

# Massive particles & unitarity cuts

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In collaboration with R. Britto

# Outline

- Introduction
- On-Shell bubbles
- Single cut & Tadpoles
- Conclusions

# Introduction

$$\mathcal{A}_{1\text{-loop}} = \sum_i \text{[diagram: square loop]} d_i + \sum_j \text{[diagram: triangle loop]} c_j$$

$$\sum_k \text{[diagram: bubble loop]} b_k + \sum_m \text{[diagram: tadpole loop]} a_m + \mathcal{R}$$

The task: Computation of the coefficients & of the rational part

- Passarino-Veltman reduction, [Passarino, Veltman '79 ...]
- Singular limits of  $\mathcal{A}_{1\text{-loop}}$  + on-shell formalism, [Bern, Dixon, Dunbar, Kosower '94,'95 ...]
  - Not only double but also multiple cuts
  - Works great in the massless case ...
    - ↪  $PP \rightarrow Z/W + 4\text{-jets}$  computed [BLACKHAT; '11]  $\rightsquigarrow$  (see Z. Bern' s talk)
  - Few issues in the massive case
    - ↪ Bypassed in numerical methods [Ossola, Papadopoulos, Pittau; '07]  $\rightsquigarrow$  (see R. Pittau' s talk)  
[Mastrolia, et al.; '10]

# Introduction

$$\mathcal{A}_{1\text{-loop}} = \sum_i \text{[diagram]} d_i + \sum_j \text{[diagram]} c_j + \sum_l \text{[diagram]} \tilde{b}_l$$
$$\sum_k \text{[diagram]} b_k + \sum_m \text{[diagram]} a_m + \mathcal{R}$$

The equation shows the decomposition of the one-loop amplitude  $\mathcal{A}_{1\text{-loop}}$  into several terms. The first row contains three terms: a sum over  $d_i$  with a square loop diagram, a sum over  $c_j$  with a triangle loop diagram, and a sum over  $\tilde{b}_l$  with a red bubble diagram. The second row contains two terms: a sum over  $b_k$  with a white bubble diagram and a sum over  $a_m$  with a blue tadpole diagram, followed by a remainder term  $\mathcal{R}$ .

The issues are related to **tadpoles** and **on-shell bubbles**

- Tree-level amplitudes become singular
- Double and single cut diverge,
- Gauge dependence arises, [Ellis *et al.* '08]
- Analytic computation of  $a_m$  tricky (no physical channel).

# Introduction

$$\mathcal{A}_{1\text{-loop}} = \sum_i \text{[diagram: box with 4 external lines]} d_i + \sum_j \text{[diagram: triangle with 3 external lines]} c_j + \sum_l \text{[diagram: bubble with 2 external lines]} \tilde{b}_l$$
$$\sum_k \text{[diagram: bubble with 4 external lines]} b_k + \sum_m \text{[diagram: tadpole with 1 external line]} a_m + \mathcal{R}$$

The issues are related to **tadpoles** and **on-shell bubbles**

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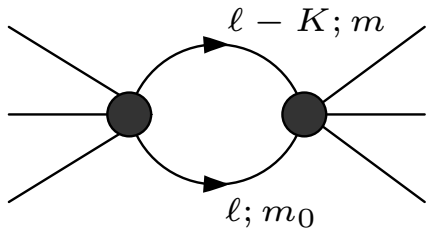
*Let's have a look at these issues!*

$$\mathcal{A}_{1\text{-loop}} = \sum_i \text{[Diagram 1]} d_i + \sum_j \text{[Diagram 2]} c_j + \sum_l \text{[Diagram 3]} \tilde{b}_l \\
+ \sum_k \text{[Diagram 4]} b_k + \sum_m \text{[Diagram 5]} a_m + \mathcal{R}$$

