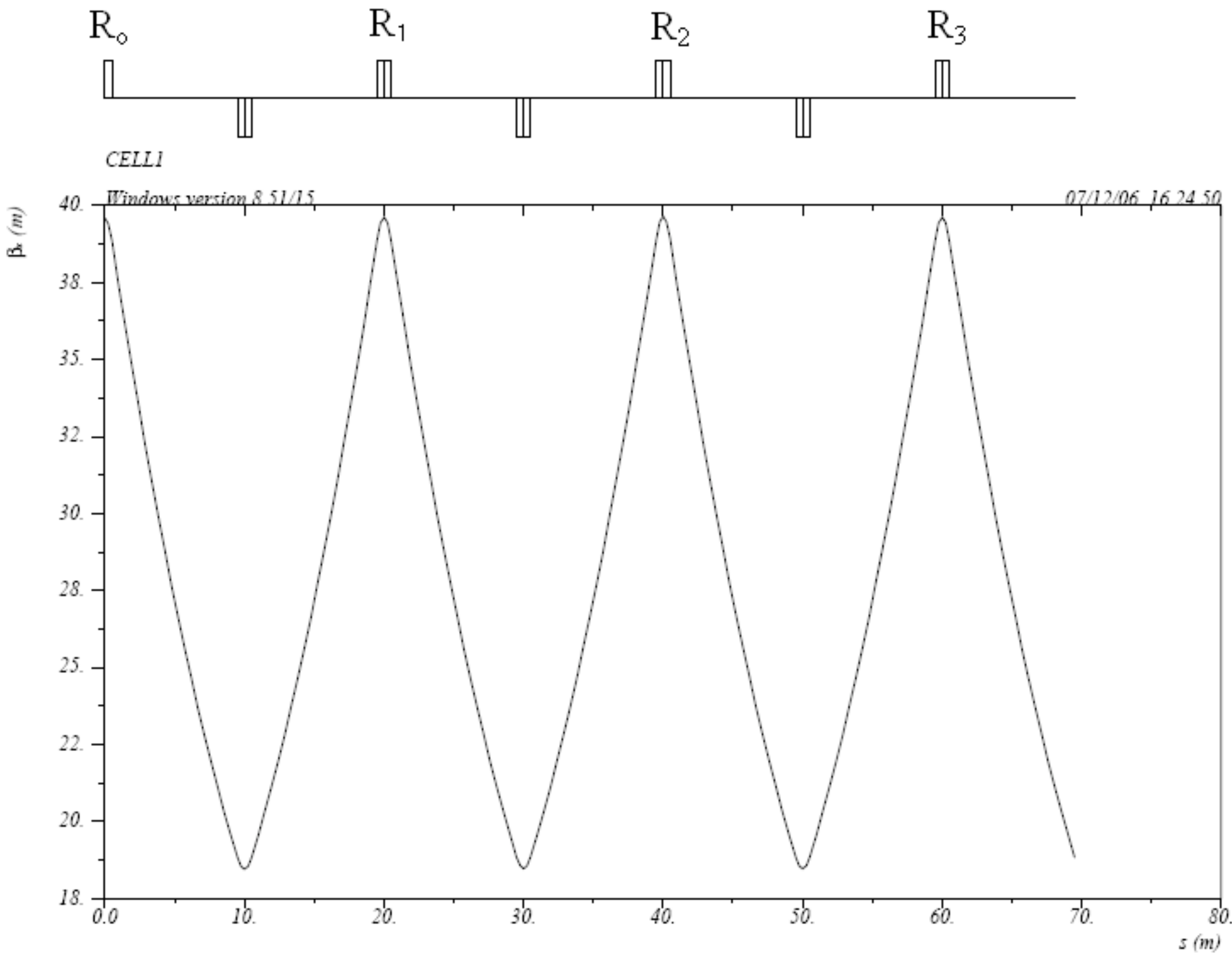
where

$$A = R_1(C_2S_3 - C_3S_2) \qquad (1\text{-}6)$$

$$B = R_2(C_3S_1 - C_1S_3) \qquad (1\text{-}7)$$

$$G = R_3(C_2S_1 - C_1S_2) \qquad (1\text{-}8)$$

Consider FODO channel with beam measurement stations after each cell:



1.1 Semi-analytical approach

Transformation matrix after each cell is :

$$\begin{vmatrix} x \\ x' \end{vmatrix} = \begin{vmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{vmatrix} \begin{vmatrix} x_o \\ x'_o \end{vmatrix} \qquad (2\text{-}1)$$

Considering centers of quadrupoles, where $\alpha = 0$, matrix elements are:

$$C = \cos \mu \qquad S = \beta \sin \mu \qquad (2\text{–}2)$$

Consider measurements afer each FODO cell:

$$\begin{aligned} C_1 &= \cos \mu & S_1 &= \beta \sin \mu \\ C_2 &= \cos 2\mu & S_2 &= \beta \sin 2\mu \\ C_3 &= \cos 3\mu & S_3 &= \beta \sin 3\mu \end{aligned} \qquad (2\text{–}3)$$

Uncertainty in beam emittance:

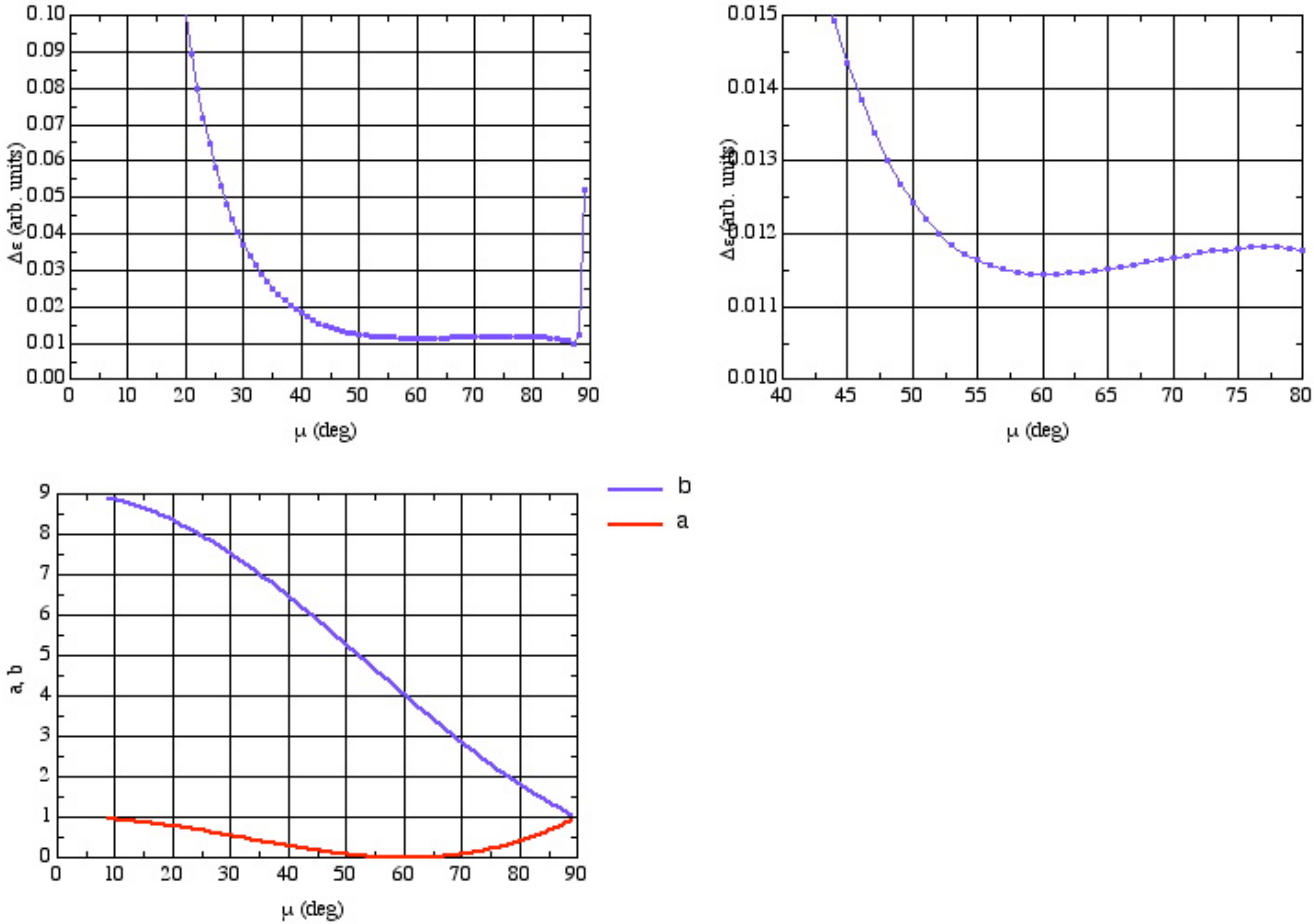
$$\Delta э = \sqrt{(\Delta э_1)^2 + (\Delta э_2)^2 + (\Delta э_3)^2} \qquad (2\text{-}4)$$

where partial errors are:

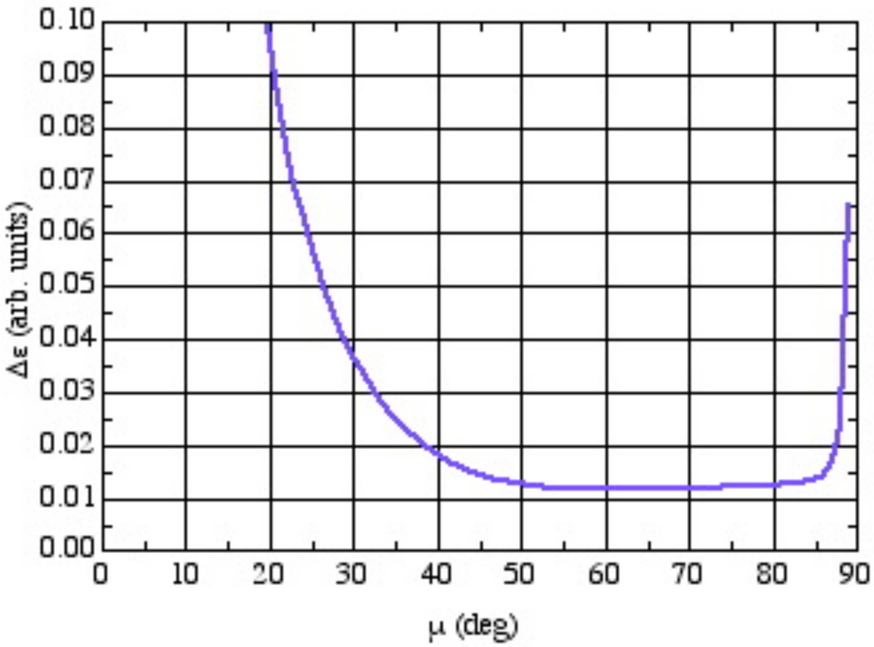
$$\Delta э_1 = э(R_1, R_2, R_3) - э(R_1+\Delta R_1, R_2, R_3)$$

