Calculating gluon one-loop amplitudes numerically

[Linear Collider Workshop 2008]

Jan Winter [a]

– Fermilab –

- *Next-to-leading order calculations*
- *Algorithm – from tree-level to one-loop amplitudes*
- ***Preliminary** results – work in progress*

[a] In collaboration with: W. Giele

http://www.sherpa-mc.de/

NLO calculations

➡ ***Lessons learned from LEP, HERA, Tevatron:***
LO predictions are fine, yet often give rough estimates only

- new experiments ➡ complex processes containing multijet final states
- correct interpretation of data ➡ accurate theoretical descriptions are required

➡ NLO: 1st real predicition of normalization of many observables
less sensitivity to unphysical input scales (μ_{F} & μ_{R})
more physics (parton merging, jet substructure, ISR, more IS parton species)

NLO calculations

➡ ***Lessons learned from LEP, HERA, Tevatron:***
LO predictions are fine, yet often give rough estimates only

- new experiments ➡ complex processes containing multijet final states
- correct interpretation of data ➡ accurate theoretical descriptions are required

➡ NLO: 1st real predicition of normalization of many observables
less sensitivity to unphysical input scales ($\mu_{\rm F}$ & $\mu_{\rm R}$)
more physics (parton merging, jet substructure, ISR, more IS parton species)

➡ ***Components of NLO calculations***

- tree-level amplitudes (LO & real radiation) + one-loop correction to Born level + subtraction terms to handle and combine singularities + phase-space generator
- computational algorithms based on Feynman diagram calculations are of exponential complexity
- the real bottleneck, virtual corrections (tensor-integral reductions generate large # of terms)
- @ tree level: algorithms of polynomial complexity exist ($\tau \sim N^{\#}$)
  recursive methods efficiently re-use recurring groups of offshell Feynman graphs
- @ loop level: unitarity-cut methods factorize one-loop into tree amplitudes
  computing time grows with # of cuts & depends on algorithm employed at tree level

Goal ➡ ***provide algorithm(s) [tools] of polynomial complexity to calculate virtual corrections***

Unitarity techniques for 1-loop amplitudes

active field of research ...

- *Britto et al.*
- *Bern et al.* – *BlackHat project and code.*
- *Ossola et al.* – *CutTools code.*
- *Ellis et al.* – *Rocket Science.*

- *This work is based on* –

  [Ellis, Giele, Kunszt, arXiv:0708.2398] 4dim method, cut-constructible part

  [Giele, Kunszt, Melnikov, arXiv:0801.2237] Ddim method, rational part

  [Giele, Zanderighi, arXiv:0805.2152] application of Ddim method to pure gluons

Decomposing one-loop amplitudes

➡ ***into a linear sum of scalar box, triangle, bubble and tadpole master integrals (cut-constructible part) and rational terms***

$$\mathcal{A}_N(\{p_i\}) = \sum_{[i_1|i_4]} d_{i_1 i_2 i_3 i_4} I^{(D)}_{i_1 i_2 i_3 i_4} + \sum_{[i_1|i_3]} c_{i_1 i_2 i_3} I^{(D)}_{i_1 i_2 i_3} + \sum_{[i_1|i_2]} b_{i_1 i_2} I^{(D)}_{i_1 i_2} + \sum_{[i_1|i_1]} a_{i_1} I^{(D)}_{i_1} + \mathcal{R}_N$$

- master integrals known in literature
- and implemented in various codes, e.g. QCDLoop [Ellis, Zanderighi] (QCDLoop.fnal.gov)
- to do: determination of the master-integral coefficients ⬅ unitarity techniques
- problem: extraction of lower-point coefficients
  "subtracting terms already included in higher-point contributions"
- solution: identify subtraction terms at the integrand level [Ossola, Papadopoulos, Pittau]
  partial fractioning of the integrand: expand over 4,3,2,1 propagator terms
  residues of pole terms contain master-integral coefficient plus finite number of spurious terms
  spurious terms vanish upon integration

➡ note that $[i_1, i_M] = 1 \leq i_1 < i_2 < \ldots < i_M \leq N$

and $$I^{(D)}_{i_1 \ldots i_M} = \int d^D \ell \frac{1}{d_{i_1} \ldots d_{i_M}}$$

➡ re-expressing the integrand

[Ellis, Giele, Kunszt]

$$\mathcal{A}_N(\{p_i\},\ell) \;=\; \frac{\mathcal{N}(\{p_i\},\ell)}{d_1 d_2 \ldots d_N} \;=$$

$$\sum_{[i_1|i_4]} \frac{\bar{d}_{i_1 i_2 i_3 i_4}(\ell)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{[i_1|i_3]} \frac{\bar{c}_{i_1 i_2 i_3}(\ell)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\bar{b}_{i_1 i_2}(\ell)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\bar{a}_{i_1}(\ell)}{d_{i_1}}$$

- solve for numerator factors:
  need to find $\ell = \ell_{i_1 \ldots i_M}$ such that $d_j(\ell_{i_1 \ldots i_M}) = 0$ for $j = i_1, \ldots, i_M$
- define $\mathrm{Res}_{i_1 \ldots i_M}(\mathcal{A}_N(\ell)) = \{d_{i_1}(\ell) \ldots d_{i_M}(\ell) \times \mathcal{A}_N(\ell)\}|_{\ell=\ell_{i_1 \ldots i_M}}$ then

$$\bar{d}_{i_1 i_2 i_3 i_4}(\ell) \;=\; \mathrm{Res}_{i_1 i_2 i_3 i_4}(\mathcal{A}_N(\ell))\,, \quad \bar{c}_{i_1 i_2 i_3}(\ell) \;=\; \mathrm{Res}_{i_1 i_2 i_3}\left(\mathcal{A}_N(\ell) - \sum_{[j_1|j_4]} \frac{\bar{d}_{j_1 j_2 j_3 j_4}(\ell)}{d_{j_1} d_{j_2} d_{j_3} d_{j_4}}\right), \ldots$$

- find parametric (most general) form of residues, removing spurious terms ➡
  box coefficient: $\bar{d}_{i_1 i_2 i_3 i_4}(\ell) = d_{i_1 i_2 i_3 i_4} + (\ell n_4)\, \tilde{d}_{i_1 i_2 i_3 i_4}$

$$\Rightarrow \quad \int d^D\ell \frac{\bar{d}_{i_1 i_2 i_3 i_4}(\ell)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} \;=\; d_{i_1 i_2 i_3 i_4} \int d^D\ell \frac{1}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} \;=\; d_{i_1 i_2 i_3 i_4}\; I_{i_1 i_2 i_3 i_4}$$

Disentangling the rational part

[GIELE, KUNSZT, MELNIKOV]

➡ ***by generalizing the unitarity method to higher dimensions***

- keep momenta and polarization vectors of external particles in 4D
- integer dimensionality for virtual particles: 2 sources, loop momentum $\to D$, spin-polarization states $\to D_s$, and ensure $D_s \geq D$
- continuation to non-integer dimensions once coefficients determined (schemes: 'tHV, FDH)

$$\mathcal{A}_N^{(D_s)}(\{p_i\},\ell) = \frac{\mathcal{N}^{(D_s)}(\{p_i\},\ell)}{d_1 d_2 \ldots d_N} =$$

$$\sum_{[i_1|i_5]} \frac{\bar{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(\ell)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\bar{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(\ell)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{[i_1|i_3]} \frac{\bar{c}^{(D_s)}_{i_1 i_2 i_3}(\ell)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\bar{b}^{(D_s)}_{i_1 i_2}(\ell)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\bar{a}^{(D_s)}_{i_1}(\ell)}{d_{i_1}}$$

- loop momentum effectively has only 4+1 components: $\mathcal{N}(\ell) = \mathcal{N}^{(D_s)}(\ell_{1..4}, -\ell_5^2 - \ldots - \ell_D^2)$
- dependence of $\mathcal{N}$ on D_s is linear: $\mathcal{N}^{(D_s)}(\ell) = \mathcal{N}_0(\ell) + (D_s - 4)\,\mathcal{N}_1(\ell)$

$$\bar{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(\ell) = \mathrm{Res}_{i_1 i_2 i_3 i_4 i_5}(\mathcal{A}_N^{(D_s)}(\ell)),\ \bar{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(\ell) = \mathrm{Res}_{i_1 i_2 i_3 i_4}\left[\mathcal{A}_N^{(D_s)}(\ell) - \sum_{[j_1|j_5]} \frac{\bar{e}^{(D_s)}_{j_1 j_2 j_3 j_4 j_5}(\ell)}{d_{j_1} d_{j_2} d_{j_3} d_{j_4} d_{j_5}}\right]$$

- parametric form of Res has larger structure ➡ some new terms not spurious, 4 new master integrals

box coefficient: $\bar{d}^{(D_s)}_{i_1\ldots i_4}(\ell) = d^{(0)}_{i_1\ldots i_4} + (\ell n_4)\, d^{(1)}_{i_1\ldots i_4} + s_e^2\,[d^{(2)}_{i_1\ldots i_4} + (\ell n_4)\, d^{(3)}_{i_1\ldots i_4}] + s_e^4\, d^{(4)}_{i_1\ldots i_4}$

