Resummation of large IR logarithms for the Thrust distribution

Pier Francesco Monni

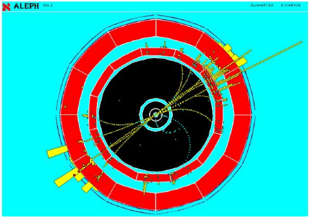
In collaboration with T. Gehrmann and G. Luisoni

Institut für Theoretische Physik

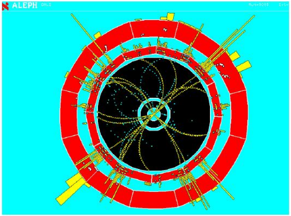
Event shapes are *IRC* safe observables which describe the topology of hadronic final states. They enjoy two peculiar features:

- *Intuitive physical picture...*
    - *e.g.* Thrust $T = \max_{\{\vec{n}_T\}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}_T|}{\sum_i |\vec{p}_i|}$

$T \simeq 1$: two jet event



$T \simeq \frac{1}{2}$: spherical event



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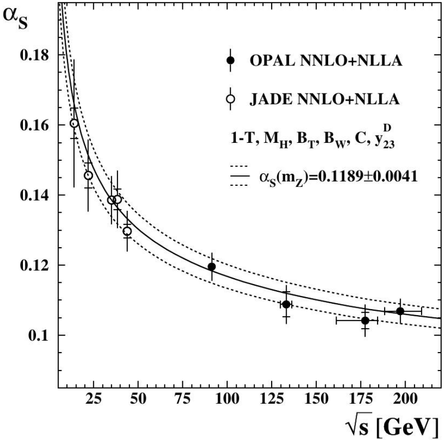
- *Intuitive physical picture...*
  - Thrust $T = \max_{\{\vec{n}_T\}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}_T|}{\sum_i |\vec{p}_i|}$
  - Heavy Jet Mass $\rho = \frac{1}{Q^2} \max(M_L^2, M_R^2)$, $\qquad M_{L/R}^2 = (\sum_i p_1)^2$
  - Broadenings $B_{L/R} = \frac{1}{2} \sum_i |\vec{p}_i \times \vec{n}_T|$
    - total $B_T = B_L + B_R$
    - wide $B_W = \max(B_L, B_R)$
  - C parameter $C = \frac{3}{2} \frac{\sum_{i,j} \left(|\vec{p}_i||\vec{p}_j| - \frac{(\vec{p}_i \cdot \vec{p}_j)^2}{|\vec{p}_i||\vec{p}_j|} \right)}{(\sum_i |\vec{p}_i|)^2}$
  - Jet resolution parameters (*e.g.* Durham/Cambridge y_3,...)
  - ...

Event shapes are *IRC* safe observables which describe the geometry of hadronic final states. They enjoy two peculiar features:

- *Intuitive physical picture...*
- *High sensitivity to QCD properties...*

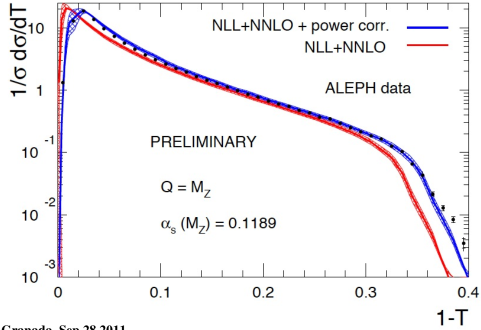
- Precise measurement of the strong coupling α_s [OPAL collaboration '11]



Event shapes are *IRC* safe observables which describe the geometry of hadronic final states. They enjoy two peculiar features:

- *Intuitive physical picture...*
- *High sensitivity to QCD properties...*

- Precise measurement of the strong coupling α_s
- Analysis of hadronisation corrections ($\sim \frac{1}{Q}$, shape function, dispersive model...)