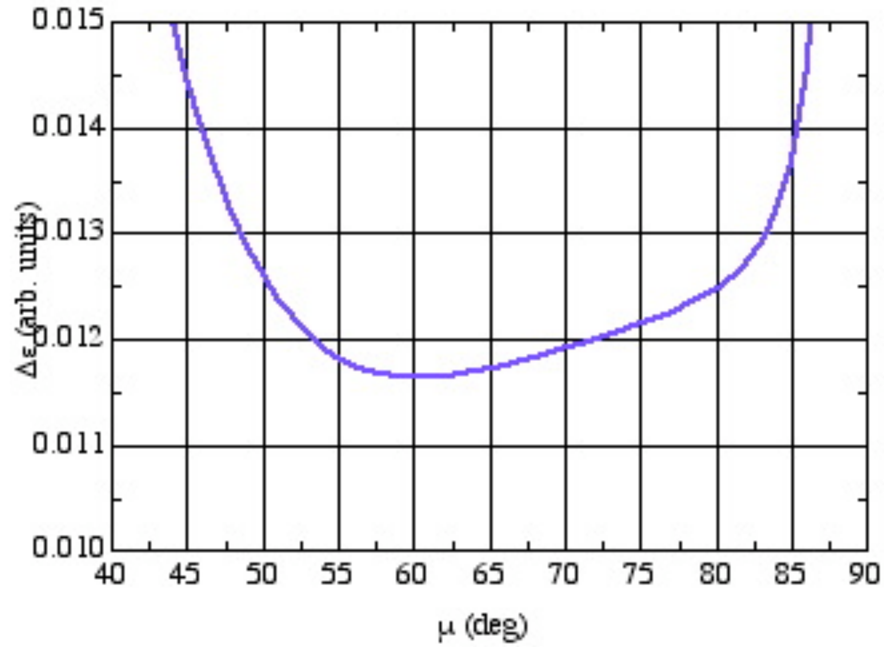
$$\Delta э_2 = э(R_1, R_2, R_3) - э(R_1, R_2+\Delta R_2, R_3) \qquad (2\text{-}5)$$

$$\Delta э_3 = э(R_1, R_2, R_3) - э(R_1, R_2, R_3+\Delta R_3)$$



Error in determination of emittance and parameters a, b as functions of phase advance per cell for $\Delta R/R = 0.01$.





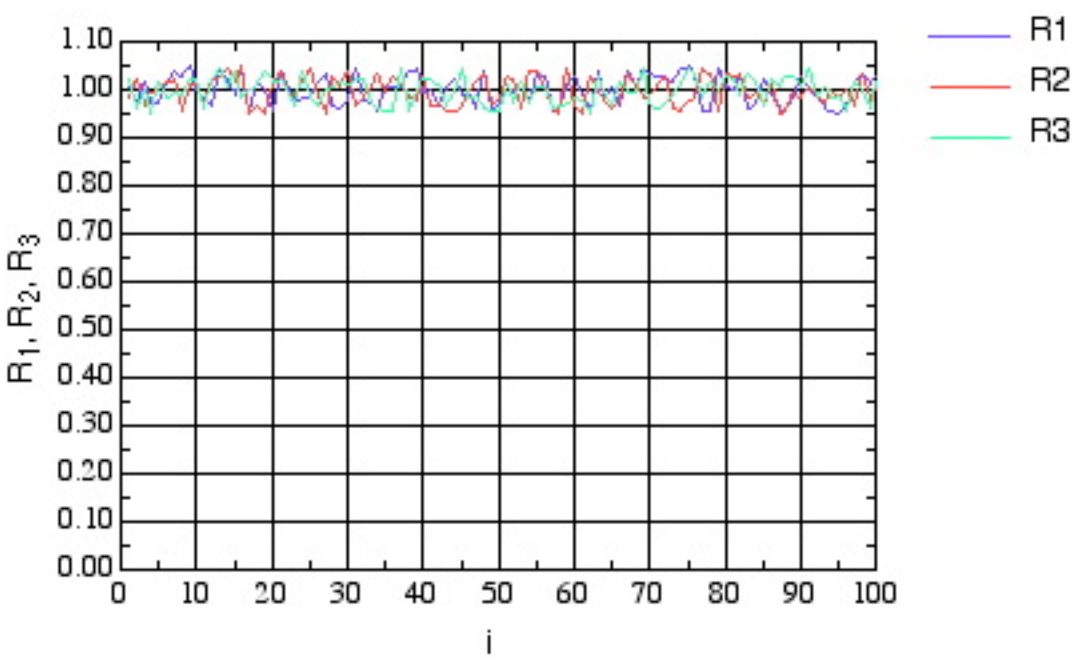
Error in determination of emittance (semi-analytical method) for $\Delta R_i/R_i$ = - 0.01 (i=1,2,3).