Generating loop momenta

➡ ***under the constraint that the inverse propagators vanish***

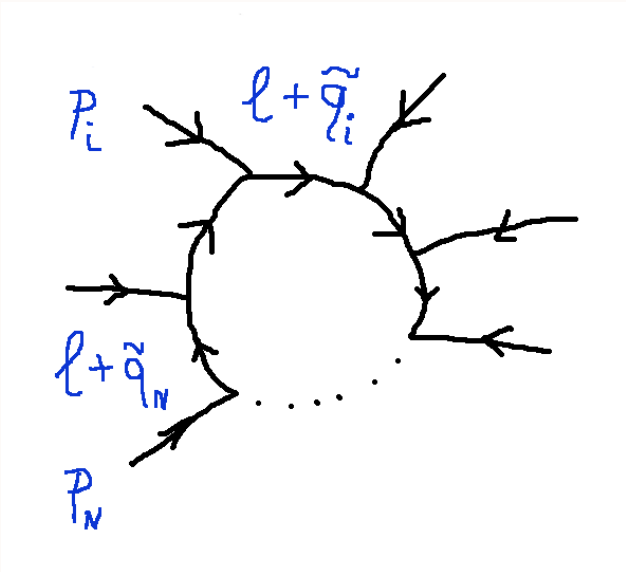
$$d_j(\ell_{i_1\ldots i_M}) = 0 \qquad \text{for} \qquad j = i_1, \ldots, i_M$$

- definition is: $d_i = d_i(\ell) = (\ell + \tilde{q}_i)^2 - m_i^2 = (\ell + q_i - q_{i_M})^2 - m_i^2$

  where $q_i = \sum_{j=1}^{i} p_j$
- cut configuration is: $i_1 \ldots i_M$
- parametrize loop momentum in D dimensions $(\alpha_i = \ell n_i)$

$$\ell_{i_1\ldots i_M} = V_{i_1\ldots i_M} + \sum_{j=M}^{D} \alpha_j n_j$$

$$\alpha_M^2 = -V_{i_1\ldots i_M}^2 + m_{i_M}^2 - \alpha_{M+1}^2 - \ldots - \alpha_D^2$$



- physical space defined by external particles, (sum of) inflow momenta ➡ $V_{i_1\ldots i_M}$
- orthogonal to physical: trivial space spanned by $n_M, \ldots, n_D$ (re-using n's for other cuts)
- make use of van Neerven–Vermaseren basis (involves calculation of (large) determinants)
- leaves enough freedom in choosing loop momentum ℓ

Parametric forms of residues

➡ ***generating the "Right-Hand-Side" (RHS) of the equation for the numerator factors***

- use freedom in choosing ℓ to find coefficients, $s_e^2 = -\sum_{j=5}^{D} \alpha_j^2$ (clever α choices possible)

$$\bar{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(\ell) \;=\; s_e^2\, e^{(0)}_{i_1 i_2 i_3 i_4 i_5}$$

$$\bar{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(\ell) \;=\; d^{(0)}_{i_1 i_2 i_3 i_4} + \alpha_4\, d^{(1)}_{i_1 i_2 i_3 i_4} + s_e^2\, [d^{(2)}_{i_1 i_2 i_3 i_4} + \alpha_4\, d^{(3)}_{i_1 i_2 i_3 i_4}] + s_e^4\, d^{(4)}_{i_1 i_2 i_3 i_4}$$

$$\bar{c}^{(D_s)}_{i_1 i_2 i_3}(\ell) \;=\; c^{(0)}_{i_1 i_2 i_3} + \alpha_3\, c^{(1)}_{i_1 i_2 i_3} + \alpha_4\, c^{(2)}_{i_1 i_2 i_3} + 4 \text{ more} + s_e^2\, [c^{(7)}_{i_1 i_2 i_3} + \alpha_3\, c^{(8)}_{i_1 i_2 i_3} + \alpha_4\, c^{(9)}_{i_1 i_2 i_3}]$$

$$\bar{b}^{(D_s)}_{i_1 i_2}(\ell) \;=\; b^{(0)}_{i_1 i_2} + \alpha_2\, b^{(1)}_{i_1 i_2} + \alpha_3\, b^{(2)}_{i_1 i_2} + \alpha_4\, b^{(3)}_{i_1 i_2} + 5 \text{ more} + s_e^2\, b^{(9)}_{i_1 i_2}$$

$$\bar{a}^{(D_s)}_{i_1}(\ell) \;=\; a^{(0)}_{i_1} + \alpha_1\, a^{(1)}_{i_1} + \alpha_2\, a^{(2)}_{i_1} + \alpha_3\, a^{(3)}_{i_1} + \alpha_4\, a^{(4)}_{i_1}$$

- solving: make X choices of ℓ to solve for X coefficients
- cut-c part: $D = D_s = 4$
- rational part: $D > 4$, (1st) $D_s = D + 1$ (2nd) $D_s = D$ to eliminate D_s dependence of LHS
  - in principle, infinite # of equations for a fixed # of unknowns ➡ Coefficients can be fitted!

Generating the Left-Hand-Side

- What is $\mathrm{Res}_{i_1 \ldots i_M}(\mathcal{A}_N^{(D_s)}(\ell))$?

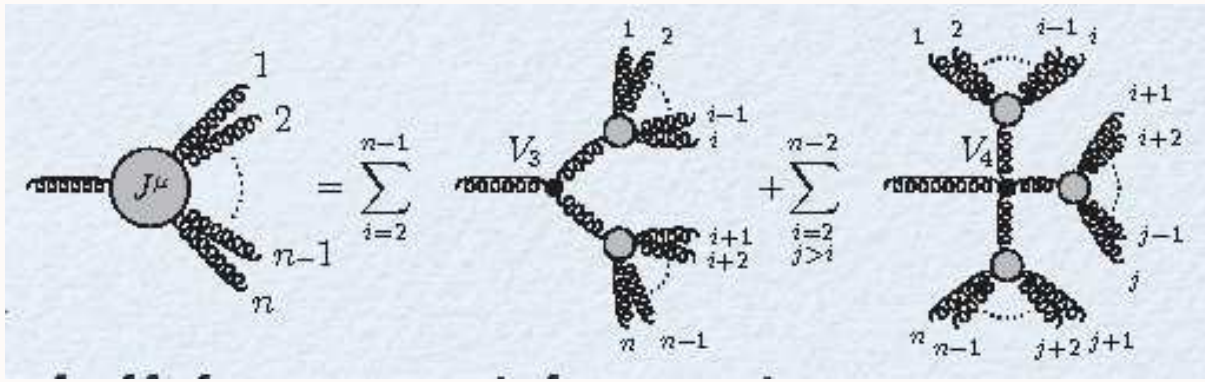
$$= \{ d_{i_1}(\ell) \ldots d_{i_M}(\ell) \times \mathcal{A}_N(\ell) \}|_{d_{i_1}(\ell)=\cdots=d_{i_M}(\ell)=0}$$

- requires calculation of factorized un-integrated one-loop amplitude
- unitarity cuts: M on-shell propagators, amplitude factorizes into M tree-level amplitudes

$$\mathrm{Res}_{i_1 \ldots i_M}(\mathcal{A}_N^{(D_s)}(\ell)) = \sum_{\{\lambda_1,\ldots,\lambda_M\}=1}^{D_s-2} \left(\prod_{k=1}^{M} \mathcal{M}^{(0)} \left(\ell_{i_k}^{(\lambda_k)}; p_{i_k+1}, \ldots, p_{i_{k+1}}; -\ell_{i_{k+1}}^{(\lambda_{k+1})} \right) \right)$$

- two D_s dimensional gluons with complex momenta and $D_s - 2$ polarization states $(\ell_{i_k} = \ell + \tilde{q}_{i_k})$
- construct polarizations following method for n vectors

- Berends–Giele recursion relations to calculate tree-level amplitudes
- very economical scheme
- LHS:
  first correct for D_s dependence, then take subtractions into account



Colour-ordered one-loop amplitude

- coefficients are now independent of dimensionality
- dimensionality can now be continued to $4-2\,\epsilon$

$$\mathcal{A}_N^{cc} = \sum_{[i_1|i_4]} d^{(0)}_{i_1 i_2 i_3 i_4} I^{(4-2\epsilon)}_{i_1 i_2 i_3 i_4} + \sum_{[i_1|i_3]} c^{(0)}_{i_1 i_2 i_3} I^{(4-2\epsilon)}_{i_1 i_2 i_3} + \sum_{[i_1|i_2]} b^{(0)}_{i_1 i_2} I^{(4-2\epsilon)}_{i_1 i_2}$$

$$\mathcal{R}_N = -\sum_{[i_1|i_4]} \frac{d^{(4)}_{i_1 i_2 i_3 i_4}}{6} + \sum_{[i_1|i_3]} \frac{c^{(7)}_{i_1 i_2 i_3}}{2} - \sum_{[i_1|i_2]} \left(\frac{(q_{i_1} - q_{i_2})^2}{6} - \frac{m^2_{i_1} + m^2_{i_2}}{2} \right) b^{(9)}_{i_1 i_2}$$

C++ code

➡ ***Another tool ...*** *Rocket (Rucola) was already launched* [GIELE, ZANDERIGHI]

- independent implementation (from scratch, no translation of Fortran routines)
- allows for independent xchecks of unitarity method and its results
- knowing the tool is knowing the methods, and knowing the details
- C++ ... different philosophy ... modularity, transparency
- allows for combination with other C++ codes ... potentially ... ***COMIX*** ... Gleisberg's automated CS subtraction ... ***Sherpa*** ...

