External field QED calculations - the method of Nikishov and Ritus et al

Anthony Hartin

Advanced QED for future colliders Workshop

Mar 3, 2009

Synopsis

- Methodology
  - Use Furry picture to include the external field

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field
  - Modified Feynman rules

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field
  - Modified Feynman rules
- Beamstrahlung example
  - Tree level with respect to the Furry Picture

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field
  - Modified Feynman rules
- Beamstrahlung example
  - Tree level with respect to the Furry Picture
  - Only consider unpolarised fermions

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field
  - Modified Feynman rules
- Beamstrahlung example
  - Tree level with respect to the Furry Picture
  - Only consider unpolarised fermions
  - Specific Volkov representation for constant crossed field

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field
  - Modified Feynman rules
- Beamstrahlung example
  - Tree level with respect to the Furry Picture
  - Only consider unpolarised fermions
  - Specific Volkov representation for constant crossed field
  - Fourier Transform of the Volkov solutions

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field
  - Modified Feynman rules
- Beamstrahlung example
  - Tree level with respect to the Furry Picture
  - Only consider unpolarised fermions
  - Specific Volkov representation for constant crossed field
  - Fourier Transform of the Volkov solutions
  - Phase Integral calculation

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field
  - Modified Feynman rules
- Beamstrahlung example
  - Tree level with respect to the Furry Picture
  - Only consider unpolarised fermions
  - Specific Volkov representation for constant crossed field
  - Fourier Transform of the Volkov solutions
  - Phase Integral calculation
  - The Beamstrahlung Transition Rate

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field
  - Modified Feynman rules
- Beamstrahlung example
  - Tree level with respect to the Furry Picture
  - Only consider unpolarised fermions
  - Specific Volkov representation for constant crossed field
  - Fourier Transform of the Volkov solutions
  - Phase Integral calculation
  - The Beamstrahlung Transition Rate
  - Comparison with Operator method

Synopsis

- Methodology
  - Use Furry picture to include the external field
  - Volkov Solution
  - Dressed momentum and mass shift
  - Propagator in the external field
  - Modified Feynman rules
- Beamstrahlung example
  - Tree level with respect to the Furry Picture
  - Only consider unpolarised fermions
  - Specific Volkov representation for constant crossed field
  - Fourier Transform of the Volkov solutions
  - Phase Integral calculation
  - The Beamstrahlung Transition Rate
  - Comparison with Operator method

Motivation

- By the Nikishov and Ritus method I refer to a series of papers beginning in 1964 in which many of the first order external field
- There were other collaborators and independent calculations as well!

Motivation

- By the Nikishov and Ritus method I refer to a series of papers beginning in 1964 in which many of the first order external field
- There were other collaborators and independent calculations as well!
- Am presenting Nikishov and Ritus method because this is the method I learnt first
- It is interesting to compare with the Operator method

Motivation

- By the Nikishov and Ritus method I refer to a series of papers beginning in 1964 in which many of the first order external field
- There were other collaborators and independent calculations as well!
- Am presenting Nikishov and Ritus method because this is the method I learnt first
- It is interesting to compare with the Operator method
- Generally, the Nikishov-Ritus method produces complicated expressions
- I want to perform calculations to first order as efficiently as possible in order to extend to higher orders

Motivation

- By the Nikishov and Ritus method I refer to a series of papers beginning in 1964 in which many of the first order external field
- There were other collaborators and independent calculations as well!
- Am presenting Nikishov and Ritus method because this is the method I learnt first
- It is interesting to compare with the Operator method
- Generally, the Nikishov-Ritus method produces complicated expressions
- I want to perform calculations to first order as efficiently as possible in order to extend to higher orders
- I work in natural units $\hbar, c = 1$. I usually write scalar products of 4-vectors (kp)
- I often refer to dimensionless quantities like $\frac{\hbar\omega}{mc^2}$ just as ω

Circularly polarised electromagnetic field

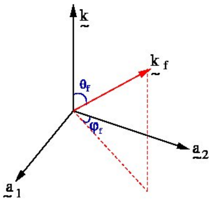

Figure: External field 3 vectors and photon scattering angles.

- Interactions involving intense lasers are characterised usually involve a circularly polarized field

$$A_\mu = a_{1_\mu} cos(kx) + a_{2_\mu} sin(kx)$$

Circularly polarised electromagnetic field

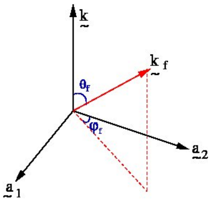

Figure: External field 3 vectors and photon scattering angles.