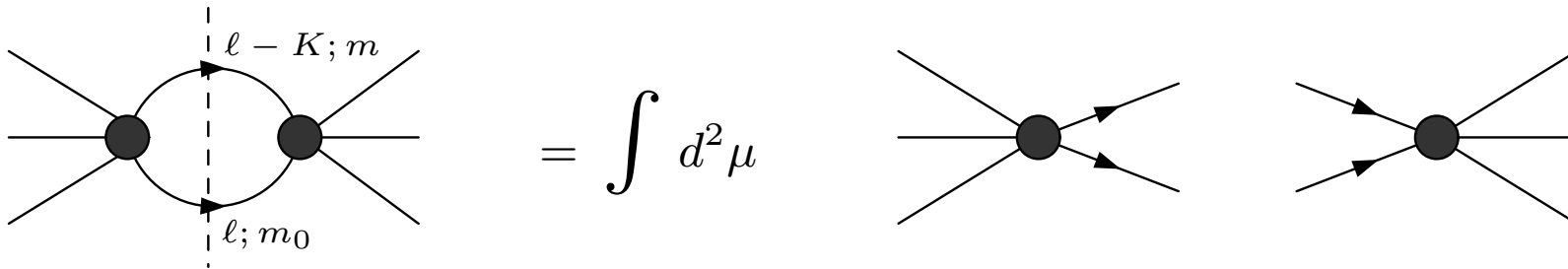
The equation defines the one-loop amplitude  $\mathcal{A}_{1\text{-loop}}$  as a sum of five types of diagrams, each multiplied by a coefficient and summed over an index. 
   
 1.  $\sum_i d_i$ : A square loop with four external lines (two on the left, two on the right).
   
 2.  $\sum_j c_j$ : A triangle loop with three external lines (one on the left, two on the right).
   
 3.  $\sum_l \tilde{b}_l$ : A bubble loop (two vertices connected by two arcs) with one external line on the left and two on the right. The diagram and coefficient are highlighted in red.
   
 4.  $\sum_k b_k$ : A bubble loop with two external lines on the left and two on the right.
   
 5.  $\sum_m a_m$ : A bubble loop with three external lines on the left and one on the right.
   
 6.  $\mathcal{R}$ : A remainder term.

*Coefficient of the on-shell bubbles*

# Double cut in a nutshell



# Double cut in a nutshell



$$[d^2 \mu \equiv d^4 \ell \delta(\ell^2 - m_0^2) \delta((\ell - K)^2 - m^2)]$$

● Integral performed via:

● Spinor integration  $d^2 \mu \rightsquigarrow \langle \lambda, d\lambda \rangle [\tilde{\lambda}, d\tilde{\lambda}]$  [Cachazo, Svrček, Witten; '04] [Britto *et al.*; '05]

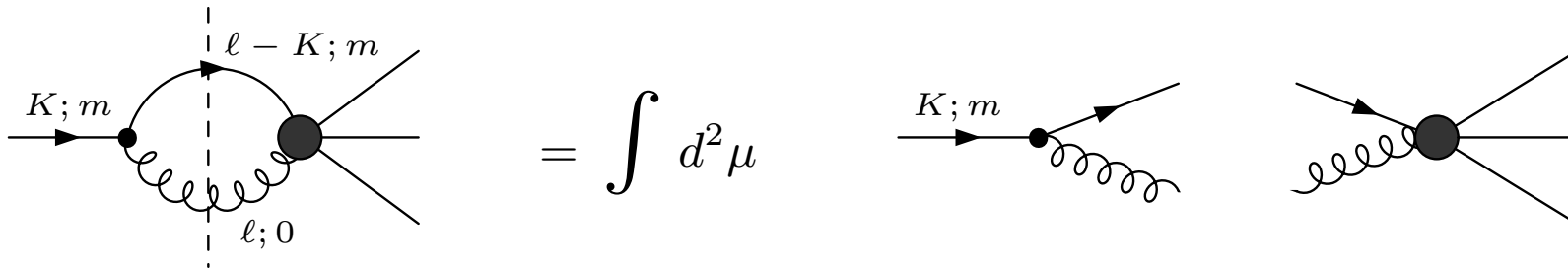
● Generalized Cauchy theorem  $d^2 \mu \rightsquigarrow dz d\bar{z}$  [Mastrolia; '09]

● Evaluate the  $B_0(K^2, m_0^2, m^2)$  coefficients [Britto *et al.*; '05] . . .