- Fixed order predictions for event shape distributions **diverge** in the dijet limit ($T \to 1$)...
  - We compute the probability of real emission constraining the event such that $\Theta(1 - T < \tau)$
  - Dijet limit $T \simeq 1 \to$ real gluon radiation forbidden
  - Imbalance between real (*constrained*) and virtual (*unaffected*) radiation leads to large logarithms in the distribution of the event shape
- The perturbative series is **poorly convergent** in the high T region due to the presence of terms $\alpha_s \log(1 - T) \simeq 1$
- For a physical prediction to be reliable we need to resum logarithmically enhanced terms

Resummation

$$R(y) = 1 + \bar{\alpha}_s \mathscr{A}(y) + \bar{\alpha}_s^2 \mathscr{B}(y) + \bar{\alpha}_s^3 \mathscr{C}(y) + \mathscr{O}(\bar{\alpha}_s^4)$$

$\bar{\alpha}_s \mathscr{A}(y)$	$\bar{\alpha}_s L$	$\bar{\alpha}_s L^2$				
$\bar{\alpha}_s^2 \mathscr{B}(y)$	$\bar{\alpha}_s^2 L$	$\bar{\alpha}_s^2 L^2$	$\bar{\alpha}_s^2 L^3$	$\bar{\alpha}_s^2 L^4$		
$\bar{\alpha}_s^3 \mathscr{C}(y)$	$\bar{\alpha}_s^3 L$	$\bar{\alpha}_s^3 L^2$	$\bar{\alpha}_s^3 L^3$	$\bar{\alpha}_s^3 L^4$	$\bar{\alpha}_s^3 L^5$	$\bar{\alpha}_s^3 L^6$

LL, NLL, NNLL, N^3LL, ...

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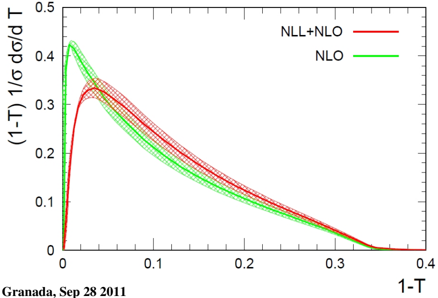
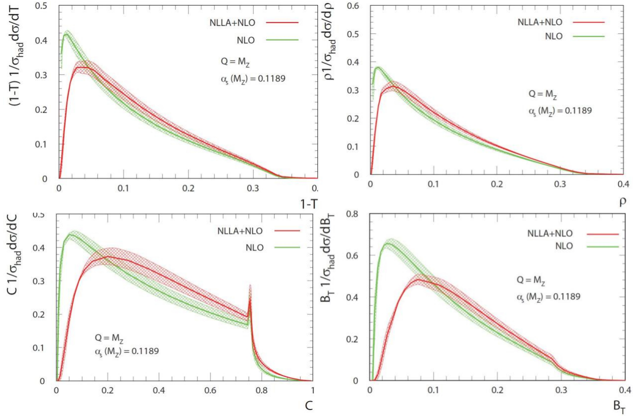
  - LL $\to \alpha^n L^{n+1}$
  - NLL $\to \alpha^n L^n$
  - $\mathrm{N^2LL} \to \alpha^n L^{n-1}$
  - ...



- CTTW approach [Catani, Trentadue, Turnock, Webber '93]
  - Based on parton branching algorithm [Catani, Marchesini, Webber '91]
  - Allows for resummation of "soft-collinear"/"hard-collinear" logarithms up to NLL
  - Not yet (fully) extended beyond NLL to include "large-angle" logarithms and hemispheres correlation...(single N^2LL application: *EEC* [De Florian, Grazzini '04])
  - Results for many standard shape observables :T, ρ, C, B_W, B_T [Catani et al.'93; Dokshitzer et al. '98]



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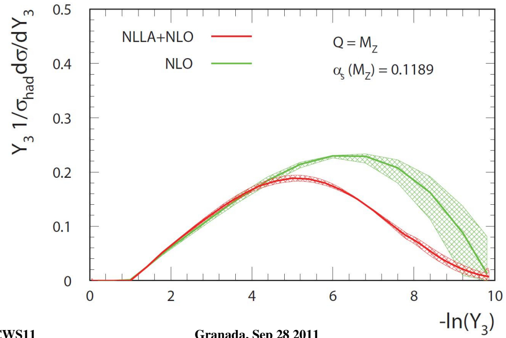
- Automated NLL resummation in momentum space : `Caesar` [Salam et al. '04]
  - automated treatment of a wide class of global observables (*e.g.* Durham y_3 jet resolution)