- Interactions involving intense lasers are characterised usually involve a circularly polarized field

$$A_\mu = a_{1_\mu} cos(kx) + a_{2_\mu} sin(kx)$$

The Lorentz condition implies $(a_1 k), (a_2 k) = 0$ so

$$(\vec{a}_1, \vec{a}_2, \vec{k}) \; ; \; |\vec{a}_1| = |\vec{a}_2| = a^2$$

Circularly polarised electromagnetic field

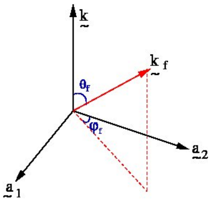

Figure: External field 3 vectors and photon scattering angles.

- Interactions involving intense lasers are characterised usually involve a circularly polarized field

$$A_\mu = a_{1_\mu} cos(kx) + a_{2_\mu} sin(kx)$$

The Lorentz condition implies $(a_1 k), (a_2 k) = 0$ so

$$(\vec{a}_1, \vec{a}_2, \vec{k}) \; ; \; |\vec{a}_1| = |\vec{a}_2| = a^2$$

- These vectors form the coordinate system
- $\frac{e^2 a^2}{m^2}$ and $\frac{\omega}{m}$ are the external field intensity and energy

Circularly polarised electromagnetic field

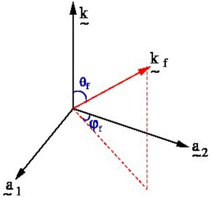

Figure: External field 3 vectors and photon scattering angles.

- Interactions involving intense lasers are characterised usually involve a circularly polarized field

$$A_\mu = a_{1_\mu} cos(kx) + a_{2_\mu} sin(kx)$$

The Lorentz condition implies $(a_1 k), (a_2 k) = 0$ so

$$(\vec{a}_1, \vec{a}_2, \vec{k}) \; ; \; |\vec{a}_1| = |\vec{a}_2| = a^2$$

- These vectors form the coordinate system
- $\frac{e^2 a^2}{m^2}$ and $\frac{\omega}{m}$ are the external field intensity and energy
- For a general 4-vector Q_μ

$$(kQ) = 0 \implies (a_1 Q)^2 + (a_2 Q)^2 = a^2 Q^2$$

External electromagnetic fields II

- Formation length is the distance travelled by a charged particle while a radiated photon moves one wavelength in front of it

External electromagnetic fields II

- Formation length is the distance travelled by a charged particle while a radiated photon moves one wavelength in front of it
- If the intense particle bunch fields in particle collider collisions are such that
  - The formation length is shorter than the bunch length

External electromagnetic fields II

- Formation length is the distance travelled by a charged particle while a radiated photon moves one wavelength in front of it
- If the intense particle bunch fields in particle collider collisions are such that
  - The formation length is shorter than the bunch length
  - The formation length is shorter than the field wavelength

External electromagnetic fields II

- Formation length is the distance travelled by a charged particle while a radiated photon moves one wavelength in front of it
- If the intense particle bunch fields in particle collider collisions are such that
  - The formation length is shorter than the bunch length
  - The formation length is shorter than the field wavelength
  - The particle is relativistic

External electromagnetic fields II

- Formation length is the distance travelled by a charged particle while a radiated photon moves one wavelength in front of it
- If the intense particle bunch fields in particle collider collisions are such that
  - The formation length is shorter than the bunch length
  - The formation length is shorter than the field wavelength
  - The particle is relativistic
- Then the field can be considered to be a constant crossed field $A_\mu = a_{1\mu}(kx)$
- The Lorentz condition $(a_1 k) = 0$

External electromagnetic fields II

- Formation length is the distance travelled by a charged particle while a radiated photon moves one wavelength in front of it
- If the intense particle bunch fields in particle collider collisions are such that
  - The formation length is shorter than the bunch length
  - The formation length is shorter than the field wavelength
  - The particle is relativistic
- Then the field can be considered to be a constant crossed field $A_\mu = a_{1\mu}(kx)$
- The Lorentz condition $(a_1 k) = 0$
- If $(kQ) = 0$ now $(a_1 Q)^2 = a^2(Q^2 - Q_y^2)$

External electromagnetic fields II

- Formation length is the distance travelled by a charged particle while a radiated photon moves one wavelength in front of it
- If the intense particle bunch fields in particle collider collisions are such that
  - The formation length is shorter than the bunch length
  - The formation length is shorter than the field wavelength
  - The particle is relativistic
- Then the field can be considered to be a constant crossed field $A_\mu = a_{1\mu}(kx)$
- The Lorentz condition $(a_1 k) = 0$
- If $(kQ) = 0$ now $(a_1 Q)^2 = a^2(Q^2 - Q_y^2)$
- For constant crossed field, $\omega \to 0$ $\frac{e^2 a^2}{m^2} \to \infty$