● . . . And triangles & boxes coefficients [Britto, Feng; '08] [Britto, Feng, Mastrolia; '08]



# Double cut & on-shell bubbles

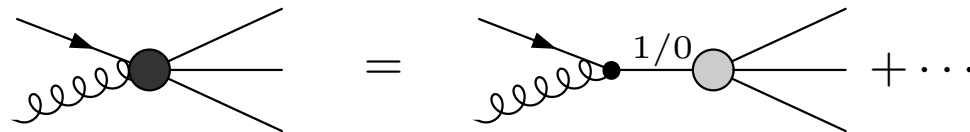


If massive external states are present:

- Double cut in their momentum enters ...

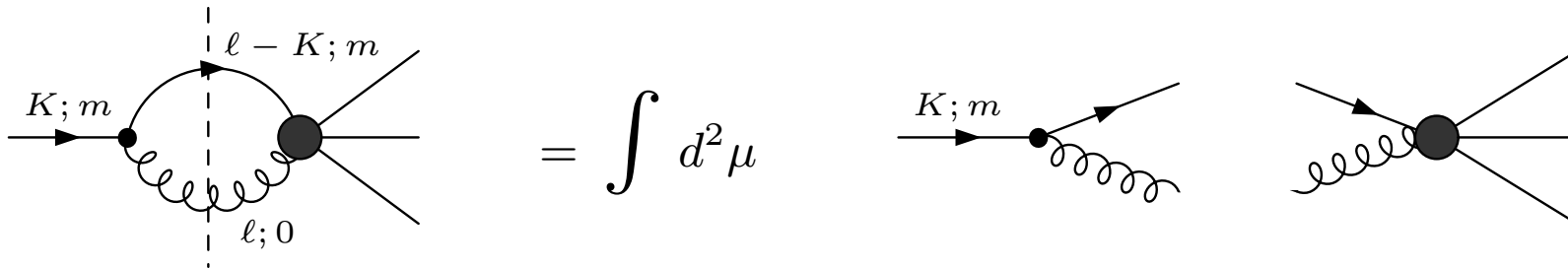
- ... And it is relevant for the coefficient of  $B_0(m^2, 0, m^2)$

- Divergent since



Problems in the 1-particle reducible diagrams (1PRD) correcting the external legs.

# Double cut & on-shell bubbles

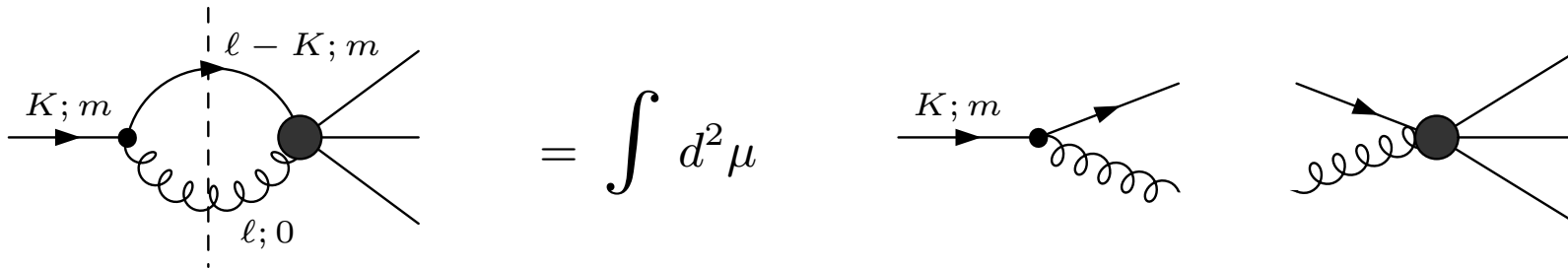


● Possible way out : [Ellis, Giele, Kunszt, Melnikov; '08]