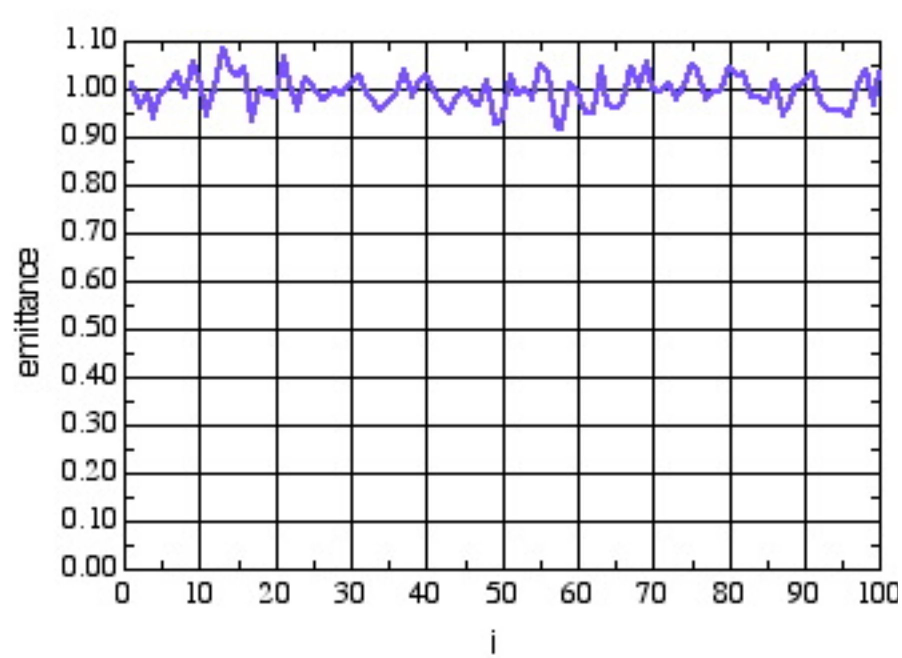
1.2 Numerical approach

For each value of phase advance μ, random variation of measured beam sizes

$$R_1 = R_1^{(0)}\,(1 + f)\,,\quad R_2 = R_2^{(0)}\,(1 + g)\,,\quad R_3 = R_3^{(0)}\,(1 + h)$$

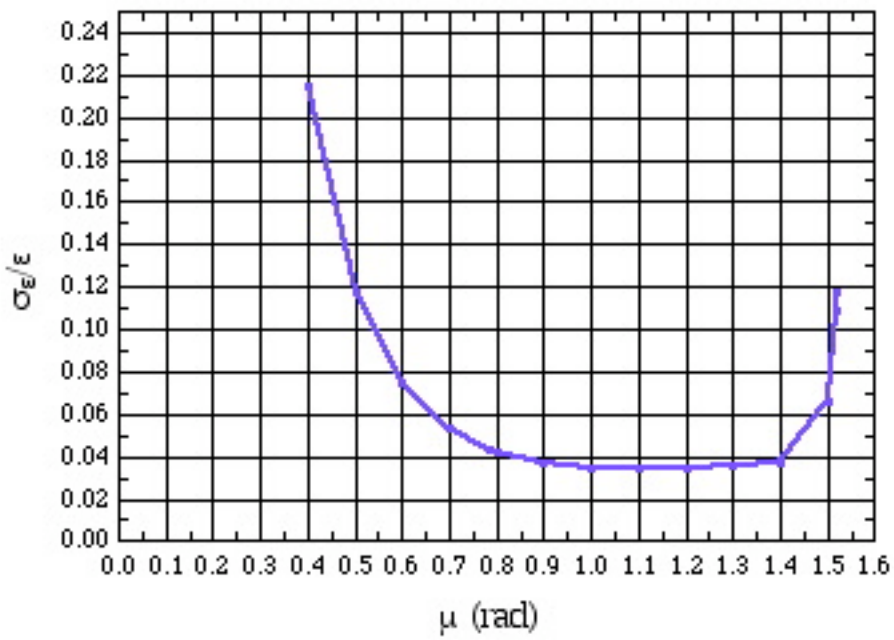
results in variation of determined value of beam emittance