External electromagnetic fields II

- Formation length is the distance travelled by a charged particle while a radiated photon moves one wavelength in front of it
- If the intense particle bunch fields in particle collider collisions are such that
  - The formation length is shorter than the bunch length
  - The formation length is shorter than the field wavelength
  - The particle is relativistic
- Then the field can be considered to be a constant crossed field $A_\mu = a_{1\mu}(kx)$
- The Lorentz condition $(a_1 k) = 0$
- If $(kQ) = 0$ now $(a_1 Q)^2 = a^2(Q^2 - Q_y^2)$
- For constant crossed field, $\omega \to 0$ $\frac{e^2 a^2}{m^2} \to \infty$
- The physically meaningful quantity is $B = |\vec{a}_1|\omega$

Volkov Solution

- The Volkov solution is a solution of the second order Dirac equation containing the external potential

$$[(p - eA)^2 - m^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}]\Psi = 0$$

Volkov Solution

- The Volkov solution is a solution of the second order Dirac equation containing the external potential

$$[(p-eA)^2 - m^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}]\Psi = 0$$

- In the Lorentz gauge $k^\mu A_\mu = 0$ we have $\sigma^{\mu\nu}F_{\mu\nu} = 2i\not{k}\not{A}'$
- Propose a general solution $\Psi = \exp(-ip.x)E_p(x)u(p)$

Volkov Solution

- The Volkov solution is a solution of the second order Dirac equation containing the external potential

$$[(p - eA)^2 - m^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}]\Psi = 0$$

- In the Lorentz gauge $k^\mu A_\mu = 0$ we have $\sigma^{\mu\nu}F_{\mu\nu} = 2i\not{k}\not{A}'$
- Propose a general solution $\Psi = \exp(-ip.x)E_p(x)u(p)$
- Substitution of the general solution yields

$$2i(kp)E_p'(x) + [e^2A^2 - 2e(Ap) - ie\not{k}\not{A}']E_p(x) = 0$$

$$E_p(x) = \exp\left(\frac{e}{2(kp)}\not{k}\not{A} - iS(x)\right)$$

$$S(x) = -i\int_0^{k.x}\left[\frac{e(Ap)}{(kp)} - \frac{e^2A^2}{2(kp)}\right]d\phi$$

Volkov Solution

- The Volkov solution is a solution of the second order Dirac equation containing the external potential

$$[(p - eA)^2 - m^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}]\Psi = 0$$

- In the Lorentz gauge $k^\mu A_\mu = 0$ we have $\sigma^{\mu\nu}F_{\mu\nu} = 2i\not{k}\not{A}'$
- Propose a general solution $\Psi = \exp(-ip.x)E_p(x)u(p)$
- Substitution of the general solution yields

$$2i(kp)E_p'(x) + [e^2A^2 - 2e(Ap) - ie\not{k}\not{A}']E_p(x) = 0$$

$$E_p(x) = \exp\left(\frac{e}{2(kp)}\not{k}\not{A} - iS(x)\right)$$

$$S(x) = -i\int_0^{k.x}\left[\frac{e(Ap)}{(kp)} - \frac{e^2A^2}{2(kp)}\right]d\phi$$

- Expand the exponential term in a series and use $\not{k}\not{k} = 0$ and the Lorentz condition $(Ak) = 0$

$$\Psi_p^V(x) = \left(1 + \frac{e\not{k}\not{A}}{2(kp)}\right)\exp(iS(x))u(p)$$

The Volkov solution for a circularly polarised field

- Substitute particular A_μ into the Volkov S function

The Volkov solution for a circularly polarised field

- Substitute particular A_μ into the Volkov S function
- For a **circularly polarised field** we naturally get an interpretation in terms of photons and a mass shift

$$S(x) = -\left[(px) + \frac{e^2a^2}{2(kp)}(kx) + \frac{e(a_1p)}{(kp)}\sin(kx) - \frac{e(a_1p)}{(kp)}\cos(kx)\right]$$

The Volkov solution for a circularly polarised field

- Substitute particular A_μ into the Volkov S function
- For a **circularly polarised field** we naturally get an interpretation in terms of photons and a mass shift

$$S(x) = -\left[(px) + \frac{e^2a^2}{2(kp)}(kx) + \frac{e(a_1p)}{(kp)}\sin(kx) - \frac{e(a_1p)}{(kp)}\cos(kx)\right]$$