THE IDEA:  $\left( \begin{array}{l} \text{1PRD are discarded} \\ \text{by the LSZ formula} \end{array} \right) \Rightarrow \left( \begin{array}{l} \text{discard them} \\ \text{from scratch} \end{array} \right)$

- 1-particle reducible contributions discarded
- Gauge dependence introduced
  - $\hookrightarrow$  Feynman gauge adopted
- Polarization sum modified
  - $\hookrightarrow \sum_{\lambda} \epsilon_{\lambda}^{\mu} \epsilon_{\lambda}^{*\nu} = -g^{\mu\nu}$

# Double cut & on-shell bubbles

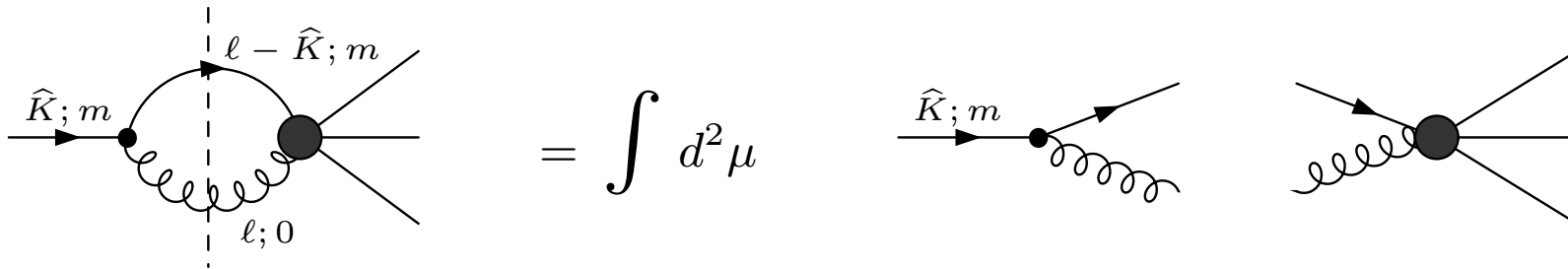


● Our method: [Britto, EM; '11]

THE IDEA:  $\left( \begin{array}{l} \text{1PRD enter the} \\ \text{double cut} \end{array} \right) \Rightarrow \left( \begin{array}{l} \text{keep them as far} \\ \text{as possible!} \end{array} \right)$

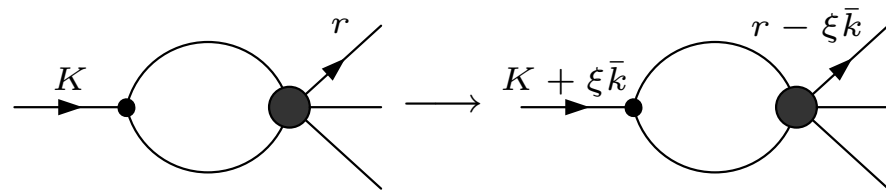
- Regularize the divergences
- Subtract 1PRD at the very end
  - ↪ Via the counterterm in the on-shell scheme

# Double cut & on-shell bubbles



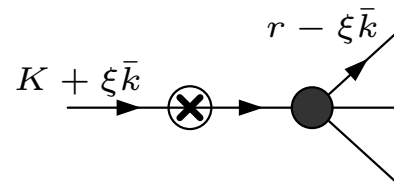
● Our method: [Britto, EM; '11]

- 1) Shift  $K \rightarrow \widehat{K}$ ,  $r \rightarrow \widehat{r}$ 
  - $\hookrightarrow$  By an amount  $\pm \xi \bar{k}$
  - $\hookrightarrow \widehat{r}$  is on shell,  $\widehat{K}$  not
  - $\hookrightarrow 1/0 \rightarrow 1/\xi$