N external gluons & their polarizations ➡ *(leading-)colour-ordered 1-loop amplitude (FDH)*

- xchecks on numbers
  - coefficients itself, poles (known analytically), final numbers (analytic and other calculations)
  - gauge invariance, choice of ℓ, dimensionality (D and D_s variation)
- accuracy and numerical stability

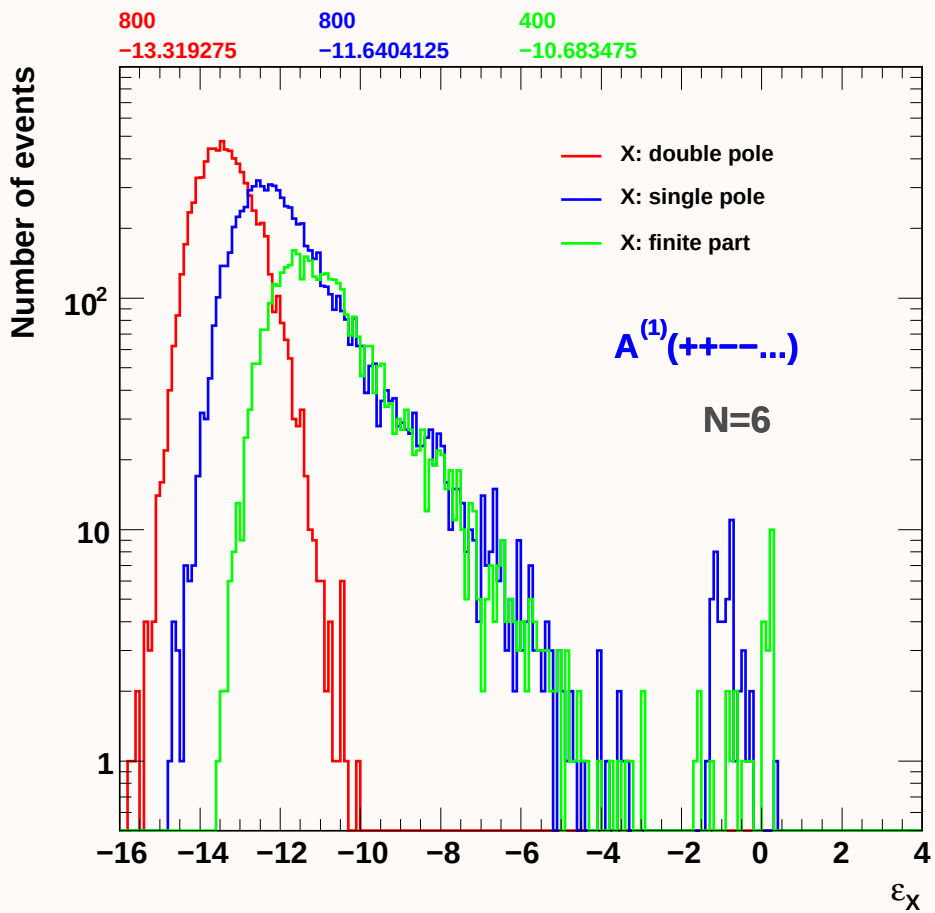
$$\varepsilon_{\mathrm{dp,sp}} = \log_{10} \frac{|\mathcal{A}^{(1)(\mathrm{dp,sp})}_{N,\mathrm{C++}} - \mathcal{A}^{(1)(\mathrm{dp,sp})}_{N,\mathrm{anly}}|}{|\mathcal{A}^{(1)(\mathrm{dp,sp})}_{N,\mathrm{anly}}|}, \qquad \varepsilon_{\mathrm{fp}} = \log_{10} \frac{2\,|\mathcal{A}^{(1)(\mathrm{fp})}_{N,\mathrm{C++}}[1] - \mathcal{A}^{(1)(\mathrm{fp})}_{N,\mathrm{C++}}[2]|}{|\mathcal{A}^{(1)(\mathrm{fp})}_{N,\mathrm{C++}}[1]| + |\mathcal{A}^{(1)(\mathrm{fp})}_{N,\mathrm{C++}}[2]|}$$

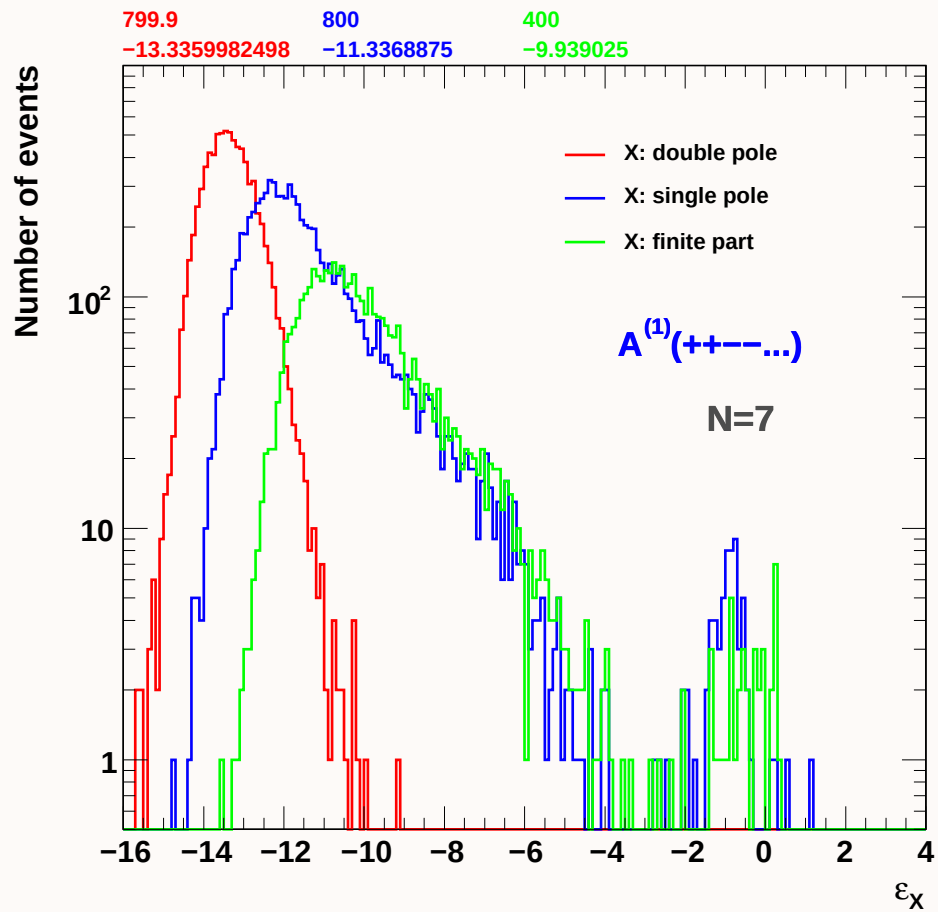
- efficiency – scaling of computing time with # of legs N $\rightarrow$ $\tau \sim N^9$

Accuracy

(preliminary) (all calculations in double precision only)

- peak positions are fine, tails seem OK, comparable to Rocket
- need to investigate on the bumpy structures for sp, fp around $\varepsilon_X = -1$ (more PSP needed!)
- losing finite-part precision with $N = 10, 11$, lost for $N = 15$
  (double precision not enough, too many large numbers involved)

