Differential Reduction Algorithms for the Laurent Expansion of Hypergeometric Functions for Feynman Diagram Calculation

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Representing Feynman Diagrams

- It would be very useful to have a general means of representing a Feynman diagram with an arbitrary number of loops and legs.
- Reduction techniques to represent a given diagram in terms of a class of more elementary integrals are required in computations.
- Since the diagrams typically diverge in 4 dimensions, an expansion must be developed in a small parameter about $d = 4$.

Hypergeometric Function Approach

- One of the most powerful representations of Feynman diagrams is in terms of **hypergeometric functions**.

- Much work has been done on finding the representation of various diagrams in terms of HG functions, and finding recursion relations among them which can be the basis for a reduction algorithm.

Hypergeometric Functions and Feynman Diagrams

- Regge proposed (45 years ago) that Feynman diagrams could be represented in terms of **HG functions**.
- The singularities of this function are coincide with the surface of Landau singularities of the Feynman diagram.
- This representation has an advantage of efficiency – for example, the 4-point massive scalar box diagram may be expressed as **192 dilogs** – or a **single HG function** of several variables. This helps to cancel spurious singularities.

[D.S. Kershaw, Phys. Rev. D8 (1973) 2708]

Generalized Hypergeometric Functions

The generalized HG function ${}_pF_q$ has expansion

$$ {}_pF_q\left(\begin{matrix} a_1, & \dots, & a_n \\ b_1, & \dots, & b_n \end{matrix} \middle| z \right) = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_j}{\prod_{k=1}^{p} (b_k)_j} \frac{z^j}{j!} $$

with $(a)_j = \Gamma(a+j)/\Gamma(a)$ the "Pochhammer symbol" and the b-parameters cannot be negative integers.

The "original" HG function is the Gauss HG function,

$$ {}_2F_1(A,B;C;z) = {}_2F_1\left(\begin{matrix} A,B \\ C \end{matrix} \middle| z \right) = \sum_{j=0}^{\infty} \frac{(A)_j (B)_j}{(C)_j} \frac{z^j}{j!} $$

Hypergeometric Functions and Feynman Diagrams

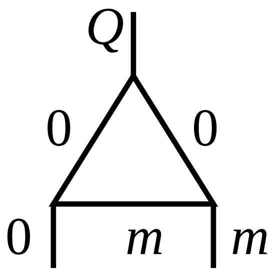
- The HG representation for one loop N-point functions was obtained in series form by Kershaw [above] and others, and the associated differential equation was constructed by Barucchi and Ponzano. [J. Math. Phys. 14 (1973) 396]

- Functions appearing in one-loop N-point functions include Appell functions and Lauricella functions.

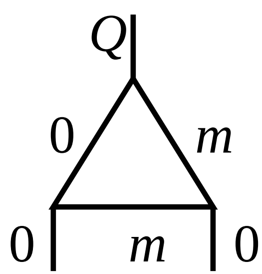
- See our recent preprint for a references to some of the historic papers and review articles: [Kalmykov, Kniehl, Ward, Yost, arXiv: 0810.3238]

Examples: Vertex Diagrams

Our recent paper contains a catalog of one-loop vertex diagrams. For example,



$$= -i\pi^{\frac{n}{2}}\left(m^2\right)^{\frac{n}{2}-3}\left\{\left(-\frac{Q^2}{m^2}\right)^{\frac{n}{2}-2}\frac{\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(2-\frac{n}{2}\right)}{\Gamma\left(3-\frac{n}{2}\right)}\,{}_2F_1\left(\begin{matrix}1,\frac{n}{2}-1\\ 3-\frac{n}{2}\end{matrix}\middle|\,\frac{Q^2}{m^2}\right) + \frac{\Gamma(n-4)}{\Gamma\left(\frac{n}{2}-1\right)}\,{}_2F_1\left(\begin{matrix}1,1\\ 5-n\end{matrix}\middle|\,\frac{Q^2}{m^2}\right)\right\}$$



$$= -i\pi^{\frac{n}{2}}\left(m^2\right)^{\frac{n}{2}-3}\frac{\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(3-\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\,{}_3F_2\left(\begin{matrix}1,1,3-\frac{n}{2}\\ \frac{n}{2},2\end{matrix}\middle|\,\frac{Q^2}{m^2}\right)$$

Feynman Diagram Computation

- The most common methods for computing Feynman diagrams are based on algebraic relations among diagrams with different numerators but common denominators, with different powers of denominators, or in different space-time dimensions.