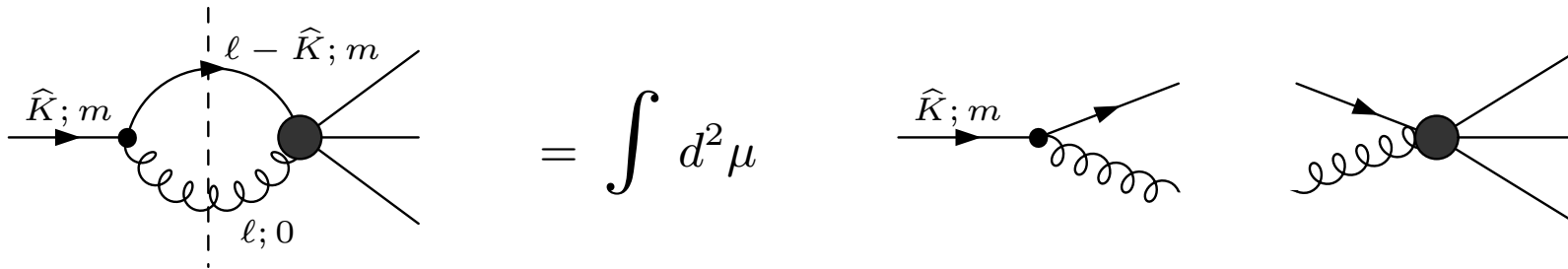


- 2) Cut in the  $\widehat{K}$  channel
  - $\hookrightarrow$  Get the  $B_0$  coefficient  $b$
- 3) Expand  $b \times B_0$  around  $\xi = 0$

- 4) Add the counterterm
  - $\hookrightarrow$  Expand around  $\xi = 0$
  - $\hookrightarrow$  Check  $1/\xi$  and  $\bar{k}$  cancellation

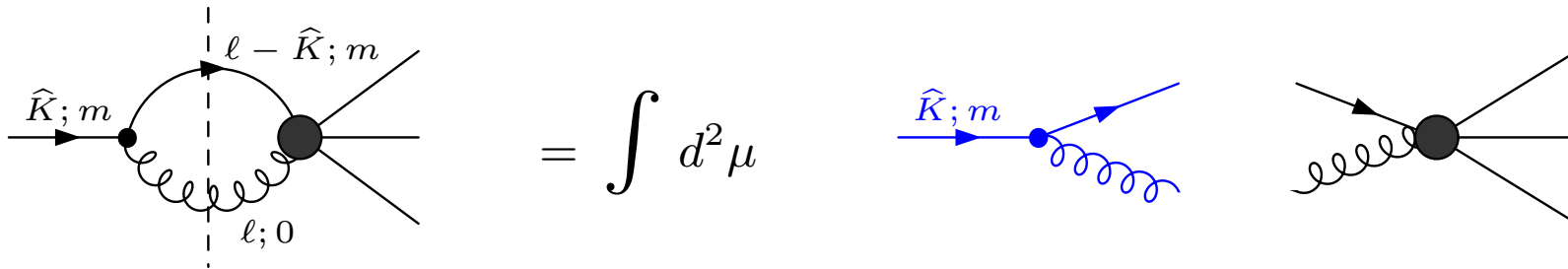


# Double cut & on-shell bubbles



CAVEAT: Gauge dependence arises!

# Double cut & on-shell bubbles



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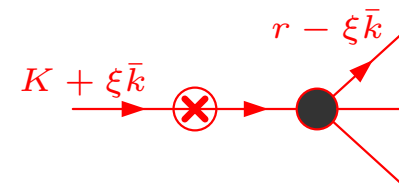
Usually  $\left\{ \begin{array}{l} \text{double cut} \\ \text{Ward identities} \end{array} \right. \Rightarrow$  gauge terms vanish (on-shellness crucial)

But: 3-point function is off shell ( $\hat{K}^2 \neq m^2$ )