- The 'dressed' 4 momentum $q_\mu \equiv p_\mu + \frac{e^2a^2}{2k_\mu}$
- The mass shift is $q^2 \equiv m^2 + e^2a^2$
- The whole Volkov solution can be expressed in terms of the dressed momentum, since

$$(a_1q) = (a_1p), (kq) = (kp)$$

The Volkov solution for a circularly polarised field

- Substitute particular A_μ into the Volkov S function
- For a **circularly polarised field** we naturally get an interpretation in terms of photons and a mass shift

$$S(x) = -\left[(px) + \frac{e^2a^2}{2(kp)}(kx) + \frac{e(a_1p)}{(kp)}\sin(kx) - \frac{e(a_1p)}{(kp)}\cos(kx)\right]$$

- The 'dressed' 4 momentum $q_\mu \equiv p_\mu + \frac{e^2a^2}{2k_\mu}$
- The mass shift is $q^2 \equiv m^2 + e^2a^2$
- The whole Volkov solution can be expressed in terms of the dressed momentum, since

$$(a_1q) = (a_1p), (kq) = (kp)$$

- A discrete Fourier Transform of $\exp(iS(x))$ gives contributions nk_μ

$$\exp(iS(x)) = \sum_{n=-\infty}^{\infty} F(n, nk_\mu)$$

The Volkov solution for a constant crossed field

- For a **constant crossed field** an interpretation in terms of external field photons is not required

$$S(x) = -[(px) + \frac{e(a_1 p)}{2(kp)}(kx)^2 - \frac{e^2 a^2}{6(kp)}(kx)^3]$$

The Volkov solution for a constant crossed field

- For a **constant crossed field** an interpretation in terms of external field photons is not required

$$S(x) = -[(px) + \frac{e(a_1 p)}{2(kp)}(kx)^2 - \frac{e^2 a^2}{6(kp)}(kx)^3]$$

- We could introduce the dressed momentum but it doesn't emerge 'naturally'

The Volkov solution for a constant crossed field

- For a **constant crossed field** an interpretation in terms of external field photons is not required

$$S(x) = -[(px) + \frac{e(a_1 p)}{2(kp)}(kx)^2 - \frac{e^2a^2}{6(kp)}(kx)^3]$$

- We could introduce the dressed momentum but it doesn't emerge 'naturally'
- Any mass shift $p^2 = m^2 + e^2a^2$ woud be problematic since $e^2a^2 \to \infty$

The Volkov solution for a constant crossed field

- For a **constant crossed field** an interpretation in terms of external field photons is not required

$$S(x) = -[(px) + \frac{e(a_1 p)}{2(kp)}(kx)^2 - \frac{e^2 a^2}{6(kp)}(kx)^3]$$

- We could introduce the dressed momentum but it doesn't emerge 'naturally'
- Any mass shift $p^2 = m^2 + e^2 a^2$ woud be problematic since $e^2 a^2 \to \infty$
- A Fourier transform gives an integration over external field energy rather than a sum of contributions

$$\exp(iS(x)) = \int dr F(r) \exp(-ir(kx))$$

Fermion propagator in the external potential

- The Volkov solution can be written as a product of Volkov E functions and the bispinor

$$\Psi_p^V(x) \equiv E_p(x)u(p)$$

Fermion propagator in the external potential

- The Volkov solution can be written as a product of Volkov E functions and the bispinor

$$\Psi_p^V(x) \equiv E_p(x)u(p)$$

- The Volkov E function can be shown to have the properties of orthogonality and completeness (Ritus Ann Phys 69 555-582 (1970), Bergou and Varro, J Phys A 13, 2823)

$$\int d^4x \overline{E}_{p_f}(x) E_{p_i}(x) = \delta(p_f - p_i)$$
$$\int d^4x E_p(x_1) \overline{E}_p(x_2) = \delta(x_1 - x_2)$$

Fermion propagator in the external potential

- The Volkov solution can be written as a product of Volkov E functions and the bispinor

$$\Psi_p^V(x) \equiv E_p(x)u(p)$$

- The Volkov E function can be shown to have the properties of orthogonality and completeness (Ritus Ann Phys 69 555-582 (1970), Bergou and Varro, J Phys A 13, 2823)

$$\int d^4x \bar{E}_{p_f}(x) E_{p_i}(x) = \delta(p_f - p_i)$$
$$\int d^4x E_p(x_1) \bar{E}_p(x_2) = \delta(x_1 - x_2)$$

- using these properties the fermion propagator in an external field can be written as the usual fermion propagator sandwiched between Volkov E functions

$$G(x_2, x_1) = \int d^4p E_p(x_2) \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \bar{E}_p(x_1)$$

Modified Feynman Rules I

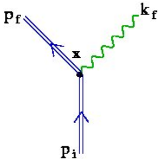

Figure: The 1st order vertex with an external field.