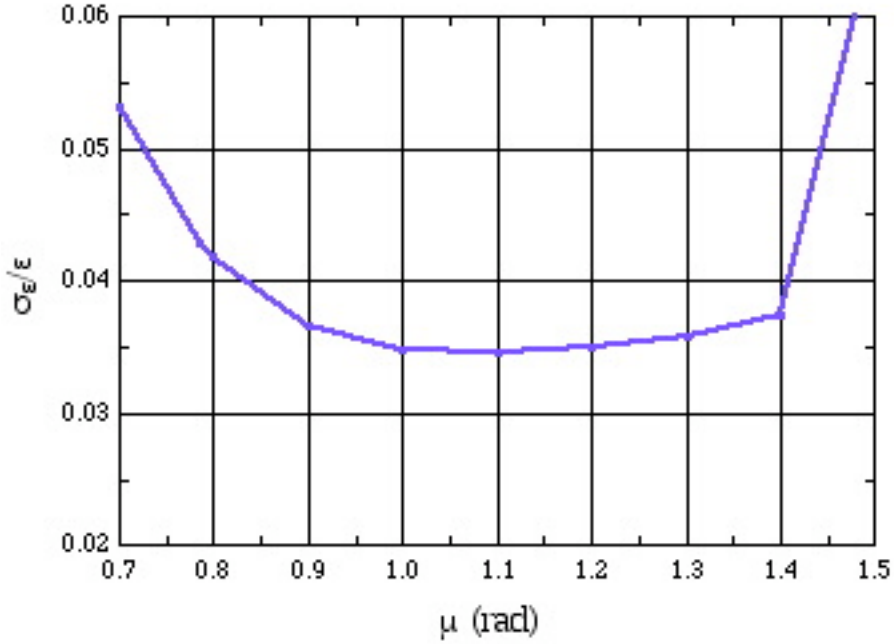




Rms value of error in beam emittance:

$$\frac{\sigma_э}{э} = \frac{1}{\bar{э}} \sqrt{\frac{1}{N} \sum_{i=1}^{N} (э_i - \bar{э})^2}$$





Error in 2D beam emittance determination (numerical method) as a function of phase advance in FODO channel with beam size measurements at every other quad ($R_1 = R_2 = R_3$).

2. 4D Beam Emittance Measurement

$\vec{\sigma}$ - matrix of 4D beam:

$$\vec{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{vmatrix} \qquad (2\text{-}1)$$

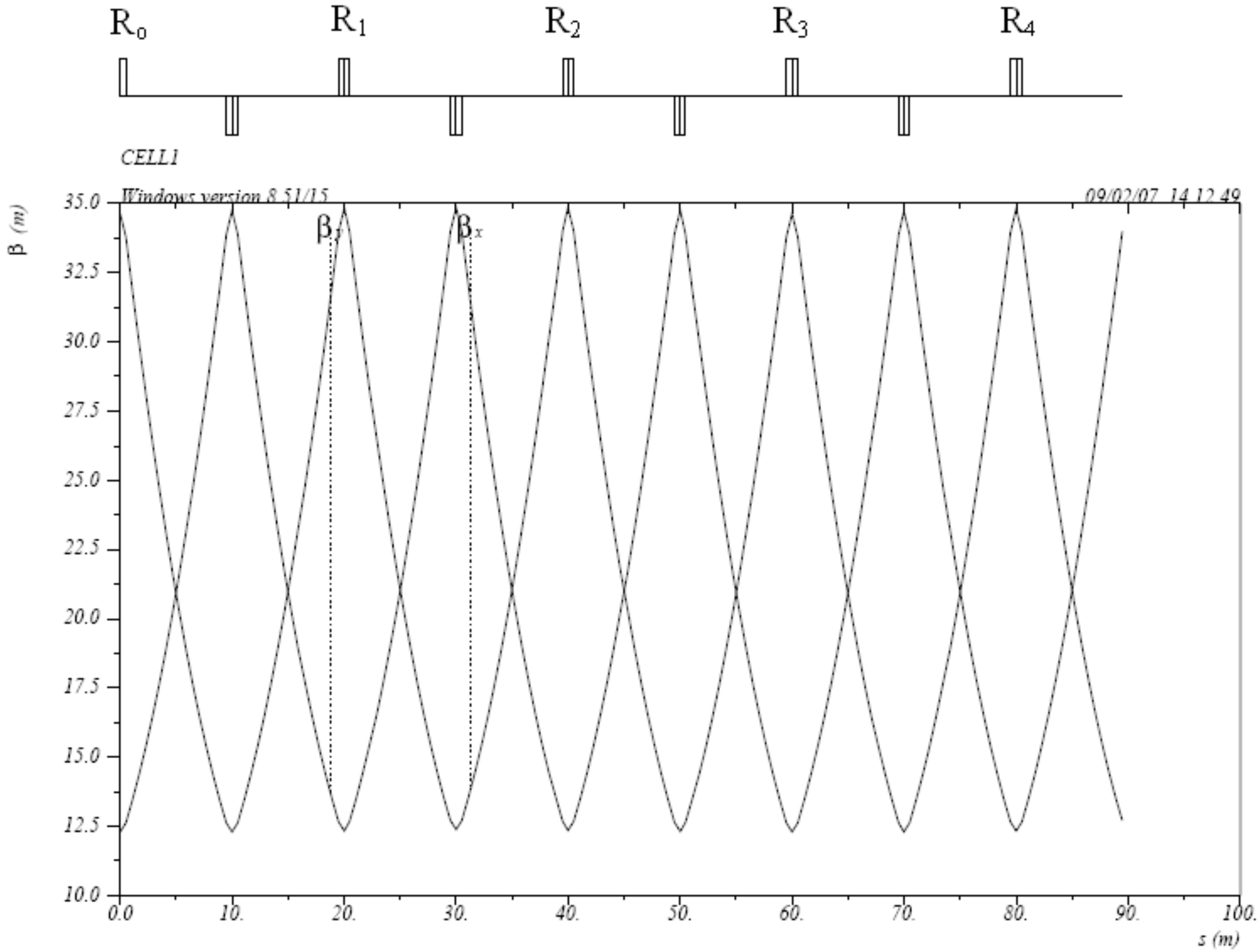
Matrix (2-1) is symmetric, $\sigma_{ij} = \sigma_{ji}$, so only 10 elements of matrix are independent. Measurements are provided for beam sizes