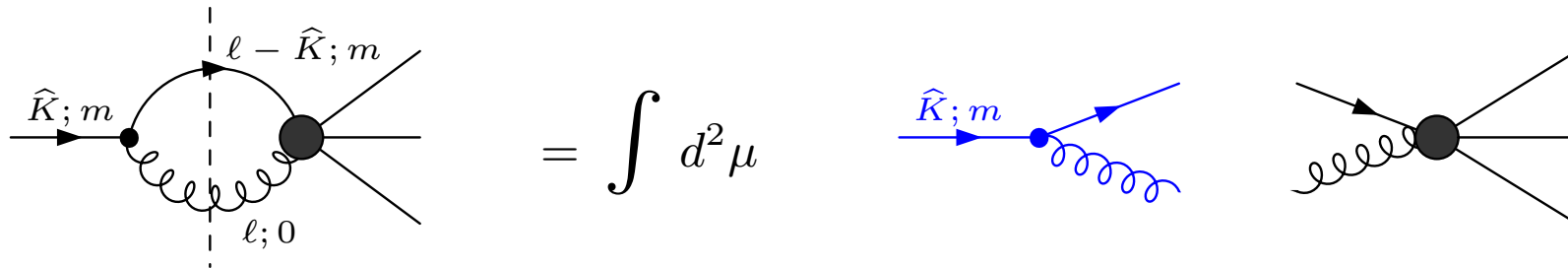
Then: residual  $q$ -dependence from  $\epsilon_{\pm}^{\mu} \dots$   
 $\hookrightarrow$  e.g.  $\epsilon_{-}^{\mu}(p) = \sqrt{2} (|p\rangle [q] + |q\rangle \langle p|) / [p, q]$

... Easy to handle

- The only term surviving the double cut
- Is cancelled modifying the **counterterm**
- Can be avoided using  $\sum_{\lambda} \epsilon_{\lambda}^{\mu} \epsilon_{\lambda}^{*\nu} = -g^{\mu\nu}$



# Double cut & on-shell bubbles



● To Summarize, our method:

- + Regularizes the divergence.
- + Uses the usual double cut.
- + Puts on shell the "complicated" tree-level ...
- ... But the "easy" one is off shell.
- Introduces a gauge dependence ...
- + ... Easy to handle!

It has been tested

- $H \rightarrow b\bar{b}$ ,  $q\bar{q} \rightarrow t\bar{t}$ ,  $gg \rightarrow t\bar{t}$ .

$$\mathcal{A}_{1\text{-loop}} = \sum_i \text{[Diagram 1]} d_i + \sum_j \text{[Diagram 2]} c_j + \sum_l \text{[Diagram 3]} \tilde{b}_l \\
+ \sum_k \text{[Diagram 4]} b_k + \sum_m \text{[Diagram 5]} a_m + \mathcal{R}$$

The equation shows the decomposition of the one-loop amplitude  $\mathcal{A}_{1\text{-loop}}$  into several terms. Each term consists of a sum over a set of diagrams multiplied by a coefficient. 
   
 -  $\sum_i d_i$ : A sum over diagrams of a square loop with four external lines (two on the left, two on the right).
   
 -  $\sum_j c_j$ : A sum over diagrams of a triangle loop with three external lines on the left and two on the right.
   
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 -  $\sum_m a_m$ : A sum over diagrams of a tadpole loop with three external lines on the left and one on the right.
   
 -  $\mathcal{R}$ : A remainder term.

*Coefficient of the tadpoles*



# Tadpole coefficients & unitarity – status

- Tadpole coefficients & unitarity:
  - Analytically under investigation . . .  
... But still merits (and needs ) further study.
  - Numerically it works fine [Ossola, Papadopoulos, Pittau; '07] [Mastrolia *et al.*; '10]

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## ● Methods available

- Kilgore' s proposal [Kilgore; '08]
  - ↪ Generalization of Forde' s method. [Forde; '07]
- Auxiliary propagator method [Britto, Feng; '09]
  - ↪ Combination of the OPP and of the double cut
- Spinor integration [Britto, EM; '10]
  - ↪ Extension of the double cut spinor integration