- More systematic approaches:
  - "USA" school [Sterman et al.; Korchemsky et al.]
    - factorisation at the Cross Section level
    - RG evolution and resummation in Laplace (Mellin) space
  - SCET school [Becher et al. ; Stewart et al.]
    - factorisation at the Lagrangian level
    - resummation in direct (momentum) space
- Several technical differences (SCET Feynman rules, collinear divergences regulators and soft-collinear subtraction, intermediate scales treatment,...)...but same framework
- They both allow for systematic resummation beyond NLL
  - Matching to NNLO [Gehrmann et al.; Weinzierl] led to
    - T : $N^{(3*)}LL+N^2LO$ (* Padé approximants for $\Gamma^{(3)}_{\text{cusp}}$, fit of the non-logarithmic soft structure) [Becher, Schwartz; Hoang et al. '08]
    - T : N^2LL+N^2LO (+ analytic computation of C_2 and $G_{3,1}$) [PFM, Gehrmann, Luisoni '11]
    - ρ : T : $N^{(3*)}LL+N^2LO$ (* Padé approximants for $\Gamma^{(3)}_{\text{cusp}}$, fit of the non-logarithmic soft structure) [Chien, Schwartz '10]
    - B_W, B_T : general factorization is now available [Becher, Neubert, Bell '11]

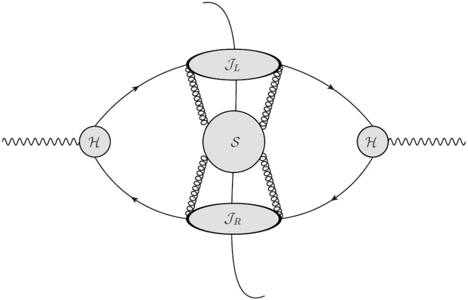
I: factorization of the observable in the dijet region (w/o recoil effects!!)

$$1-T \simeq \frac{\bar{k}^2}{Q^2}+\frac{k^2}{Q^2}+\frac{\bar{q}\cdot\bar{n}}{Q}+\frac{q\cdot n}{Q} \to \Theta(1-T<\tau) = \frac{1}{2\pi i}\int_C \frac{d\nu}{\nu} e^{\nu\tau} e^{-\frac{\bar{k}^2}{Q^2}\nu} e^{-\frac{k^2}{Q^2}\nu} e^{-\left(\frac{\bar{q}\cdot\bar{n}}{Q}+\frac{q\cdot n}{Q}\right)\nu}$$

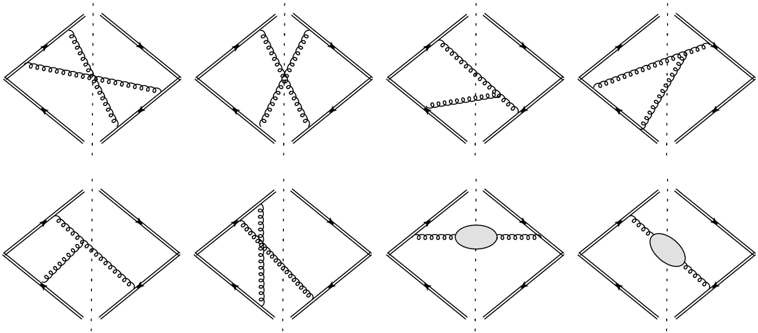
II: factorisation of the cross section ($\nu_0 = e^{-\gamma_E}$) [Collins et al.; Berger et al.][Fleming et al.; Schwartz]

$$R_T(\tau) = \mathscr{H}\left(\frac{Q^2}{\mu^2},\alpha_s(\mu)\right)\int_C \frac{d\nu}{2\pi i\nu} e^{\nu\tau}\, \tilde{\mathscr{J}}_n\left(\frac{\nu_0 Q^2}{\nu\mu^2},\alpha_s(\mu)\right)\tilde{\mathscr{J}}_{\bar{n}}\left(\frac{\nu_0 Q^2}{\nu\mu^2},\alpha_s(\mu)\right)\tilde{\mathrm{S}}\left(\frac{\nu_0^2 Q^2}{\nu^2\mu^2},\alpha_s(\mu)\right)+\mathscr{O}(\tau)$$

- single quark leg connecting $\mathscr{H}$ to $\mathscr{J}_{L,R}$
  - trivial in axial gauge
  - longitudinally polarised gluons are still present in Feynman gauge (decoupled by Ward Identities)
- soft gluons factorise
- graphs involving soft gluon exchange between S and $\mathscr{H}$ are suppressed or cancel