Modified Feynman Rules I

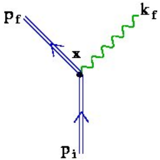

Figure: The 1st order vertex with an external field.

- Photon lines are unaffected, fermion lines represent the Volkov solutions
- Fermion bispinors are unchanged, so spin sums as usual

Modified Feynman Rules I

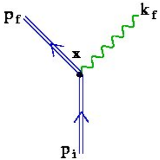

Figure: The 1st order vertex with an external field.

- Photon lines are unaffected, fermion lines represent the Volkov solutions
- Fermion bispinors are unchanged, so spin sums as usual
- Can treat the Volkov E functions as adjacent to the vertex
- So only a modified vertex is necessary

$$\Gamma^e_\mu = \int d^4x \bar{E}_{p_f}(x)\gamma_\mu E_{p_i}(x) \exp(-i(k_f x))$$

Modified Feynman Rules I

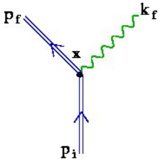

Figure: The 1st order vertex with an external field.

- Photon lines are unaffected, fermion lines represent the Volkov solutions
- Fermion bispinors are unchanged, so spin sums as usual
- Can treat the Volkov E functions as adjacent to the vertex
- So only a modified vertex is necessary

$$\Gamma^e_\mu = \int d^4x \bar{E}_{p_f}(x)\gamma_\mu E_{p_i}(x)\exp(-i(k_f x))$$

- Substitute the Volkov E functions

$$(k.x)^n \exp(iS(x))\ ;\ n = 0, 1, 2$$

$$S(x) = -\int_0^{(kx)} \left[\frac{e(Ap)}{(kp)} - \frac{e^2A^2}{2(kp)}\right] d\phi$$

Modified Feynman Rules II

- To simplify the x dependence take the Fourier Transform

Modified Feynman Rules II

- To simplify the x dependence take the Fourier Transform
- lets assume a constant crossed external field, so it will be a continuous Fourier Transform

$$(kx)^n \exp(iS(x)) = \int dr\ F_n(r) \exp(-ir(kx))$$

$$F_n(r) = \int_{-\infty}^{\infty} dt\ t^n \exp(irt + iS(t))$$

- Lets call the functions F, auxillary functions

Modified Feynman Rules II

- To simplify the x dependence take the Fourier Transform
- lets assume a constant crossed external field, so it will be a continuous Fourier Transform

$$(kx)^n \exp(iS(x)) = \int dr\ F_n(r) \exp(-ir(kx))$$

$$F_n(r) = \int_{-\infty}^{\infty} dt\ t^n \exp(irt + iS(t))$$

- Lets call the functions F, auxillary functions
- Our modified vertex in momentum space is then

$$\Gamma_\mu = (2\pi)^4 \int dr E_p(r) \gamma_\mu E_{p_i}(r) \delta^4(p_f + k_f - p_i - rk)$$

Explicit modified vertex

- When squaring a matrix function, we need to simplify products of F functions
- The F function are explicitly

$$F_n(r) = \int_{-\infty}^{\infty} dt \exp(irt + i(aP)t^2 + i\tfrac{1}{3}Qt^3)$$
$$P^\mu = \frac{e}{2}\left(\frac{p_f^\mu}{(kp_f)} - \frac{p_i^\mu}{(kp_i)}\right) \ ; \ Q = \frac{e^2a^2}{2}\frac{(kkf)}{(kp_i)(kp_f)}$$

Explicit modified vertex

- When squaring a matrix function, we need to simplify products of F functions
- The F function are explicitly

$$F_n(r) = \int_{-\infty}^{\infty} dt \exp(irt + i(aP)t^2 + i\tfrac{1}{3}Qt^3)$$
$$P^\mu = \frac{e}{2}\left(\frac{p_f^\mu}{(kp_f)} - \frac{p_i^\mu}{(kp_i)}\right) \; ; \; Q = \frac{e^2a^2}{2}\frac{(kkf)}{(kp_i)(kp_f)}$$