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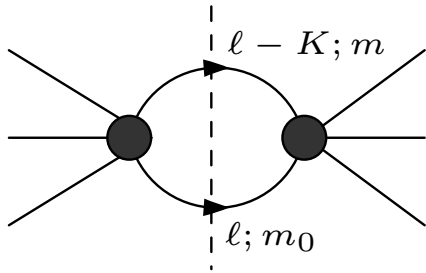
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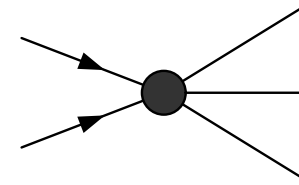
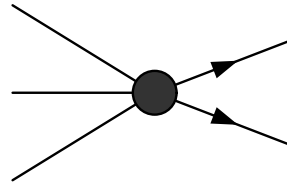
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*Let' s have a closer look to single cut spinor integration . . .*

# Double cut revisited



$$= \int d^2 \mu$$



## Double cut revisited

$$\Delta_2 [\mathcal{I}] \equiv \int d^4 \ell [\mathcal{I} D_0 D_1] \delta^+ (D_0) \delta^+ (D_1)$$

Definitions  $D_0 = \ell^2 - m_0^2$ ;  $D_1 = (\ell - K)^2 - m^2$ ;

# Double cut revisited

$$\Delta_2 [\mathcal{I}] = \int dz d\bar{z} \frac{[\mathcal{I} D_0 D_1]}{(1 + z\bar{z})^2}$$

Definitions  $D_0 = \ell^2 - m_0^2$ ;  $D_1 = (\ell - K)^2 - m^2$ ;

## 1) Reparametrizations

i)  $\ell^\mu = \tilde{\ell}^\mu + \xi K^\mu$  [Anastasiou *et al.* '07]

ii)  $\tilde{\ell}^\mu = t(p^\mu + \alpha q^\mu + z\epsilon_1^\mu + \bar{z}\epsilon_2^\mu)$  [Mastrolia '09]

# Double cut revisited

$$\Delta_2 [\mathcal{I}] = \lim_{\Lambda \rightarrow \infty} \left[ \int_0^{2\pi} d\beta \mathcal{F}(\Lambda, \beta) \right] - \pi \sum_{\text{poles } z_j} \text{Res}\{\mathcal{F}, z_j\}$$

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## 2) Generalized Cauchy formula

i)  $z = \Lambda e^{i\beta}$   $\bar{z} = \Lambda e^{-i\beta}$

ii)  $\int \mathcal{F} \rightarrow 0$ , only  $\text{Res}\{\mathcal{F}\}$  matters

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*What about single cuts?*



# Single cut integration

$$\Delta_1 [\mathcal{I}] \equiv \int d^4 \ell [\mathcal{I} D_0] \delta^+ (D_0)$$

Definitions  $D_0 = \ell^2 - m_0^2$ ;

# Single cut integration

$$\Delta_1 [\mathcal{I}] = \int dt \left\{ \lim_{\Lambda \rightarrow \infty} \int_0^{2\pi} d\beta \mathcal{G}(\Lambda, \beta) - \pi \sum_{\text{poles } z_j} \text{Res}\{\mathcal{G}, z_j\} \right\}$$

Definitions  $D_0 = \ell^2 - m_0^2$ ;

## 1) Reparametrizations

- i)  $\ell^\mu = \tilde{\ell}^\mu + \xi K^\mu$  ( $K$  arbitrary)
- ii)  $\tilde{\ell}^\mu = t(p^\mu + \alpha q^\mu + z\epsilon_1^\mu + \bar{z}\epsilon_2^\mu)$
- iii) Let  $K^2 \rightarrow \infty$

## 2) Generalized Cauchy formula

- i)  $z = \Lambda e^{i\beta}$      $\bar{z} = \Lambda e^{-i\beta}$
- ii) Both  $\int \mathcal{G}$  and  $\text{Res}\{\mathcal{G}\}$  enter