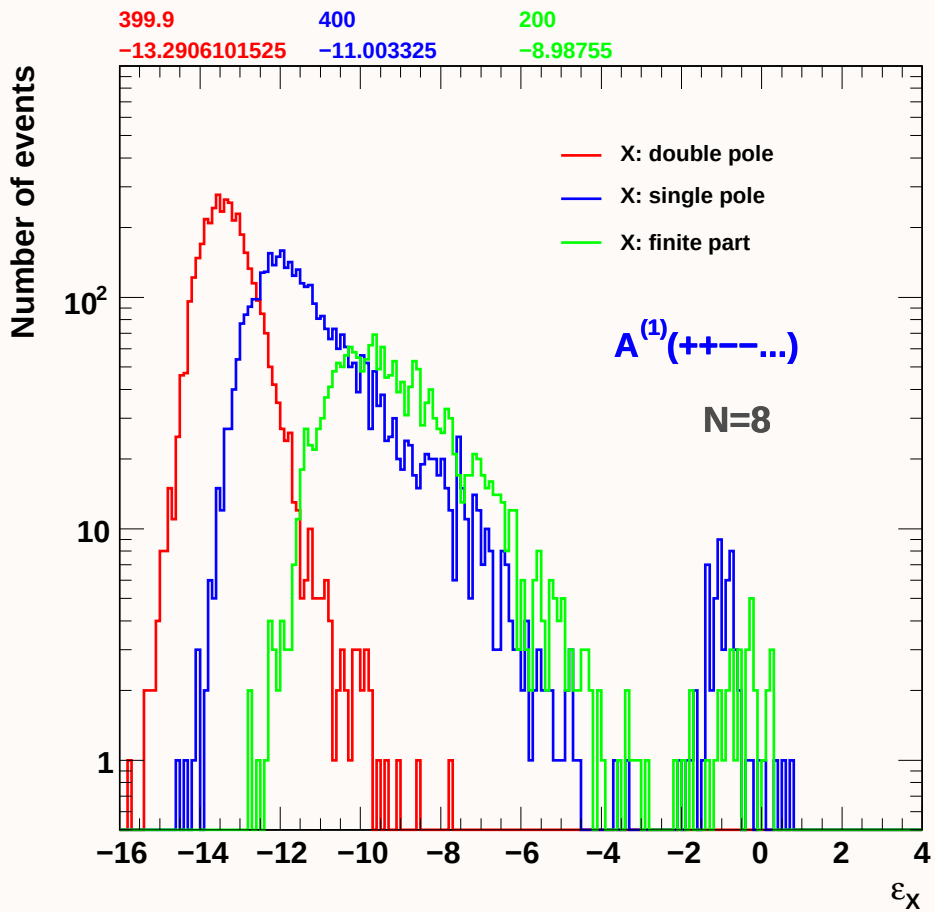


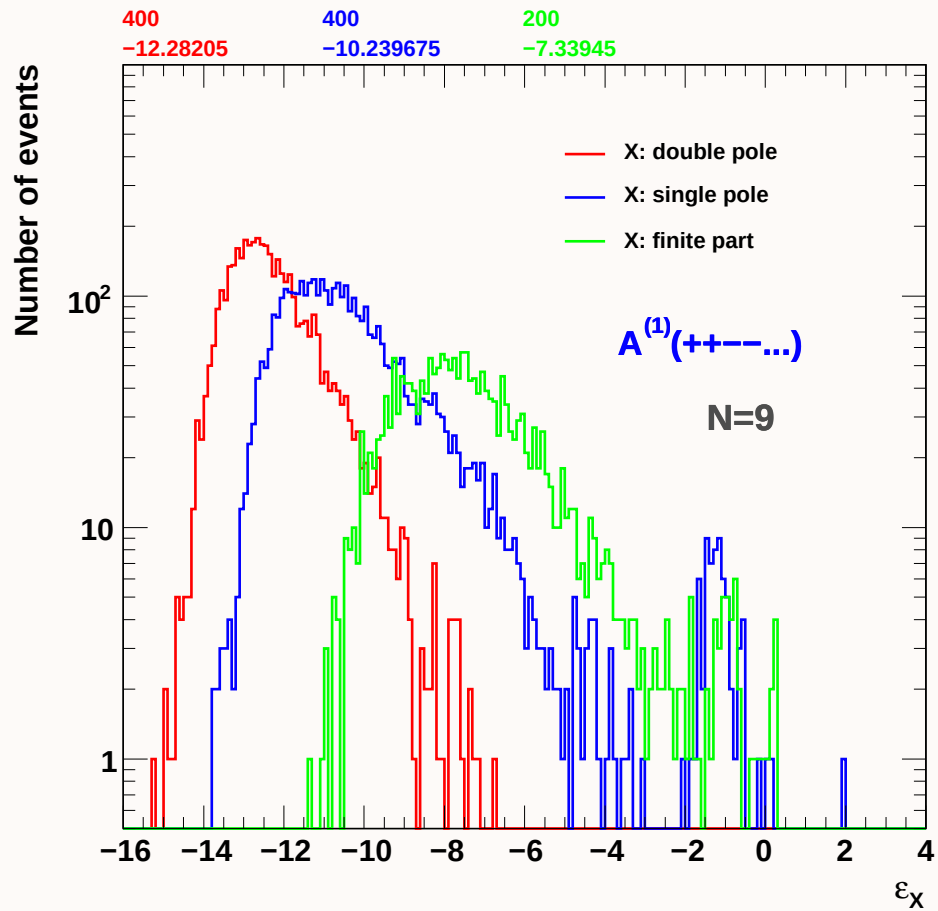


Accuracy

(preliminary) (all calculations in double precision only)

- peak positions are fine, tails seem OK, comparable to Rocket
- need to investigate on the bumpy structures for sp, fp around $\varepsilon_X = -1$ (more PSP needed!)
- losing finite-part precision with $N = 10, 11$, lost for $N = 15$
  (double precision not enough, too many large numbers involved)

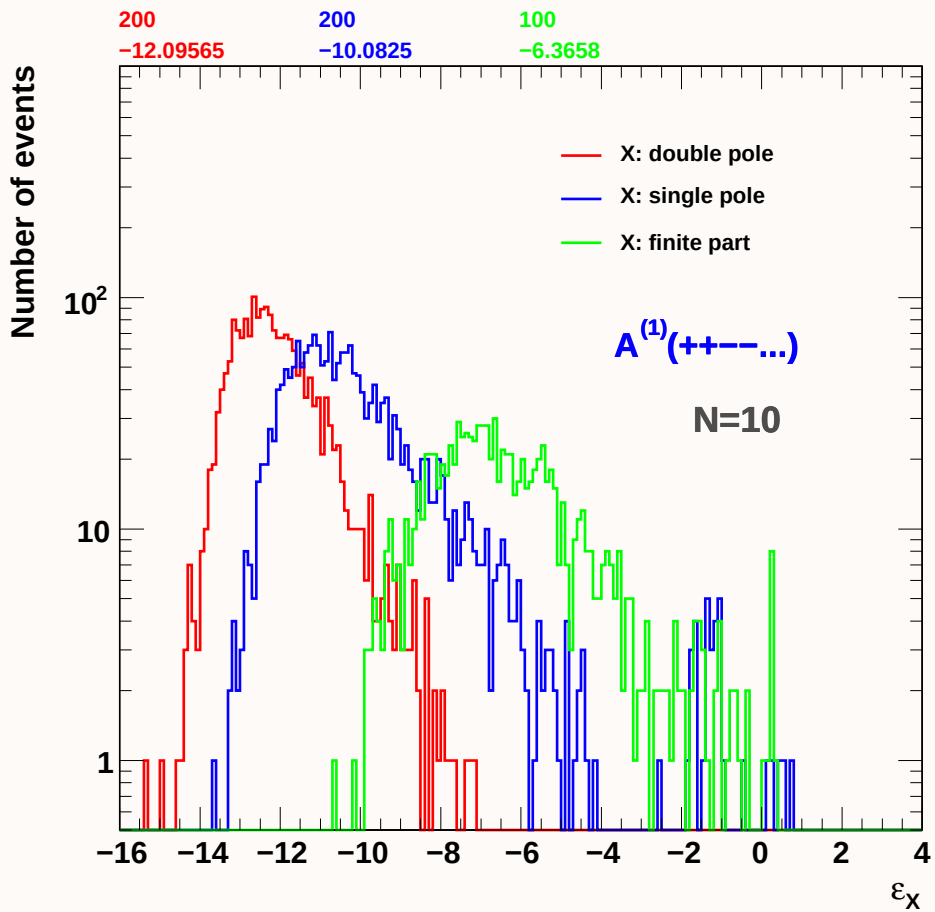


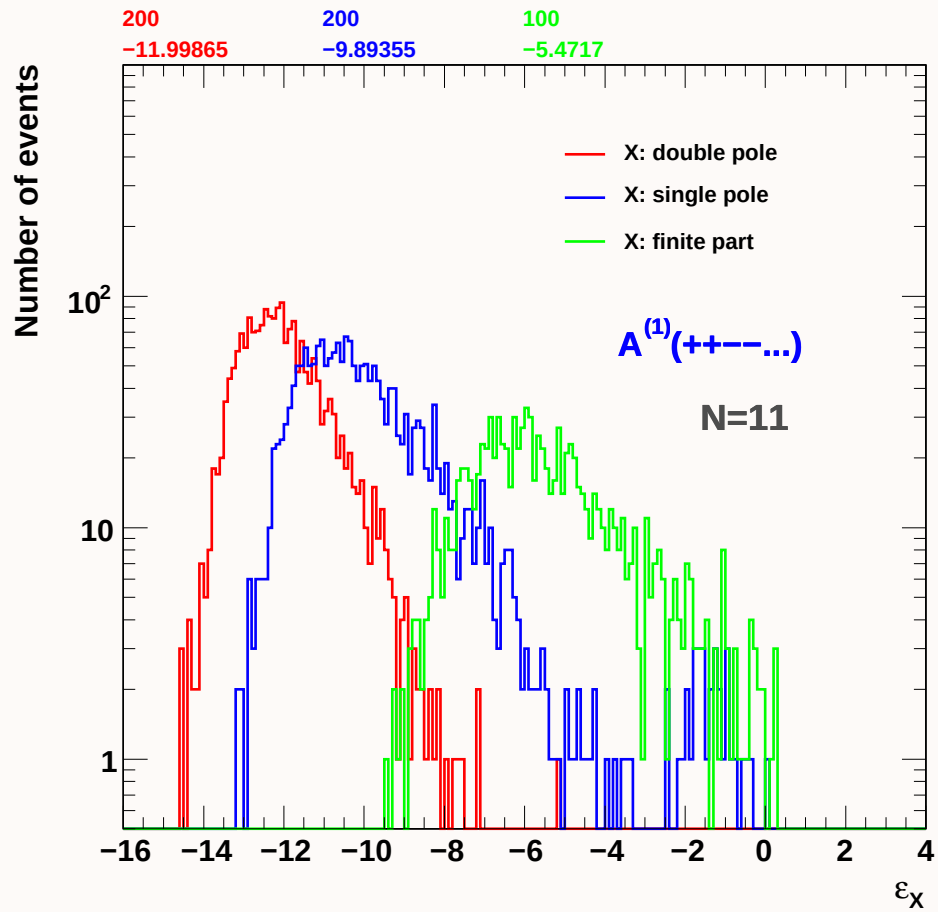


Accuracy

(preliminary) (all calculations in double precision only)

- peak positions are fine, tails seem OK, comparable to Rocket
- need to investigate on the bumpy structures for sp, fp around $\varepsilon_X = -1$ (more PSP needed!)
- losing finite-part precision with $N = 10, 11$, lost for $N = 15$
  (double precision not enough, too many large numbers involved)