- These algebraic relations are the basis for reduction algorithms, which reduce a general diagram in some class to a restricted set of master integrals which may be implemented numerically.

Feynman Diagram Computation

- HG functions can be used in a similar manner in computations.
- The algebraic and differential relations among HG functions with shifted arguments can be used to construct a reduction algorithm.
- For example, all one-loop N-point diagrams can be represented in terms of HG functions of $N–1$ variables, and can be reduced to a set of master integrals which are in turn related by a difference equation in the dimension d.

[Fleischer, Jegerlehner, Tarasov, Nucl. Phys. B 672 (2003) 303]

Epsilon Expansions

The HG function can be expanded in powers of the parameter ε. The terms in this expansion multiply poles $1/\varepsilon^n$ from UV and IR divergences. Higher-order terms are needed in the expansion for higher-loop graphs. To be useful, the coefficients of the ε expansion are needed analytically.

This means that a HG function, *e.g.* ${}_pF_{p-1}\left(\vec{A};\vec{B};z\right)$, must be expanded about its parameters, so that $\vec{A}\to\vec{A}+\vec{a}\varepsilon,\ \vec{B}\to\vec{B}+\vec{b}\varepsilon$, resulting in a Laurent series

$${}_pF_{p-1}\left(\vec{A}+\vec{a}\varepsilon;\vec{B}+\vec{b}\varepsilon;z\right)={}_pF_{p-1}\left(\vec{A};\vec{B};z\right)+\sum_{k\neq 0}\varepsilon^k L_{\vec{a},\vec{b},k}(z)$$

Three Hypergeometric Approaches

Three approaches have been taken toward HG representations of Feynman diagrams:

1. **Integral representations**
2. **Series representations**
3. **Differential representation**

Let's consider briefly what each of these mean and what has been done with them.

Integral Representation

Integrals leading to HG functions include Euler integrals

$$\Phi\left(\vec{\alpha},\vec{\beta},P\right)=\int_{\Sigma}\prod_i P\left(x_1,\cdots,x_k\right)^{\beta_i} x_1^{\alpha_1}\cdots x_k^{\alpha_k}\, dx_1\cdots dx_k$$

with the P's a set of Laurent polynomials,

and Mellin-Barnes integrals

$$\Phi\left(a_{js},b_{kr},c_i,d_j,\gamma,\vec{x}\right)=\int_{\gamma+i\Re} dz_1\cdots dz_1\,\frac{\prod_{j=1}^{p}\Gamma\left(\sum_{s=1}^{m}a_{js}z_s+c_j\right)}{\prod_{k=1}^{q}\Gamma\left(\sum_{r=1}^{m}b_{kr}z_r+d_k\right)}x_1^{-z_1}\cdots x_m^{-z_m}$$

with a, b, c, d real.

Integral Representation

The Euler integral representation has been used to obain the all-order ε expansion of Gauss HG functions in terms of Nielsen polylogarithms

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

Recently, we have derived similar results using the differential equation approach, and will discuss them later in the talk.

Series Representation

A Laurent series in r variables

$$\Phi(\vec{x}) = \sum C(\vec{m})\, x_1^{m_1} \cdots x_r^{m_r}$$

is **hypergeometric** if for each i, the ratio $C(\vec{m}+\vec{e}_i)/C(\vec{m})$ is a rational function in the multi-index $\vec{m}$, with $\vec{e}_i = (0,\cdots,0,1,0,\ldots 0)$.

i^{th} place

This is actually a particular type of HG series called a **Horn series**.

Series Representation

The Horn-Type HG series can be shown to satisfy a system of differential equations of the form

$$Q_j\left(\sum_{k=1}^r x_k \frac{\partial}{\partial x_k}\right)\frac{\Phi(\vec{x})}{x_j} = P_j\left(\sum_{k=1}^r x_k \frac{\partial}{\partial x_k}\right)\Phi(\vec{x}), \qquad j = 1, \cdots, r$$

with polynomials P_j, Q_r satisfying

$$\frac{C(\vec{m}+\vec{e}_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})}$$

Series Representation

The series approach has been studied much. Some useful implementations have appeared, which have been used in high-order calculations:

- **HypExp** – Huber & Maître: all-orders expansion of HG functions about integer values of parameters in Mathematica
- **XSummer** – Moch & Uwer: expansions of transcendental functions and symbolic summation in FORM.

In both cases, the **nested sum** representation plays an important role.

Differential Representation

A differential representation is based on the differential (or difference) relations among a class of HG functions.