# Single cut integration

$$\Delta_1 \left[ \frac{1}{D_0} \right] = \lim_{\Lambda \rightarrow \infty} \frac{\pi K^2}{2} \Lambda^2 \left( \int dt t \right) + \mathcal{O}(\Lambda)$$

Definitions  $D_0 = \ell^2 - m_0^2$ ;

Tadpole integral  $\int d^4 \ell \frac{1}{D_0} \longrightarrow \mathcal{I} = \frac{1}{D_0}$

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- Single cuts of a tadpole  $\sim \Lambda^2$
- Bubble, triangle, box, behave differently  
e.g. for a bubble  $\Delta_1[1/(D_0 D_1)] \sim \log(\Lambda^2)$
- STRATEGY: Find the tadpole coefficient of  $\int \mathcal{I}$  selecting the  $\Lambda^2 t$ -terms of  $\Delta_1[\mathcal{I}]$
- CAVEAT: spurious terms enter! i.e.  $\int \mathcal{I} = 0 \not\Rightarrow \Delta_1[\mathcal{I}] = 0$

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*The simplest example could be helpful*

# Single cut integration

$$\mathcal{I} = \frac{(2R_1 \cdot \ell)}{D_0 D_1},$$

Definitions  $D_0 = \ell^2 - m_0^2$ ;  $D_1 = (\ell - K_1)^2 - m_0^2$ ;

We want  $a(0)$ , the coefficient of  $A_0(m_0^2)$ .

1) OPP decomposition ( $K_1 \perp$  to  $n, \ell_7, \ell_8$ )

$$\mathcal{I}_{2,1} = \frac{1}{D_0} a(0) + \quad (\text{terms in } n, \ell_7, \ell_8)$$

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- 1) OPP decomposition ( $K_1 \perp$  to  $n, \ell_7, \ell_8$ )
- 2) Take the  $\Lambda^2 t$  part of the cut (recall the strategy!)

$$\Delta_1 [\mathcal{I}_{2,1}] = \Delta_1 \left[ \frac{1}{D_0} \right] a(0) + \Delta_1 \left[ (\text{terms in } n, \ell_7, \ell_8) \right]$$

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- 2) Take the  $\Lambda^2 t$  part of the cut (recall the strategy!)
- 3) Compute the single cut

$$0 = \frac{q^\mu}{(q \cdot K_1)} \left[ K_{1\mu} a(0) + R_{1\mu} + (\text{terms } \perp \text{ to } K_1^\mu) \right]$$



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- 3) Compute the single cut
- 4) Decompose  $R_1^\mu = \alpha_1 K_1^\mu + \alpha_n n^\mu + \alpha_7 \ell_7^\mu + \alpha_8 \ell_8^\mu$

$$0 = \frac{q^\mu}{(q \cdot K_1)} \left[ K_{1\mu} (a(0) + \alpha_1) \quad + \quad (\text{terms } \perp \text{ to } K_1^\mu) \right]$$

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$$0 = \frac{q^\mu}{(q \cdot K_1)} \left[ K_{1\mu} (a(0) + \alpha_1) + (\text{terms } \perp \text{ to } K_1^\mu) \right]$$

$$a(0) = -\alpha_1 = -\frac{2K_1 \cdot R_1}{K_1^2}$$

# Single cut integration

- To Summarize, the single spinor integration:
  - + Allows the computation of the tadpole coefficients . . .
  - + . . . And of the bubbles as well.
  - + Works when the Gram determinant vanishes.
  - Involves spurious terms.
  - Is not easy to automate.

# Conclusions & Outlook

## CONCLUSIONS

- Unitarity with massive particles is tricky
- Our method for the on-shell bubbles coefficients
  - Combines nicely unitarity & LSZ formula
  - Treats easily gauge dependence
- Single cut spinor integration
  - can compute tadpole coefficients (and not only! )
  - Requires to deal with the spurious terms
  - Ok when the Gram determinan vanishes

## OUTLOOK

- Check the efficiency of our method for the on-shell bubbles
- Further study of the tadpoles via unitarity techniques