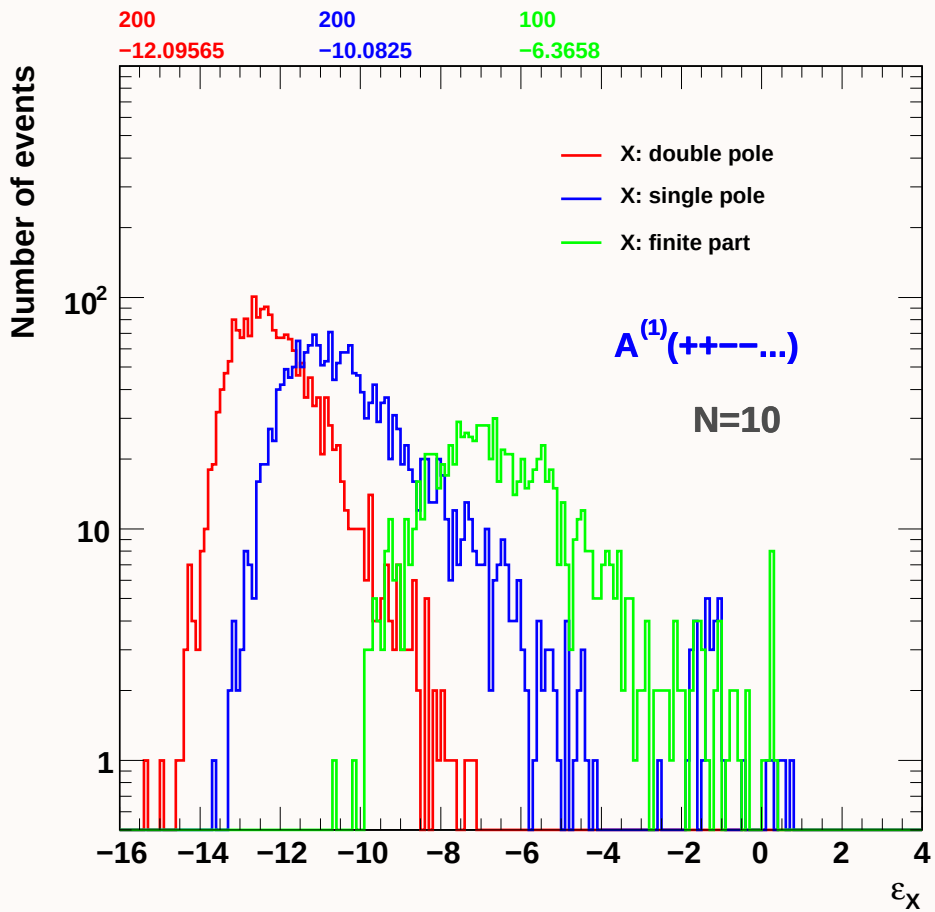


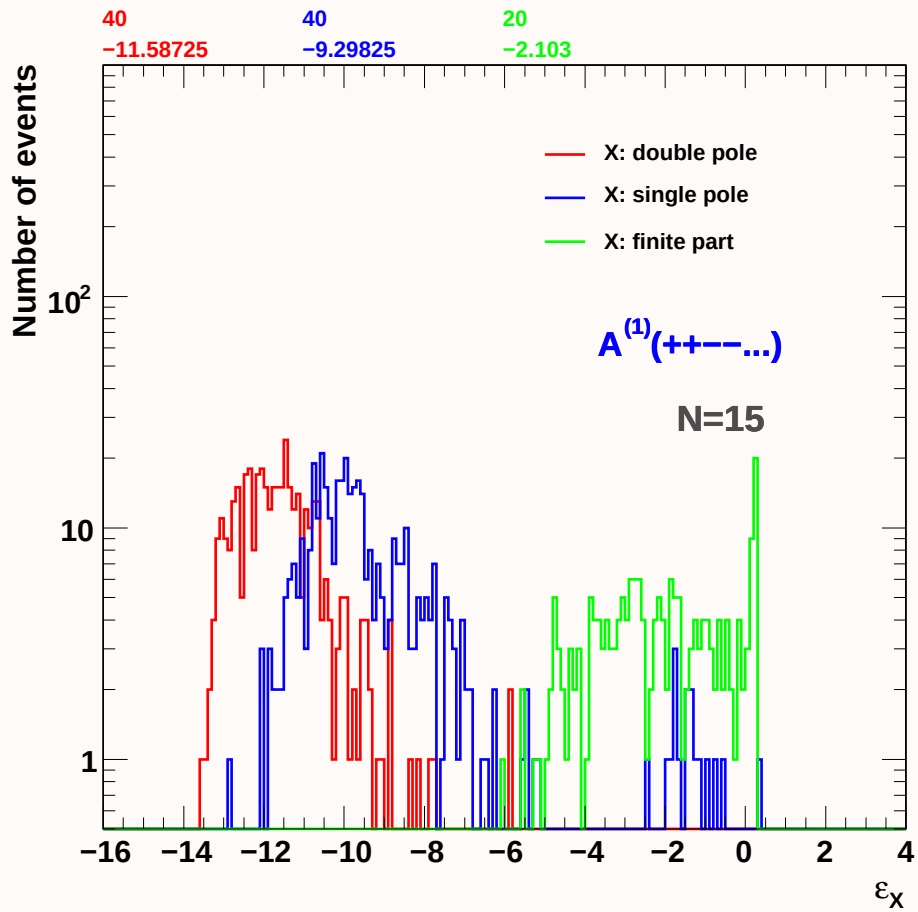


Accuracy

(preliminary) (all calculations in double precision only)

- peak positions are fine, tails seem OK, comparable to Rocket
- need to investigate on the bumpy structures for sp, fp around $\varepsilon_X = -1$ (more PSP needed!)
- losing finite-part precision with $N = 10, 11$, lost for $N = 15$
  (double precision not enough, too many large numbers involved)

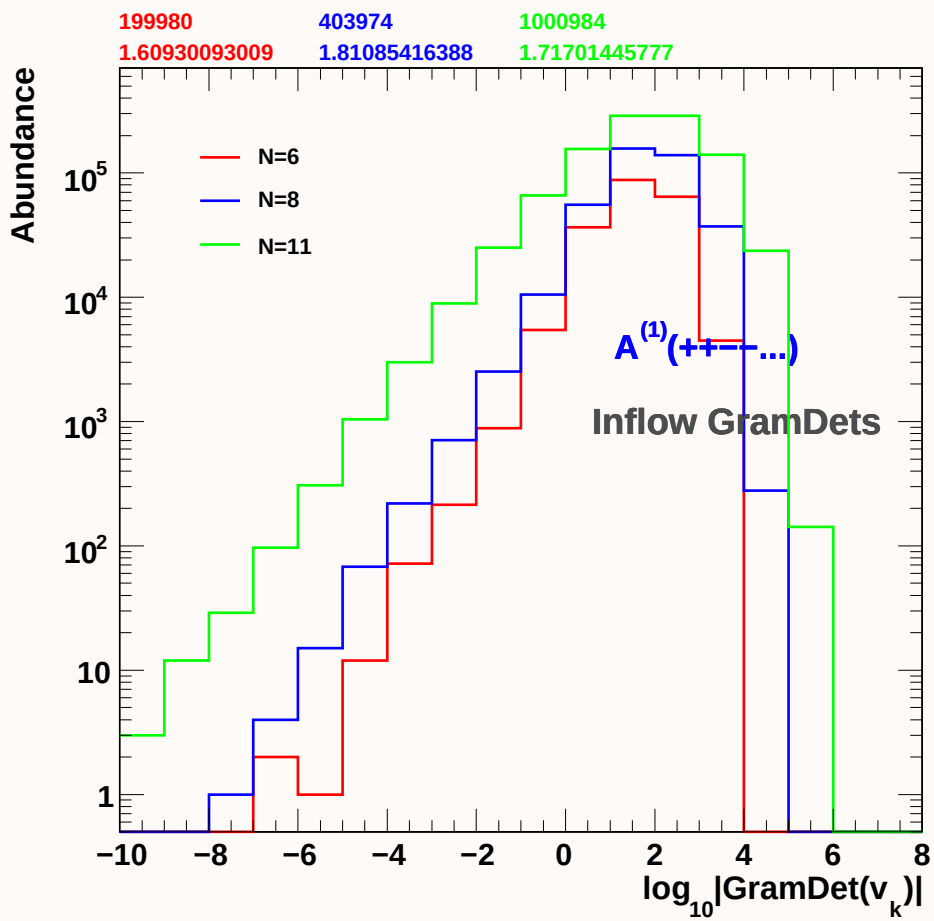


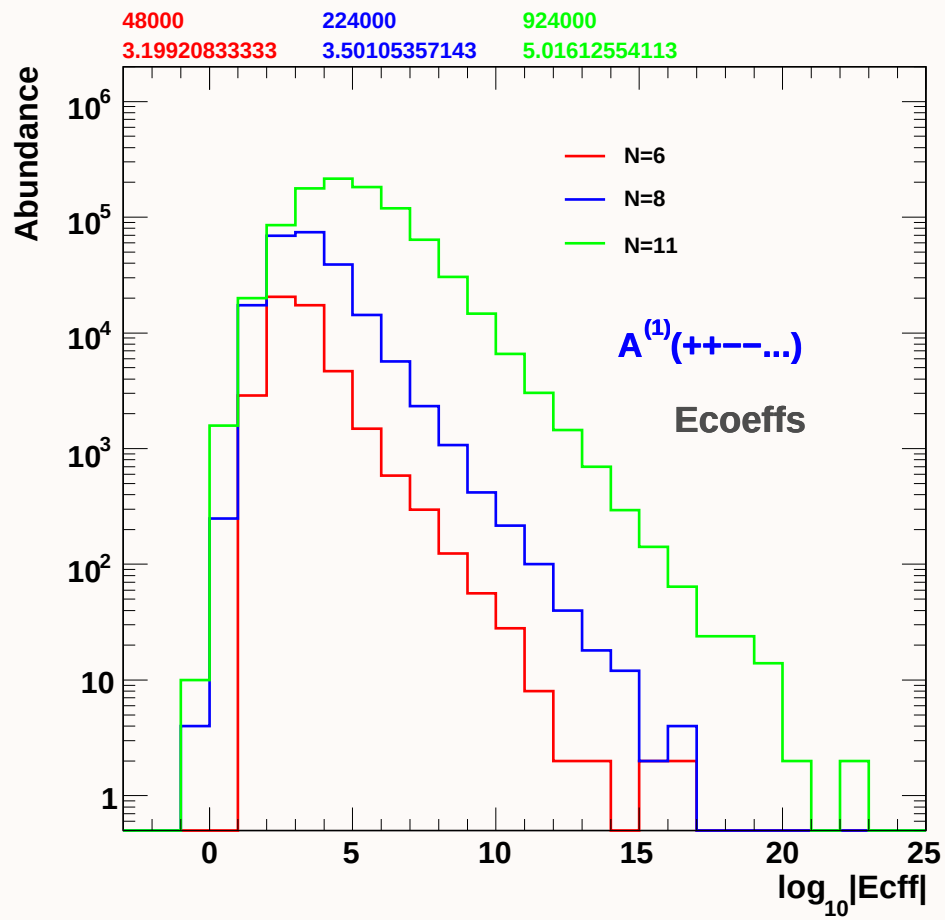


Accuracy

(preliminary) (all calculations in double precision only)