- $\tilde{S}(\frac{v_0^2 Q^2}{v^2 \mu^2}, \alpha_s(\mu))$
  - In emitting soft gluons, the hard collinear quarks behave as classical relativistic particles $\rightarrow$ interactions with soft gluons are factorised into eikonal lines
  - Includes both soft and soft-collinear corrections
  - Known at $\mathcal{O}(\alpha_s^2)$, [PFM, Gehrmann, Luisoni; Hornig et al.; Kelley et al. '11]



+ mirror symmetric

Computation of the *Soft subprocess*

$$\tilde{\mathrm{S}}\left(\frac{\nu_0^2 Q^2}{\nu^2\mu^2},\alpha_s(\mu)\right) = \frac{1}{N_C}\int d\tau e^{-\nu\tau}\sum_{k_{eik}}\langle 0|\Phi_{\bar{n}}^\dagger(0)\Phi_n^\dagger(0)|k_{eik}\rangle\,\mathscr{J}_{cut}(\tau)\langle k_{eik}|\Phi_n(0)\Phi_{\bar{n}}(0)|0\rangle$$

- two loop *cusp* and *soft* anomalous dimensions:

$$\Gamma_{\rm cusp}(\alpha_s) = \frac{\alpha_s}{\pi}C_F + \frac{\alpha_s^2}{\pi^2}C_F\left(C_A\left(\frac{67}{36}-\frac{\pi^2}{12}\right)-\frac{5}{9}T_F n_F\right)+\mathscr{O}(\alpha_s^3)$$
$$\Gamma_{\rm soft}(\alpha_s) = -\frac{\alpha_s^2}{\pi^2}C_F\left(T_F n_F\left(\frac{14}{27}-\frac{\pi^2}{36}\right)+C_A\left(-\frac{101}{54}+\frac{11}{144}\pi^2+\frac{7}{4}\zeta_3\right)\right)+\mathscr{O}(\alpha_s^3)$$

- *non-logarithmic* part of the two loop soft subprocess

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{\pi^2}\left(\frac{\pi^4}{32}C_F^2 + C_F T_F n_F\left(\frac{5}{81}+\frac{77\pi^2}{216}-\frac{13\zeta_3}{18}\right)+C_A C_F\left(-\frac{535}{324}-\frac{871\pi^2}{864}+\frac{7\pi^4}{120}+\frac{143\zeta_3}{72}\right)\right)$$
$$= \frac{\alpha_s^2(\mu)}{(4\pi)^2}\left(48.7045\,C_F^2 - 56.4989\,C_F C_A + 43.3905\,C_F T_F n_F\right)$$

- Analytical determination of C_2 and $G_{3,1}$
- Previously fitted using the generator `EVENT2` [Becher, Schwartz; Hoang, Kluth; Chien, Schwartz]

- $\tilde{S}(\frac{\nu_0^2 Q^2}{\nu^2 \mu^2}, \alpha_s(\mu))$
    - In emitting soft gluons, the hard collinear quarks behave as classical relativistic particles $\rightarrow$ interactions with soft gluons are factorised into eikonal lines
    - Includes both soft and soft-collinear corrections
    - Known at $\mathcal{O}(\alpha_s^2)$, [PFM, Gehrmann, Luisoni; Hornig et al.; Kelley et al.]
- $\tilde{\mathscr{J}}_n(\frac{\nu_0 Q^2}{\nu \mu^2}, \alpha_s(\mu))$
    - Describes the decay of a massless quark into a jet of collinear particles moving along the n direction (known at $\mathcal{O}(\alpha_s^2)$, [Becher, Neubert])
    - Defined as the cut quark propagator in the axial gauge
    - **Caution**: double counting with the soft subprocess and the $\bar{n}$-collinear jet must be avoided!
        - "non-light-like" Wilson lines + explicit subtraction of soft-collinear contributions
        - SCET Feynman rules + (where necessary) zero bin subtraction
- $\mathscr{H}(\frac{Q^2}{\mu^2}, \alpha_s(\mu))$
    - Hard virtual corrections ($\sim \delta(1-T) \rightarrow$ constant contribution at large ν)
    - Defined such that

$$\mathscr{H}\, \mathscr{J}_{\rm n} \otimes \mathscr{J}_{\bar{\rm n}} \otimes {\rm S} = \text{fixed order} + \mathcal{O}(\tau)$$