- After suitable change of variables we eliminate the t^2 term and the coefficient Q in the t^3 and the result is

$$F_0(r) = Q^{-\frac{1}{3}}\text{Ai}(z)\exp(-irQ^{-1}(aP))e^{-ir\frac{(aP)}{Q}}$$
$$F_1(r) = Q^{-\frac{2}{3}}[Q^{-\frac{2}{3}}(aP)\text{Ai}(z) - \text{i Ai'}(z)]e^{-ir\frac{(aP)}{Q}}$$
$$F_2(r) = Q^{-\frac{4}{3}}[(2Q^{-1}(aP)^2 - r)\text{Ai}(z) + \text{i } 2Q^{-\frac{1}{3}}(aP)\text{Ai'}(z)]e^{-ir\frac{(aP)}{Q}}$$

where $z = Q^{-\frac{1}{3}}(r - Q^{-1}(aP))$

Explicit modified vertex

- When squaring a matrix function, we need to simplify products of F functions
- The F function are explicitly

$$F_n(r) = \int_{-\infty}^{\infty} dt \exp(irt + i(aP)t^2 + i\tfrac{1}{3}Qt^3)$$
$$P^\mu = \tfrac{e}{2}\left(\frac{p_f^\mu}{(kp_f)} - \frac{p_i^\mu}{(kp_i)}\right) \; ; \; Q = \tfrac{e^2a^2}{2}\frac{(kkf)}{(kp_i)(kp_f)}$$

- After suitable change of variables we eliminate the t^2 term and the coefficient Q in the t^3 and the result is

$$F_0(r) = Q^{-\frac{1}{3}}\mathrm{Ai}(z)\exp(-irQ^{-1}(aP))e^{-ir\frac{(aP)}{Q}}$$
$$F_1(r) = Q^{-\frac{2}{3}}[Q^{-\frac{2}{3}}(aP)\mathrm{Ai}(z) - \mathrm{i}\ \mathrm{Ai'}(z)]e^{-ir\frac{(aP)}{Q}}$$
$$F_2(r) = Q^{-\frac{4}{3}}[(2Q^{-1}(aP)^2 - r)\mathrm{Ai}(z) + \mathrm{i}\ 2Q^{-\frac{1}{3}}(aP)\mathrm{Ai'}(z)]e^{-ir\frac{(aP)}{Q}}$$
where $z = Q^{-\frac{1}{3}}(r - Q^{-1}(aP))$

- Finally the modified vertex in a constant crossed field is

$$\Gamma^e_\mu = (2\pi)^4 \int dr \left[\gamma_\mu F_0(r) + \tfrac{e}{2}\left(\frac{\not{a}\not{k}\gamma_\mu}{(kp_f)} - \frac{\gamma_\mu\not{a}\not{k}}{(kp_i)}\right)F_1(r) + \frac{e^2a^2\not{k}\gamma_\mu\not{k}}{4(kp_i)(kp_f)}F_2(r)\right]$$

Beamstrahlung Transition Rate I

- The square of the matrix element and spin sums proceed as usual, resulting in a trace over gamma matrices
- Transition rate includes two integrations r, r' over contributions from the external field

$$dW = \frac{1}{(2\pi)^2} \frac{1}{8\epsilon_i \epsilon_f \omega_f} \sum_{if} |\langle p_f k_f | iM | p_i \rangle|^2 \, dr dr' \, d^3\vec{p}_f d^3\vec{k}_f$$

$$.\delta(p_i + rk - p_f - k_f)\delta(p_i + r'k - p_f - k_f)$$

Beamstrahlung Transition Rate I

- The square of the matrix element and spin sums proceed as usual, resulting in a trace over gamma matrices
- Transition rate includes two integrations r, r' over contributions from the external field

$$dW = \frac{1}{(2\pi)^2}\frac{1}{8\epsilon_i\epsilon_f\omega_f}\sum_{if}|\langle p_f k_f|iM|p_i\rangle|^2\, dr dr'\, d^3\vec{p}_f d^3\vec{k}_f$$

$$.\delta(p_i + rk - p_f - k_f)\delta(p_i + r'k - p_f - k_f)$$

- The phase integral part and the integrations over r, r' can be performed at the same time