- range of numbers increases with N – Gram dets of external gluons and $e^{(0)}_{ijklm}$ coefficients may become small and large, respectively

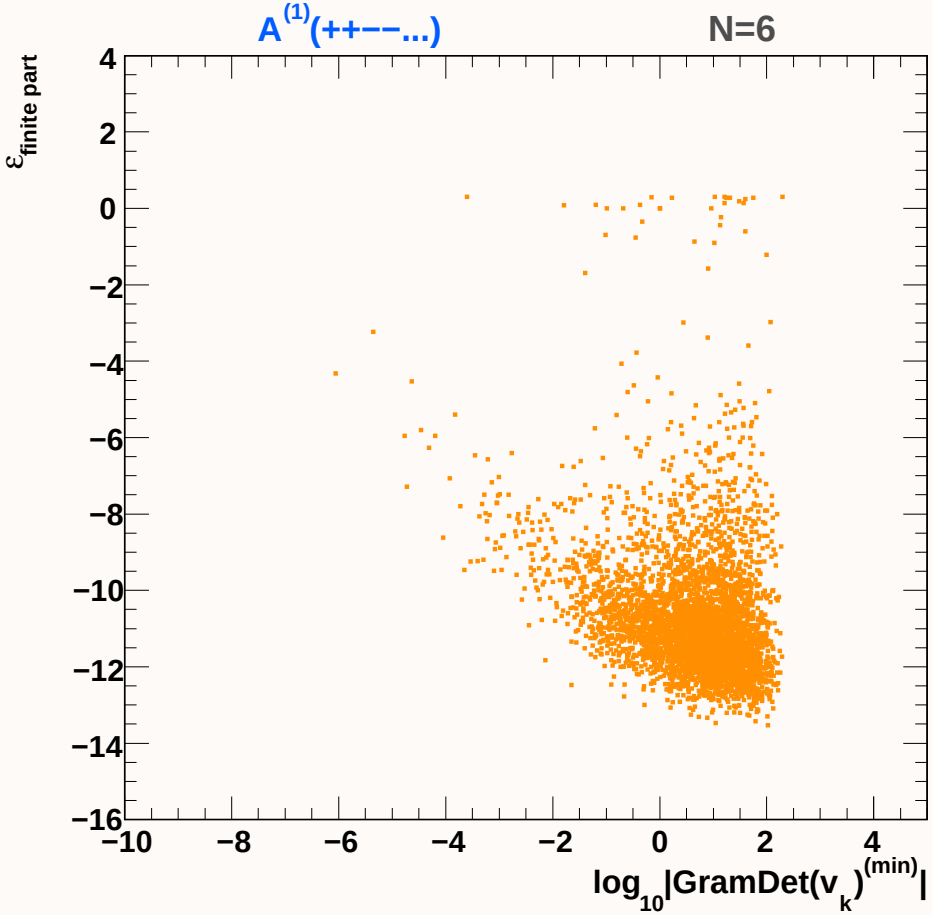


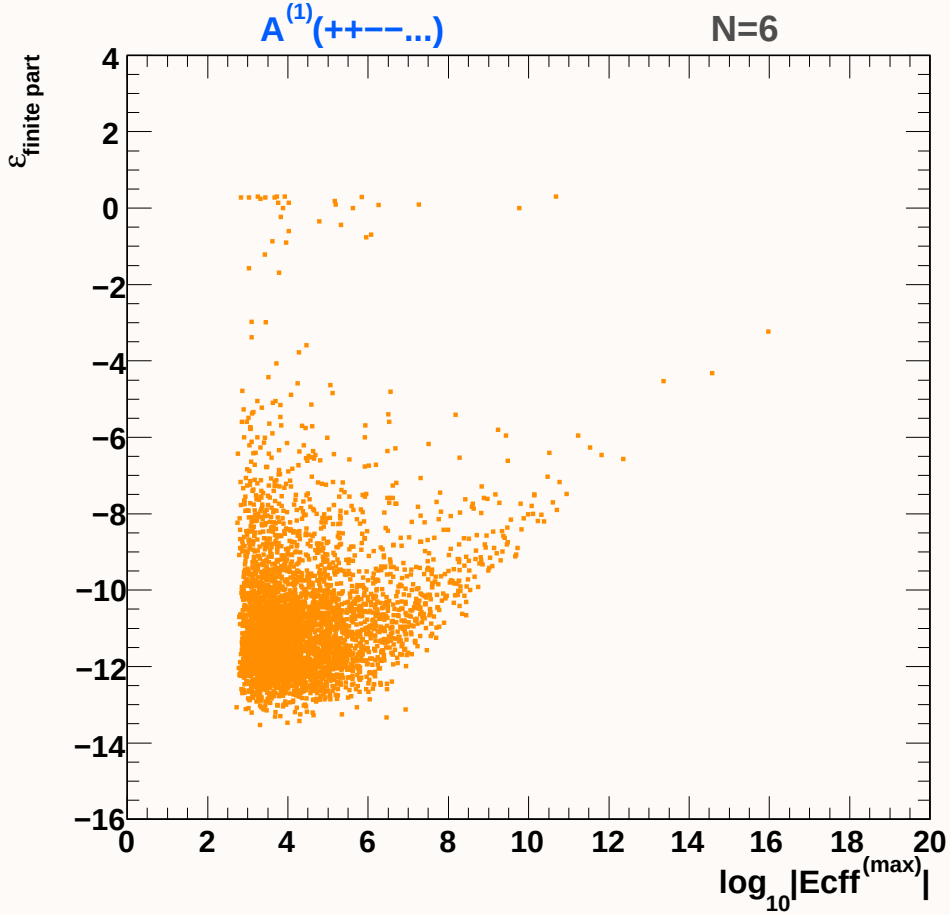


Correlations

(preliminary) (all calculations in double precision only)

- precision of finite term partly correlated with smallness/largeness of Gram dets/coefficients
- still other denominators that can become small
- e.g. the leftover d_j in the subtraction terms (even when coefficients are not large)