$$\langle x^2 \rangle = \sigma_{11}, \ \langle y^2 \rangle = \sigma_{33}, \ \langle xy \rangle = \sigma_{13} \qquad (2–2)$$

at different locations . Explicit transformation for σ_{11}, σ_{33}, σ_{13} is:

$$\begin{vmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \end{vmatrix} = \begin{vmatrix} R_{11}^2 & 2R_{11}R_{12} & 2R_{11}R_{13} & 2R_{11}R_{14} & R_{12}^2 & 2R_{12}R_{13} & 2R_{12}R_{14} & R_{13}^2 & 2R_{13}R_{14} & R_{14}^2 \\ R_{31}^2 & 2R_{31}R_{32} & 2R_{31}R_{33} & 2R_{31}R_{34} & R_{32}^2 & 2R_{32}R_{33} & 2R_{34}R_{32} & R_{33}^2 & 2R_{33}R_{34} & R_{34}^2 \\ R_{11}R_{31} & R_{12}R_{31}+R_{11}R_{32} & R_{13}R_{31}+R_{11}R_{33} & R_{31}R_{14}+R_{11}R_{34} & R_{12}R_{32} & R_{13}R_{32}+R_{12}R_{33} & R_{32}R_{14}+R_{12}R_{34} & R_{13}R_{33} & R_{14}R_{33}+R_{13}R_{34} & R_{14}R_{34} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{24} \\ \sigma_{33} \\ \sigma_{34} \\ \sigma_{44} \end{vmatrix}_0 \qquad (2\text{-}3)$$

To determine 10 independent values of σ-matrix, we take 3 x 3 + 1 =10 equations from 4 independent measurement stations.



FODO channel with four beam measurement stations after each cell

Lattice parameters:

Cell period	20 m
Beam Energy	250 GeV
Quadrupole length	0.5 m
Quadrupole strength, K_1	0.5

Example: Phase advance μ =45^o

Measured values: $\sigma_{11}^{(1)} = \sigma_{11}^{(2)} = \sigma_{11}^{(3)} = \sigma_{11}^{(4)} = 2.397600E\text{-}12$

$\sigma_{33}^{(1)} = \sigma_{33}^{(2)} = \sigma_{33}^{(3)} = \sigma_{33}^{(4)} = 1.070000E\text{-}10$

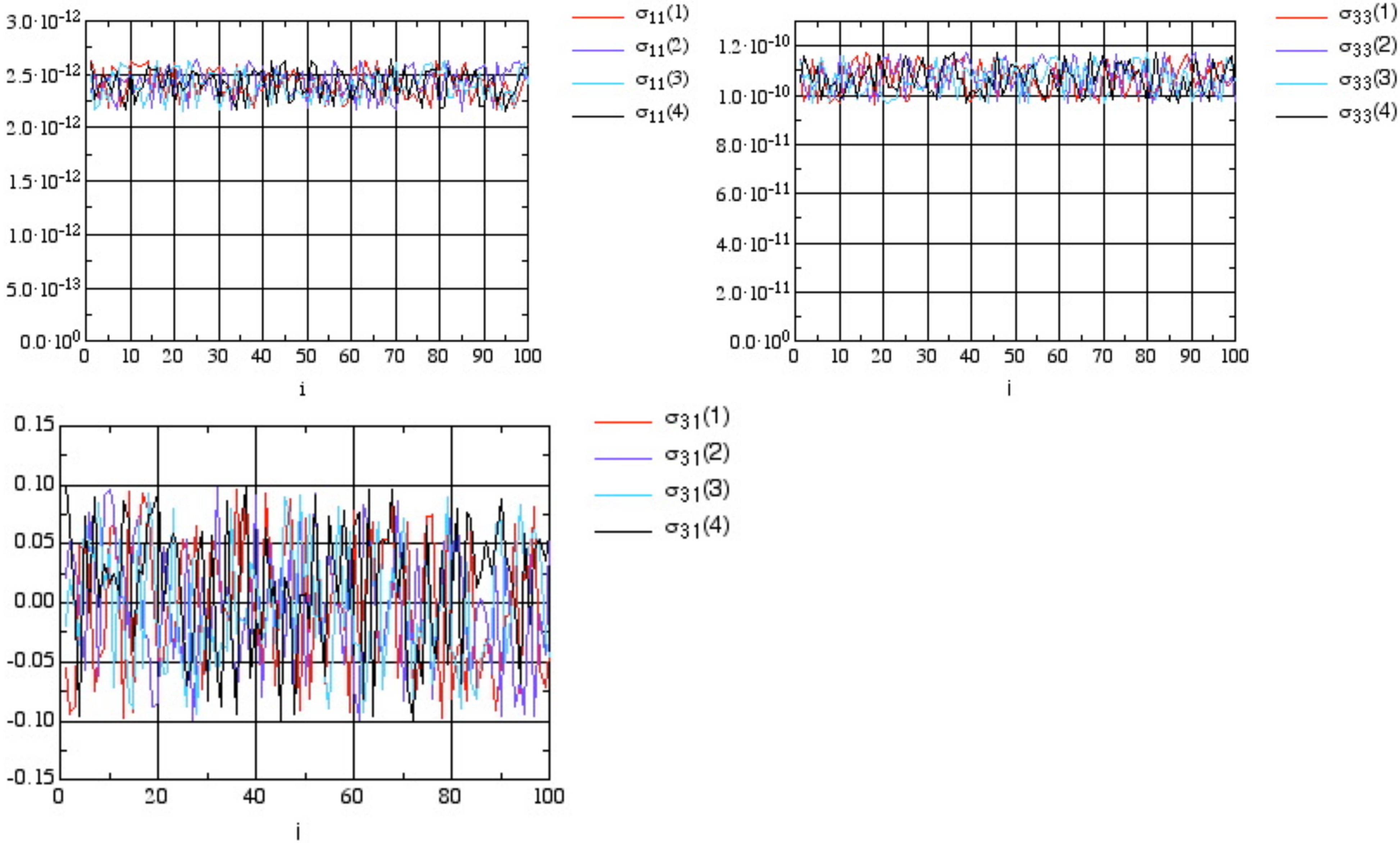
$\sigma_{13}^{(1)} = \sigma_{13}^{(2)} = \sigma_{13}^{(3)} = \sigma_{13}^{(4)} = 0$