Beamstrahlung Transition Rate I

- The square of the matrix element and spin sums proceed as usual, resulting in a trace over gamma matrices
- Transition rate includes two integrations r, r' over contributions from the external field

$$dW = \frac{1}{(2\pi)^2} \frac{1}{8\epsilon_i \epsilon_f \omega_f} \sum_{if} |\langle p_f k_f | iM | p_i \rangle|^2 \; dr dr' \; d^3\vec{p}_f d^3\vec{k}_f$$

$$.\delta(p_i + rk - p_f - k_f)\delta(p_i + r'k - p_f - k_f)$$

- The phase integral part and the integrations over r, r' can be performed at the same time
- Use an identity to introduce another delta function

$$\int \frac{1}{2\epsilon_f} \; d^3\vec{p_f} \equiv \int \delta(p_f^2 - m^2) \; d^4 p_f$$

Beamstrahlung Transition Rate I

- The square of the matrix element and spin sums proceed as usual, resulting in a trace over gamma matrices
- Transition rate includes two integrations r, r' over contributions from the external field

$$dW = \frac{1}{(2\pi)^2} \frac{1}{8\epsilon_i \epsilon_f \omega_f} \sum_{if} |\langle p_f k_f | iM | p_i \rangle|^2 \; dr dr' \; d^3\vec{p}_f d^3\vec{k}_f$$

$$.\delta(p_i + rk - p_f - k_f)\delta(p_i + r'k - p_f - k_f)$$

- The phase integral part and the integrations over r, r' can be performed at the same time
- Use an identity to introduce another delta function

$$\int \frac{1}{2\epsilon_f} \, d^3\vec{p_f} \equiv \int \delta(p_f^2 - m^2) \; d^4 p_f$$

- Integrations over p_f then r give $r \to \frac{(p_i k_f)}{(k p_f)}$

Beamstrahlung Transition Rate I

- The square of the matrix element and spin sums proceed as usual, resulting in a trace over gamma matrices
- Transition rate includes two integrations r, r' over contributions from the external field

$$dW = \frac{1}{(2\pi)^2} \frac{1}{8\epsilon_i \epsilon_f \omega_f} \sum_{if} |\langle p_f k_f | iM | p_i \rangle|^2 \, dr dr' \, d^3\vec{p}_f d^3\vec{k}_f$$

$$.\delta(p_i + rk - p_f - k_f)\delta(p_i + r'k - p_f - k_f)$$

- The phase integral part and the integrations over r, r' can be performed at the same time
- Use an identity to introduce another delta function

$$\int \frac{1}{2\epsilon_f} \, d^3\vec{p_f} \equiv \int \delta(p_f^2 - m^2) \, d^4 p_f$$

- Integrations over p_f then r give $r \rightarrow \frac{(p_i k_f)}{(k p_f)}$
- We are left with integrations $d\vec{k}_f$ and r'

Beamstrahlung Transition Rate II

- We have as the beamstrahlung transition rate
  $dW \propto Q^{-2/3} \frac{d\vec{k}_f}{\omega_f (kp_f)} [\mathrm{Ai}^2(z) - e^2 a^2 Q^{-2/3} (2 + \frac{u^2}{1+u})$
  $.(z\mathrm{Ai}^2(z) + \mathrm{Ai'}^2(z))] e^{-ir' f(k_f^x)}$
  $u = \frac{(kk_f)}{(kp_f)} \equiv g(k_f^z) \; ; \; z = h(k_f^y)$
- Solution Cartesian coordinates gives the simplest expressions

Beamstrahlung Transition Rate II

- We have as the beamstrahlung transition rate
$$dW \propto Q^{-2/3}\frac{d\vec{k}_f}{\omega_f(kp_f)}[\mathrm{Ai}^2(z) - e^2a^2Q^{-2/3}(2+\frac{u^2}{1+u})$$
$$.(z\mathrm{Ai}^2(z) + \mathrm{Ai'}^2(z))]e^{-ir'f(k_f^x)}$$
$$u = \frac{(kk_f)}{(kp_f)} \equiv g(k_f^z)\ ;\ z = h(k_f^y)$$
- Solution Cartesian coordinates gives the simplest expressions
- Integration over k_f^x gives a delta function with respect to r'
- Consequently, integration of r' can be performed
- Integration over k_f^y reduces products of Airy functions to a single Airy function using

$$\int \frac{dt}{\sqrt{t}}\mathrm{Ai}^2(t+a) = \frac{1}{2}\int_{2^{2/3}a}^{\infty}\mathrm{Ai}(y)dy$$