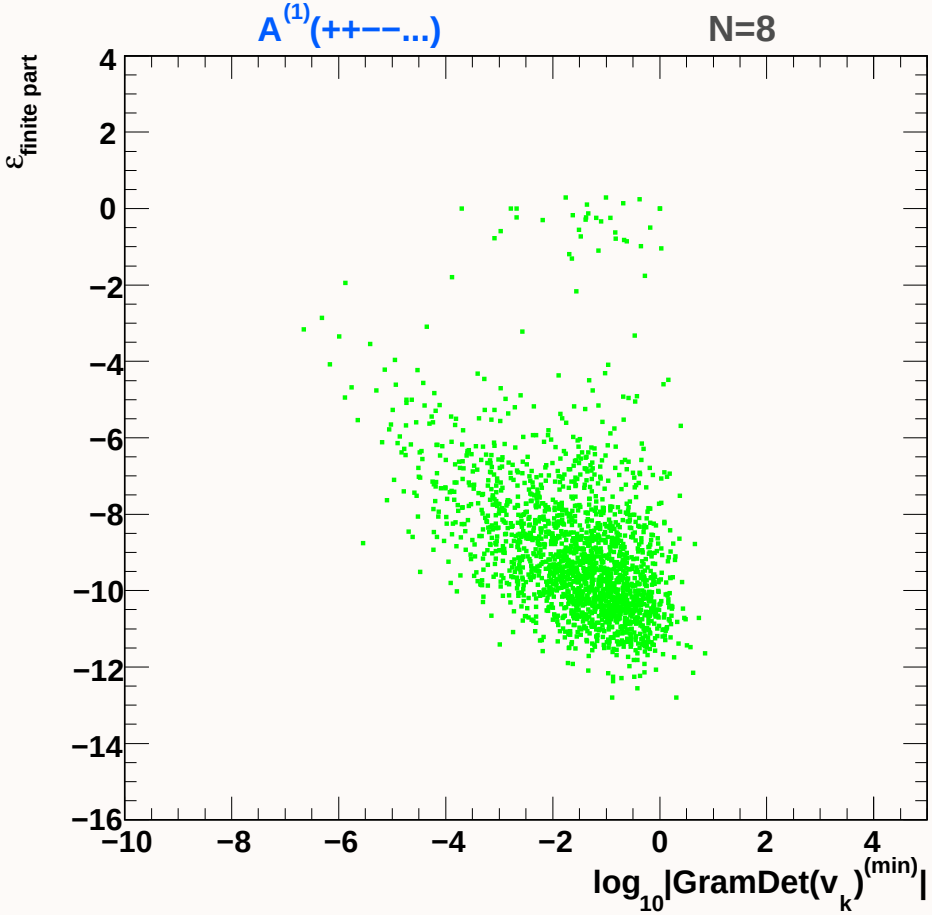


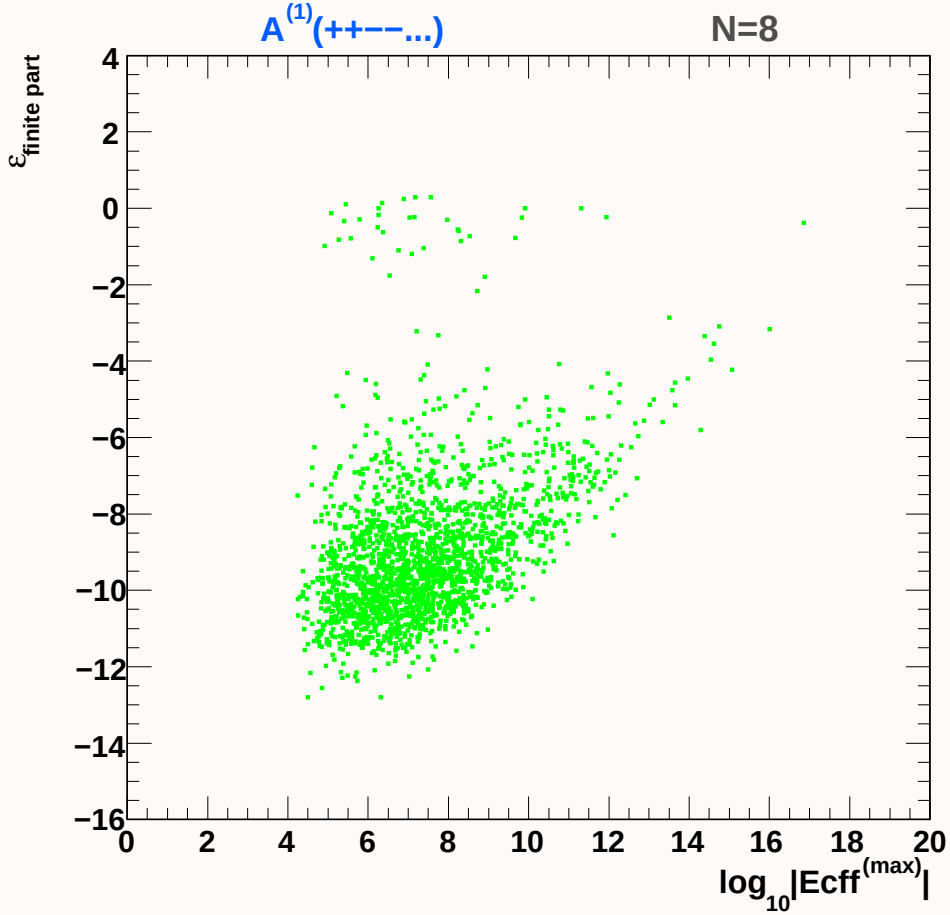


Correlations

(preliminary) (all calculations in double precision only)

- precision of finite term partly correlated with smallness/largeness of Gram dets/coefficients
- still other denominators that can become small
- e.g. the leftover d_j in the subtraction terms (even when coefficients are not large)

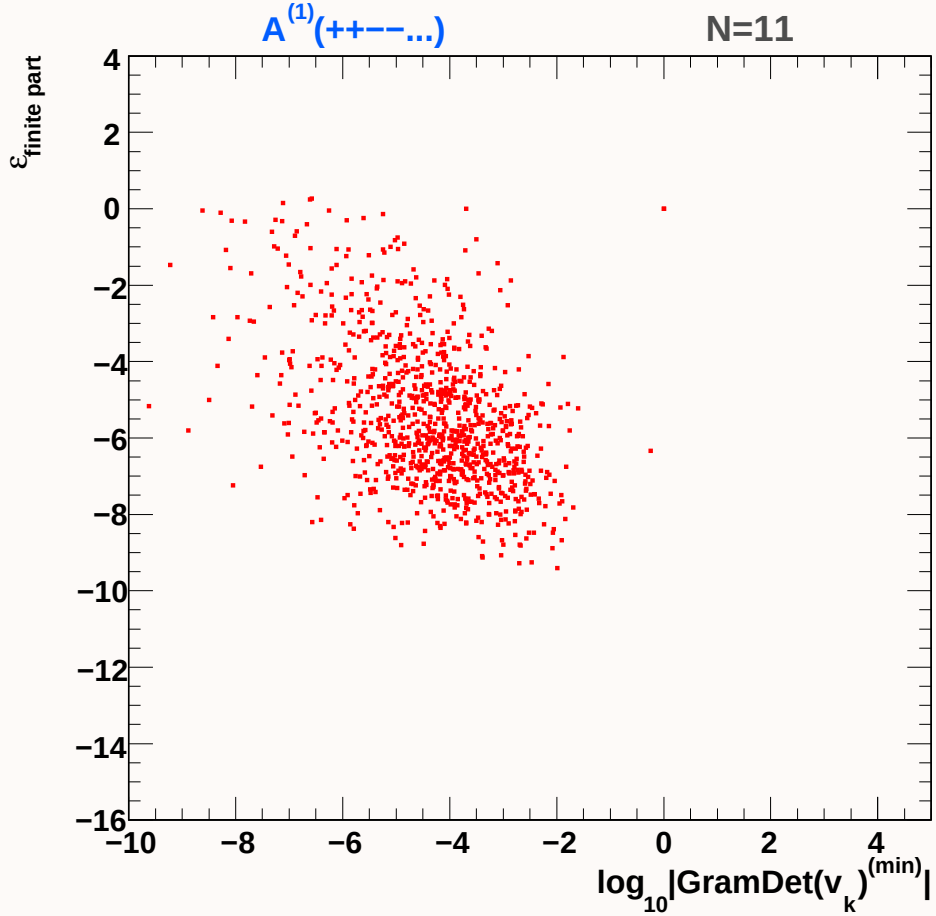


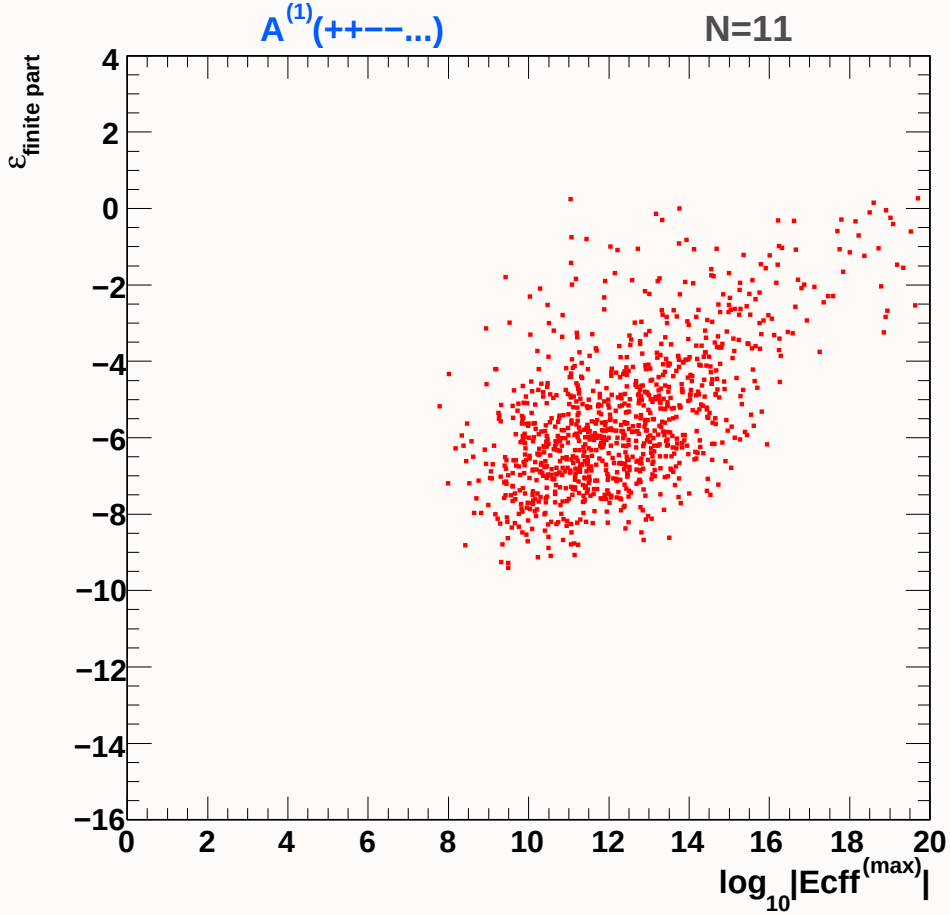


Correlations

(preliminary) (all calculations in double precision only)

- precision of finite term partly correlated with smallness/largeness of Gram dets/coefficients
- still other denominators that can become small
- e.g. the leftover d_j in the subtraction terms (even when coefficients are not large)