Determined values of σ – matrix

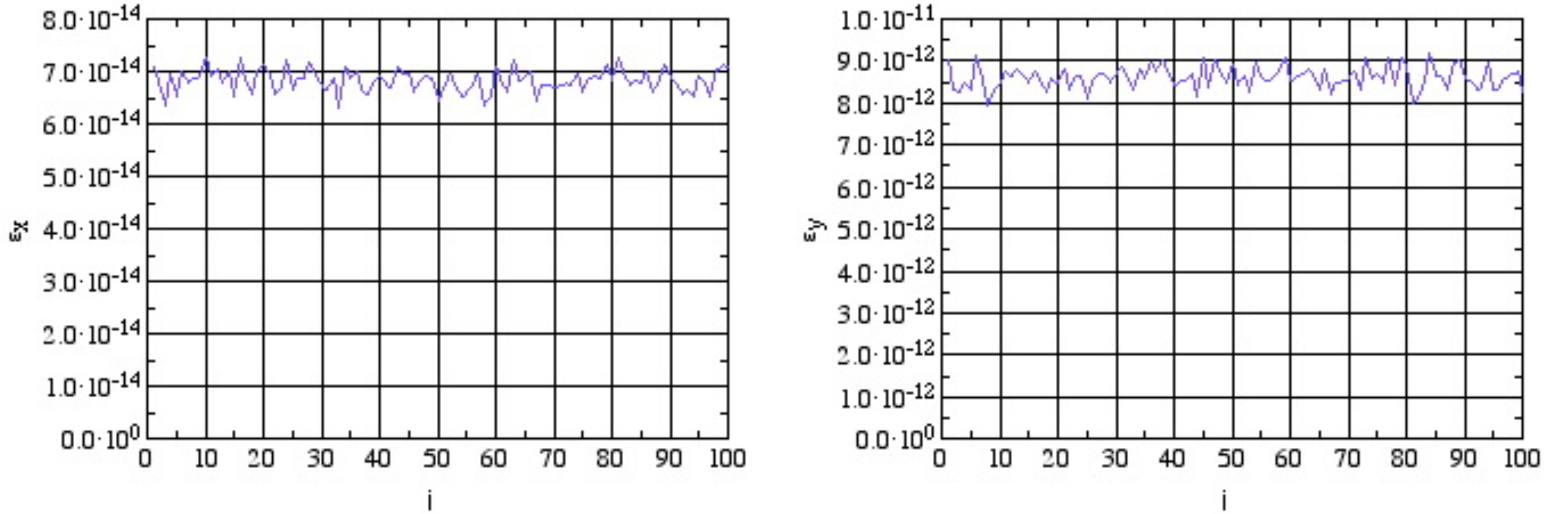
	Exact	Numerical
$\sigma_{11}^{(0)}$	2.397600E-12	2.397600E-12
$\sigma_{12}^{(0)}$	0	0
$\sigma_{13}^{(0)}$	0	0
$\sigma_{14}^{(0)}$	0	0
$\sigma_{22}^{(0)}$	1.5676182 E-15	1.567620E-15
$\sigma_{23}^{(0)}$	0	0
$\sigma_{24}^{(0)}$	0	0
$\sigma_{33}^{(0)}$	1.070000E-10	1.070000E-10
$\sigma_{34}^{(0)}$	0	0
$\sigma_{44}^{(0)}$	3.50975e-13	3.509757E-13

Determined values of beam emittance:

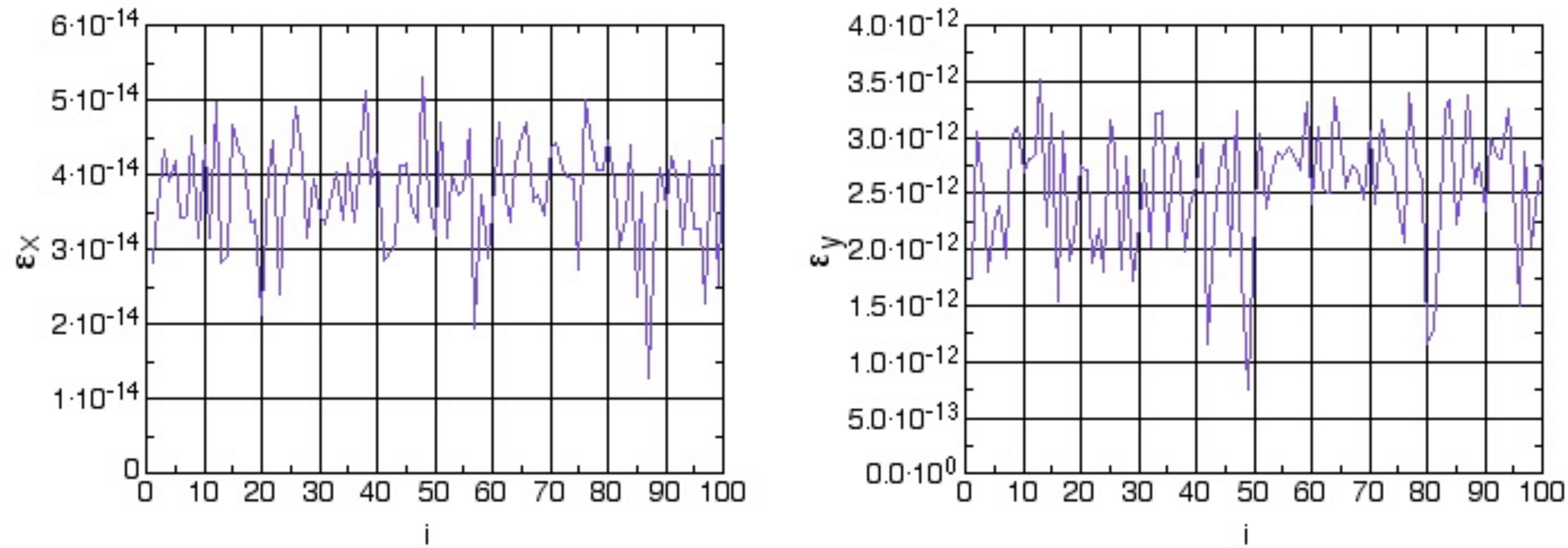
	Exact	Numerical
$Э_x = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$	6.1306782e-14	6.130682E-14
$Э_y = \sqrt{\sigma_{33}\sigma_{44} - \sigma_{34}^2}$	6.1281668e-12	6.128164E-12



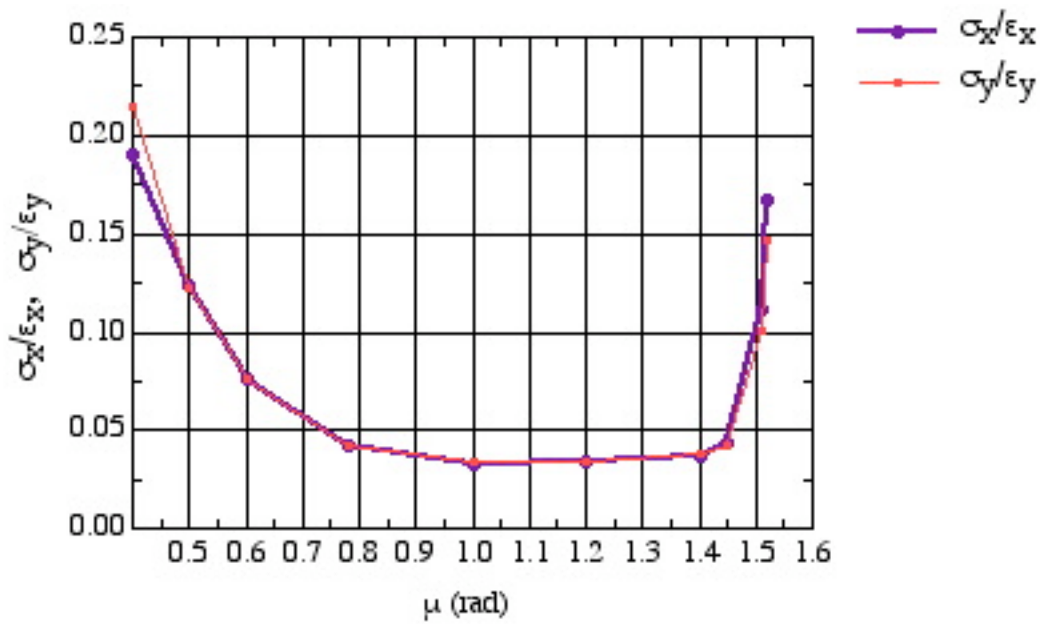
Variation of measured values of σ_{11}, σ_{33}, σ_{31} at four different locations.



Variation of determined values of beam emittances at $\mu = 60^o$.



Variation of determined values of beam emittances at $\mu = 22.9^o$.



Error in 4D beam emittance determination (numerical method) as a function of phase advance in FODO channel with beam size measurements after each cell.

Conclusion

Minimal value of error in emittance determinantion in FODO channel for uncoupled beam is reached for the value of phase advance between measurement stations $\mu = 60^o$.

Next step: analysis of coupled beam emittance determination