For example, a HG function of the form

$$\Phi(\vec{z},\vec{w},\mathbf{W}) = \sum_{k_1,\cdots,k_n=0}^{\infty} \left(\prod_{a=1}^{m} \frac{1}{z_a + \sum_{b=1}^{n} W_{ab} k_b} \right) \prod_{j=1}^{n} \frac{x_j^{k_j}}{k_j!}$$

satisfies

$$\frac{\partial}{\partial x_j} \Phi(\vec{z},\vec{x},\mathbf{W}) = \Phi(\vec{z} + \vec{W}_j, \vec{x}, \mathbf{W}), \qquad \frac{\partial}{\partial z_i} \left(z_i \Phi + \sum_{j=1}^{n} W_{ij} x_j \frac{\partial \Phi}{\partial x_j} \right) = 0$$

where $\mathbf{W}$ is an rxm matrix and $\vec{W}_j$ is its j^{th} row.

Differential Representation

- Davydychev began applying the differential approach to constructing the ε expansion [Phys. Rev. D61 (2000) 087701]
- The key to this approach is that HG functions satisfy certain differential equations, *e.g.*

$$\left[z\prod_{i=1}^{p}\left(z\frac{d}{dz}+A_i \right) - z\frac{d}{dz}\prod_{k=1}^{p-1}\left(z\frac{d}{dz}+B_k-1 \right) \right] {}_pF_{p-1}\left(\vec{A};\vec{B};z\right) = 0$$

- A differential equation for the coefficients of the ε expansion can be derived directly from this equation without reference to the series or integral representations, by expanding [⋯] in ε.

Differential Representation

Constructing an iterated solution has an advantage, in principle, over the series approach:

- Each term in the ε expansion is related to previously derived terms.
- There is no need to work with an increasingly large collection of independent sums at each new order.

Differential Representation

We have been considering the approach of constructing an iterated solution to the differential equation satisfied by the HG function.

I will be summarizing some recent results on constructing all-order ε expansions for certain classes of HG functions in papers by Kalmykov, Ward, Yost:

- JHEP 0702 (2007) 040 hep-th/0612240

  Gauss HG functions, Integer, ½-Integer parameters
- JHEP 0711 (2007) 009 arXiv:0707.3654

  Generalized HG functions, Integer parameters

Transcendental Functions for Epsilon Expansions

The ε expansion can introduce new transcendental functions that must be implemented in the calculation. One of the goals of our work has been to classify the functions needed to construct an all-order e expansion of certain classes of HG functions.

In particular, the **multiple polylogarithms**

$$\mathrm{Li}_{k_1,k_2,\ldots,k_n}(z_1, z_2, \ldots, z_n) = \sum_{m_1>m_2>\cdots>m_n>0} \frac{z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

have proven useful for representing the coefficients of the ε expansions of a large class of HG functions.

The Case of Integer Parameters

- In the case when the parameters A, B, C are integers, the ε expansion may be written in terms of harmonic polylogarithms.

  [Remiddi and Vermaseren, Int. J. Mod. Phys. A15 (2000), 725]

Harmonic polylogarithms are a special case of multiple polylogarithms:

$$\mathrm{Li}_{k_1,\cdots,k_n}(z) = \sum_{m_1>m_2>\cdots>m_n>0} \frac{z^{m_1}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}} = \mathrm{H}(\vec{m}; z)$$

with vector $\vec{m}$ given by $(\underbrace{0,0,\cdots,0}_{k_1-1\text{ times}},1,\underbrace{0,0,\cdots,0}_{k_2-1\text{ times}},1,\cdots,\underbrace{0,0,\cdots,0}_{k_n-1\text{ times}},1)$.

Generalization

- There is no proof that all the terms in the ε expansion can be represented in terms of only harmonic polylogarithms.
- There are, in fact, known examples that cannot be expressed in terms of harmonic polylogarithms with a simple argument.
- For HG functions of the **Gauss** type, we have been able to prove a representation in terms of harmonic polylogarithms.

Gauss Hypergeometric Functions

Some Feynman diagrams giving rise to Gauss HG Functions include

- one-loop propagator diagrams with arbitrary masses and momenta
- two loop bubble diagrams with arbitrary masses
- one-loop massless vertex diagrams with three nonzero external momenta.

Theorem for Gauss HG Functions

We proved the following theorem:

The ε expansion of a Gauss HG function

$$ {}_2F_1\left(A+a\varepsilon, B+b\varepsilon; C+c\varepsilon; z\right) $$

with A, B, C integers or half-integers may be expressed in terms of harmonic polylogarithms with polynomial coefficients.

In the process, we obtained a constructive procedure to calculate all terms in the expansion iteratively.