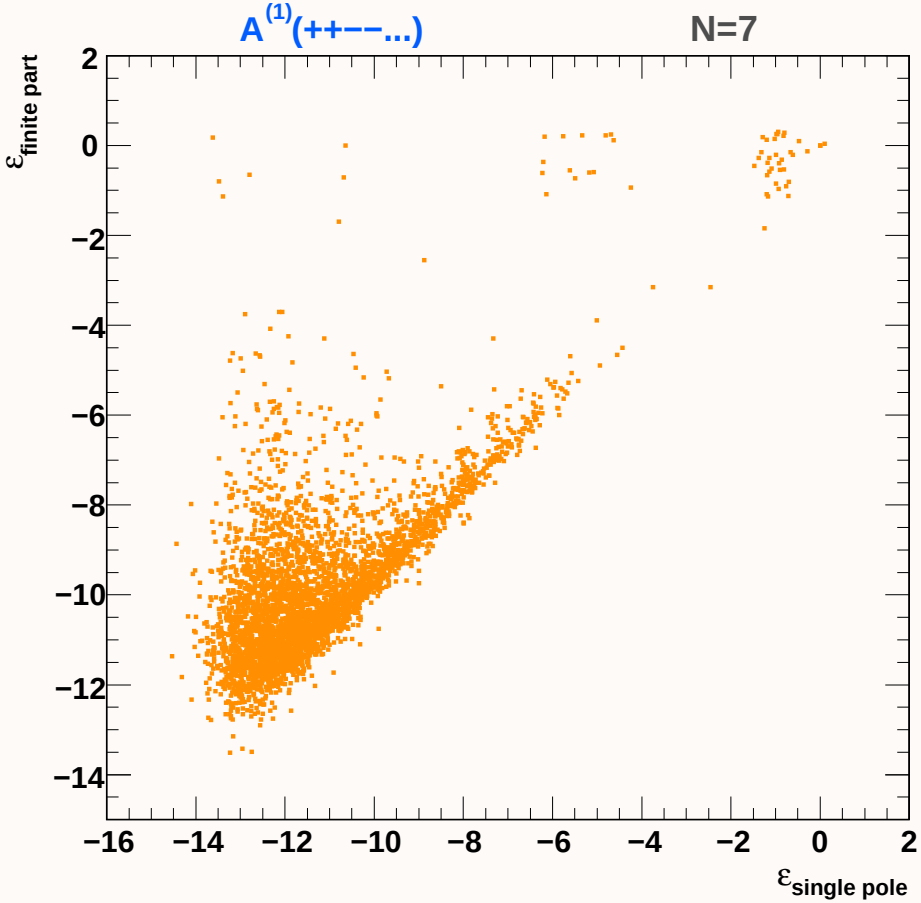


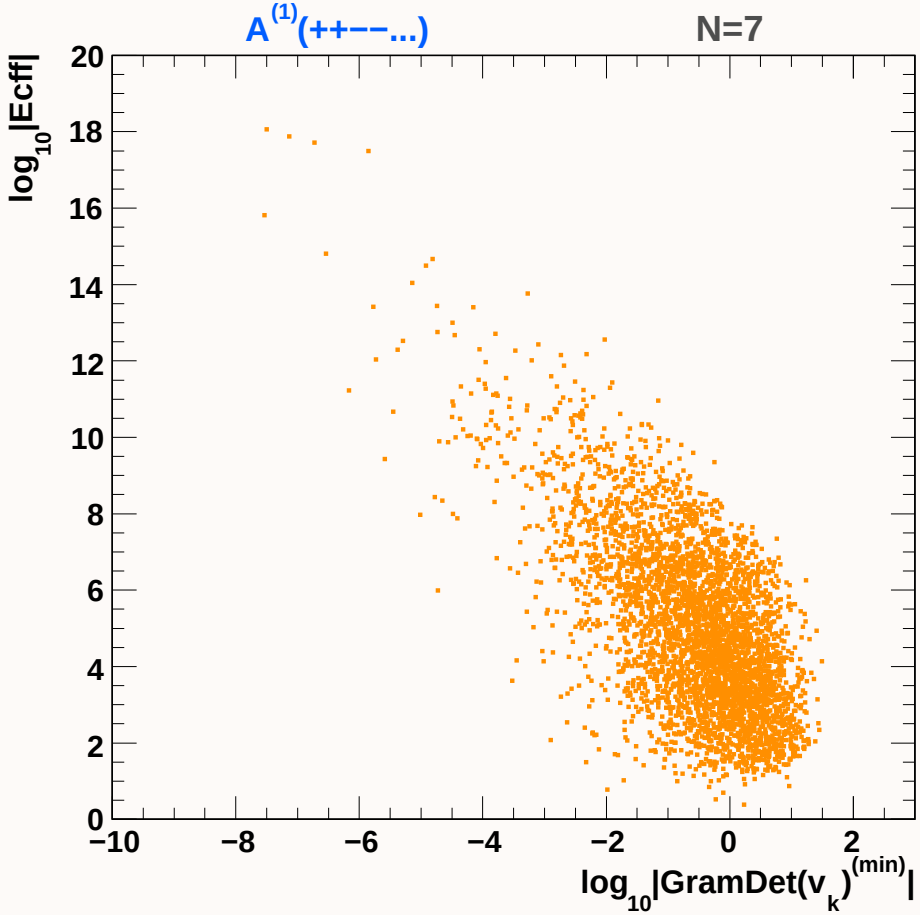


Correlations

(preliminary) (all calculations in double precision only)

- left: correlation between single-pole and finite-part accuracy
- right: which $e^{(0)}_{ijklm}$ coefficient occurs when external-gluon Gram det is minimal?

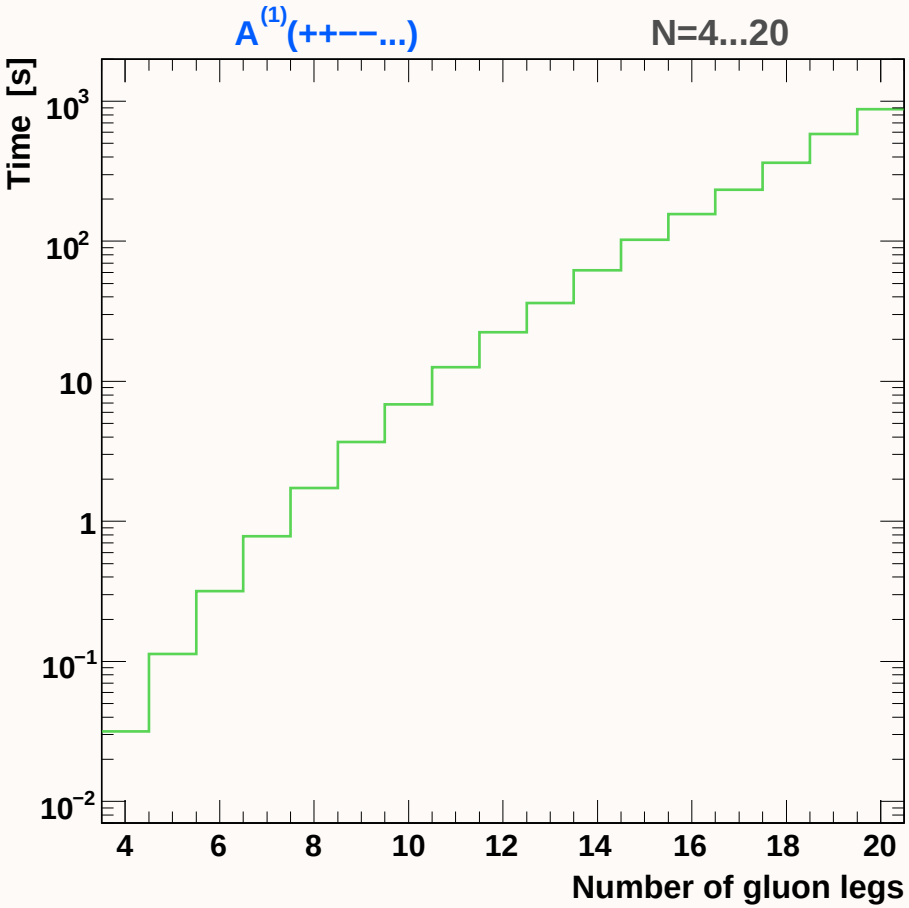


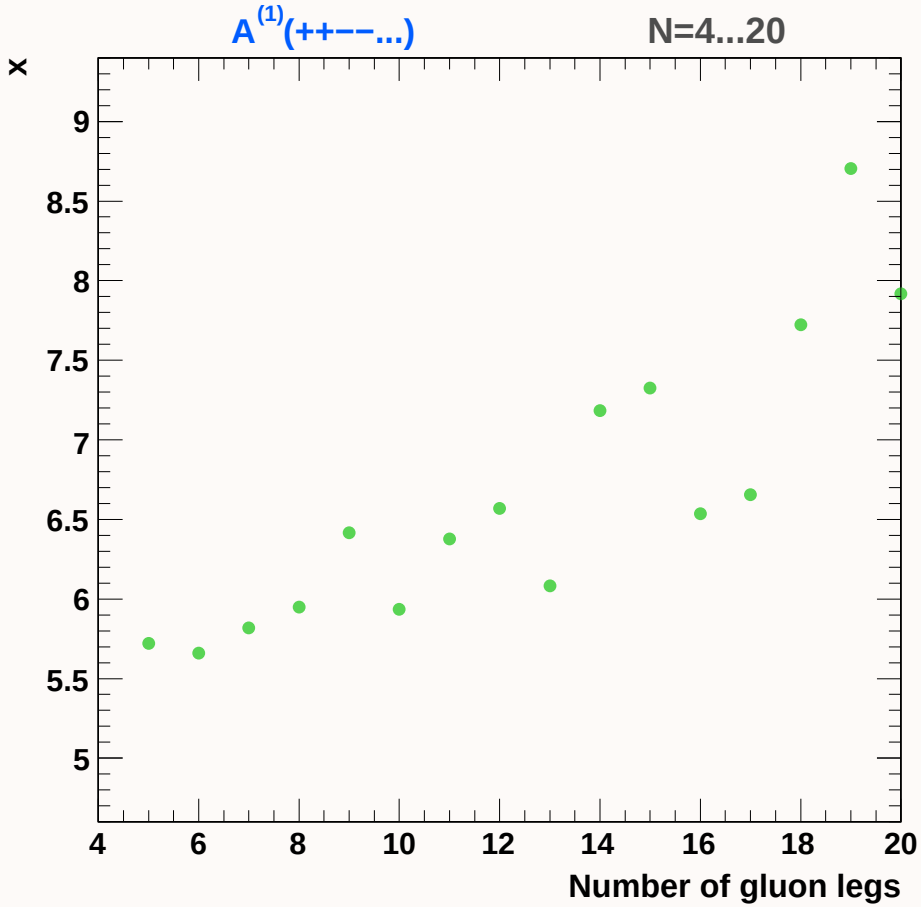


Speed of the calculation

(preliminary) (all calculations in double precision only)

- check for algorithm of polynomial complexity ($\tau \sim N^x$)
- check fractions: $x = \ln \frac{\tau_{N+1}}{\tau_N} / \ln \frac{N+1}{N}$





Basic tool is set up and running

... there's much more to do ...!!!