Resummed cross section in Laplace space (from NLL to N^{2}LL):

$$R_T(\tau) = \mathcal{H}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu)\right) \int_C \frac{d\nu}{2\pi i \nu} e^{\nu\tau} \tilde{\mathcal{J}}^2\left(1, \alpha_s\left(\sqrt{\frac{\nu_0}{\nu}}Q\right)\right) \tilde{S}\left(1, \alpha_s\left(\frac{\nu_0 Q}{\nu}\right)\right)$$
$$\times \exp\left\{ -2 \int_{\frac{\nu_0}{\nu}}^1 \frac{du}{u} \left(\int_{u^2Q^2}^{uQ^2} \frac{dk^2}{k^2} \mathcal{A}(\alpha_s(k^2)) + \mathcal{B}(\alpha_s(uQ^2)) \right) \right\} + \mathcal{O}(\tau)$$

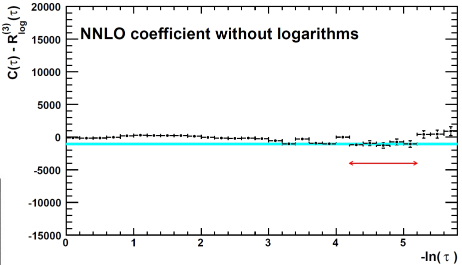
- $\mathcal{A}(\alpha_s) = \Gamma_{\text{cusp}}(\alpha_s) - \beta(\alpha_s)\frac{\partial \Gamma_{\text{soft}}(\alpha_s)}{\partial \alpha_s} = \sum_{k\geq 1} A^{(k)} \left(\frac{\alpha_s}{\pi}\right)^k$ $\quad \mathcal{B}(\alpha_s) = \Gamma_{\text{soft}}(\alpha_s) + \Gamma_{\text{coll}}(\alpha_s) = \sum_{k\geq 1} B^{(k)} \left(\frac{\alpha_s}{\pi}\right)^k$
- Altarelli-Parisi splitting function $P_{qq}(\alpha_s, z) = 2\frac{\Gamma_{\text{cusp}}(\alpha_s)}{(1-z)_+} + 2\mathcal{B}(\alpha_s)\delta(1-z) + \ldots$, [Korchemsky] (known at $\mathcal{O}(\alpha_s^3)$, [Vogt, Vermaseren, Moch])
- Beyond NLL:
  - Large angle soft emission
  - Interplay between Logarithms and Constants (breakdown of exponentiation)

Resummed cross section in Thrust space (from NLL to N^{2}LL):

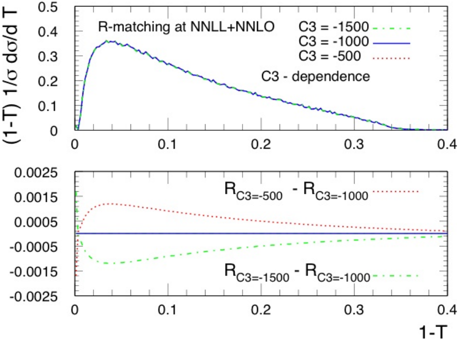
$$R_T(\tau) = \left(1 + \sum_{k=1}^{3} C_k \left(\frac{\alpha_s}{2\pi}\right)^k\right) e^{\mathscr{F}(\alpha_s(Q^2), \log\frac{1}{\tau})} \frac{1}{\Gamma(1-\gamma(\lambda))} \left[1 + \frac{\alpha_s}{\pi}\beta_0 f_2'(\lambda)\psi^{(0)}(1-\gamma(\lambda)) + \frac{1}{2}\frac{\alpha_s}{\pi}\beta_0\gamma'(\lambda)\Gamma(1-\gamma(\lambda))\frac{d^2}{d\gamma^2(\lambda)}\frac{1}{\Gamma(1-\gamma(\lambda))} + \frac{\alpha_s}{\pi}C_F\left(\gamma_E\left(\frac{3}{2}-\gamma_E\right) - \frac{\pi^2}{6}\right)\right] + \mathscr{O}(\tau)$$