Reduction Algorithm

The proof begins with the observation that any Gauss HG function can be written as a linear combination of two others with parameters differing from the original parameters by an integer.

- Specifically,

$$P(a,b,c,z)\,{}_2F_1\left(a+I_1, b+I_2; c+I_3; z\right) = \left\{Q_1(a,b,c,z)\frac{d}{dz} + Q(a,b,c,z)\right\}\,{}_2F_1\left(a,b;c;z\right)$$

with a, b, c arbitrary parameters, I_1, I_2, I_3 integers, and P, Q_1, Q_2 polynomials in the parameters and argument z.

Reduction Algorithm

In this way, the given HG function can be reduced to a combination of five basis functions and their first derivatives:

$$ {}_2F_1(a, b\varepsilon; 1+c\varepsilon; z), \quad {}_2F_1(a\varepsilon, b\varepsilon; \tfrac{1}{2}+c\varepsilon; z), $$

$$ {}_2F_1(\tfrac{1}{2}+a\varepsilon, b\varepsilon; 1+c\varepsilon; z), \quad {}_2F_1(\tfrac{1}{2}+a\varepsilon, b\varepsilon; \tfrac{1}{2}+c\varepsilon; z), \quad {}_2F_1(\tfrac{1}{2}+a\varepsilon, \tfrac{1}{2}+b\varepsilon; \tfrac{1}{2}+c\varepsilon; z) $$

In fact, it is known that only the first two are algebraically independent, so to prove the theorem, it is sufficient to consider only these two basis functions and show that they can be expressed as harmonic polylogarithms.

Outline of Proof

- The proof proceeds by writing a differential equation satisfied by the basis HG functions, and expanding the solution in powers of ε^n.
- The coefficients of these powers can then be constructed iteratively and recognized as harmonic polylogarithms.
- Obtaining the k^{th} coefficient requires knowledge of the previous ones, in this construction.

Gauss HG Functions: More General

The ε expansions of the functions

$$ {}_2F_1\left(I_1+\tfrac{p}{q}+a\varepsilon, I_2+b\varepsilon; I_3+c\varepsilon; z\right), \qquad {}_2F_1\left(I_1+a\varepsilon, I_2+b\varepsilon; I_3+\tfrac{p}{q}+c\varepsilon; z\right), $$

$$ {}_2F_1\left(I_1+\tfrac{p}{q}+a\varepsilon, I_2+b\varepsilon; I_3+\tfrac{p}{q}+c\varepsilon; z\right), \qquad {}_2F_1\left(I_1+\tfrac{p}{q}+a\varepsilon, I_2+\tfrac{p}{q}+b\varepsilon; I_3+\tfrac{p}{q}+c\varepsilon; z\right) $$

with I_1, I_2, I_3, p, q integers can be expressed in terms of multiple polylogarithms whose arguments are q^{th} roots of unity and another variable that is an algebraic function of z, with coefficients that are ratios of polynomials. For integers or half-integers, harmonic polylogarithms are sufficient.

[KWY, JHEP 2 (2007) 040 + M. Yu. Kalmykov, B. A. Kniehl, arXiv:0807.0567 (Nucl. Phys. B)]

Theorem for Generalized HG Functions

The ε expansions of the functions

$$ {}_pF_{p-1}\left(\vec{A}+\vec{a}\varepsilon, \tfrac{p}{q}+I; \vec{B}+\vec{b}\varepsilon; z\right), \qquad {}_pF_{p-1}\left(\vec{A}+\vec{a}\varepsilon, \vec{B}+\vec{b}\varepsilon; I+\tfrac{p}{q}; z\right), $$

with A_i, B_i, I, p, q integers can be expressed in terms of multiple polylogarithms whose arguments are powers of q^{th} roots of unity and another variable that is an algebraic function of z, with coefficients that are ratios of polynomials.

[KWY JHEP 10 (2007) 048, JHEP 11 (2007) 009
+ M. Yu. Kalmykov, B. A. Kniehl, arXiv:0807.0567 (Nucl. Phys. B)]

Outlook

- This is just a very brief introduction to HG function approach to Feynman diagrams.
- One goal is to combine the results into a software package based on the differential equation representation. Bytev, Kalmykov, and Kniehl have recently constructed a Mathematica implementation for the HG functions ${}_pF_{p-1}$, and F_1, ... F_4 called **HYPERDIRE**.
- Conversely, mathematicians have been using results motivated by Feynman diagrams to discover new relations among HG functions and related functions. This is a fertile area of interaction between mathematics and physics.