Beamstrahlung Transition Rate II

- We have as the beamstrahlung transition rate
  $dW \propto Q^{-2/3} \frac{d\vec{k}_f}{\omega_f(kp_f)}[\mathrm{Ai}^2(z) - e^2a^2Q^{-2/3}(2 + \frac{u^2}{1+u})$
  $.(z\mathrm{Ai}^2(z) + \mathrm{Ai'}^2(z))]e^{-ir'f(k_f^x)}$
  $u = \frac{(kk_f)}{(kp_f)} \equiv g(k_f^z) \; ; \; z = h(k_f^y)$
- Solution Cartesian coordinates gives the simplest expressions
- Integration over k_f^x gives a delta function with respect to r'
- Consequently, integration of r' can be performed
- Integration over k_f^y reduces products of Airy functions to a single Airy function using

$$\int \frac{dt}{\sqrt{t}} \mathrm{Ai}^2(t+a) = \frac{1}{2} \int_{2^{2/3}a}^{\infty} \mathrm{Ai}(y)dy$$

- Shift dk_z shifted to du

$$W = \frac{\alpha m^2}{\pi\sqrt{3}\epsilon_i} \int_0^{\infty} \frac{du}{(1+u)^2} \left[\int_x (u)^{\infty} K_{5/3}(y)dy - \frac{u^2}{1+u} K_{2/3}(x(u)) \right]$$

The Quantum Beamstrahlung expression

I want to compare finally the beamstrahlung Transition Rate obtained using the Nikishov-Ritus and Operator methods

The Quantum Beamstrahlung expression

I want to compare finally the beamstrahlung Transition Rate obtained using the Nikishov-Ritus and Operator methods

$$dW \propto \frac{du}{(1+u)^2}\left[\int_x^\infty K_{5/3}(y)dy - \frac{u^2}{1+u}K_{2/3}(x)\right]$$

$$x(NR) = \frac{2}{3ea(kp_i)}\frac{\omega_f}{\epsilon_i - \omega_f}$$

$$x(Op) = \frac{2}{3ea(kp_i)}\frac{(kk_f)}{(kpi)-(kkf)}$$

The Quantum Beamstrahlung expression

I want to compare finally the beamstrahlung Transition Rate obtained using the Nikishov-Ritus and Operator methods

$$dW \propto \frac{du}{(1+u)^2}\left[\int_x^\infty K_{5/3}(y)dy - \frac{u^2}{1+u}K_{2/3}(x)\right]$$

$$x(NR) = \frac{2}{3ea(kp_i)}\frac{\omega_f}{\epsilon_i - \omega_f}$$

$$x(Op) = \frac{2}{3ea(kp_i)}\frac{(kk_f)}{(kpi)-(kkf)}$$

- The Transition Rates (W) agree since the integration variable is just a variable from 0 to ∞ in any case
- The Differential Transition Rates (dW) however are not the same
- In the limit of ultra-relativistic fermion, the radiation angle is very small

$$\frac{(kk_f)}{(kpi)-(kkf)} \rightarrow \frac{\omega_f}{\epsilon_i - \omega_f}$$

Summary

- Work in the Furry picture
- Use Volkov solutions for fermions in an external field
- Write the propagator in terms of Volkov E functions

Summary

- Work in the Furry picture
- Use Volkov solutions for fermions in an external field
- Write the propagator in terms of Volkov E functions
- Use usual Feynman rules with a modified vertex

Summary

- Work in the Furry picture
- Use Volkov solutions for fermions in an external field
- Write the propagator in terms of Volkov E functions
- Use usual Feynman rules with a modified vertex
- Take Fourier Transform to simplify the dependence on space-time variables
- For constant crossed field, no mass shift, dressed momentum
  - no mass shift
  - no dressed momentum
  - no interpretation in terms of external field photons

Summary

- Work in the Furry picture
- Use Volkov solutions for fermions in an external field
- Write the propagator in terms of Volkov E functions
- Use usual Feynman rules with a modified vertex
- Take Fourier Transform to simplify the dependence on space-time variables
- For constant crossed field, no mass shift, dressed momentum
  - no mass shift
  - no dressed momentum
  - no interpretation in terms of external field photons
- Method applied to Beamstrahlung process
- I want the most effecient simplification to apply to higher orders

Summary

- Work in the Furry picture
- Use Volkov solutions for fermions in an external field
- Write the propagator in terms of Volkov E functions
- Use usual Feynman rules with a modified vertex
- Take Fourier Transform to simplify the dependence on space-time variables
- For constant crossed field, no mass shift, dressed momentum
    - no mass shift
    - no dressed momentum
    - no interpretation in terms of external field photons
- Method applied to Beamstrahlung process
- I want the most effecient simplification to apply to higher orders
- Expressions simplified to functions of single Airy functions
- Comparison of Transition Rates between Operator and Nikishov-Ritus show agreement for ultra-relativistic fermions