- $\mathscr{F}(\alpha_s(Q^2), L) = Lf_1(\frac{\alpha_s}{\pi}\beta_0 L) + f_2(\frac{\alpha_s}{\pi}\beta_0 L) + \frac{\alpha_s}{\pi}\beta_0 f_3(\frac{\alpha_s}{\pi}\beta_0 L) + G_{3,1}\frac{\alpha_s^3}{(2\pi)^3}L + \mathscr{O}(\alpha_s^4 L^2)$
  $\gamma(\lambda) = f_1(\lambda) + \lambda f_1'(\lambda)$, $\quad \lambda = \frac{\alpha_s}{\pi}\beta_0 \log\frac{1}{\tau}$
- C_2, f_3 and $G_{3,1}$ analytically known!
- The presence of the Landau pole does not affect the inversion of the Laplace transform and it is directly mapped onto the Thrust space
- Numerical agreement with SCET based result after proper treatment of their formula (exponentiation of constants, subleading terms...)

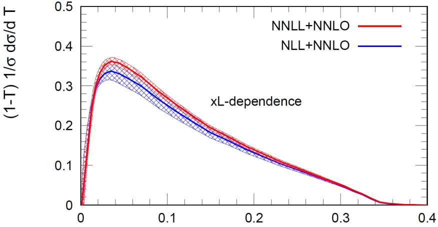
- We compare two different matching schemes (R vs $Log(R)$)
- The former requires the $\mathcal{O}(\alpha_s^3)$ constant term C_3
- C_3 unknown!
  - we fit it using `EERAD3`
  - good subtraction of the logarithms
  - high numerical instabilities below $L \sim -5.75$ $\rightarrow$ high systematic error $\sim 50\%$!

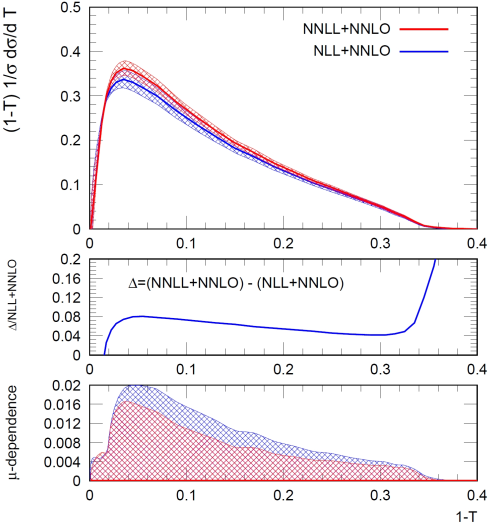


- $C_3 = -1050 + \pm 180(\text{stat}) \pm 500(\text{syst})$
- R matching scheme improves the prediction in the 3 jet region with respect to the $Log(R)$ scheme



- comparison to NLL+N^{2}LO [Gehrmann, Luisoni, Stenzel]
- renormalization scale dependence reduces from $\sim 6\%$ to $\sim 4\%$
- resummation scale dependence reduces from $\sim 7.2\%$ to $\sim 2.7\%$ in the peak region
- better description of the 3 jets region
- multi-jet region dominated by fixed order prediction





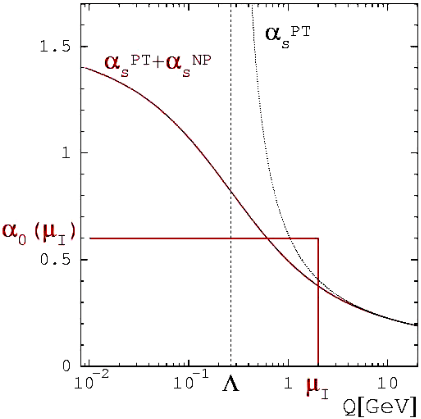
Dispersive Model, [Dokshitzer et al. '96]

- Assume integrable α_s at low Q (existence of an effective coupling α_s^{eff})
- Subtraction of renormalon ambiguity
- The leading correction amounts to a shift...

$$\frac{\mathrm{d}\sigma(y)}{\mathrm{d}y} \to \frac{\mathrm{d}\sigma(y-a_y\mathrm{P})}{\mathrm{d}y}$$

$$\alpha_0(\mu_I) = \frac{1}{\mu_I}\int_0^{\mu_I} \alpha_s^{\text{eff}}(k)\mathrm{d}k$$

$$\mathrm{P} = \frac{4C_F}{\pi^2}\mathscr{M}\frac{\mu_I}{Q}\left\{\alpha_0(\mu_I) - \left[\alpha_s(\mu_R) + \alpha_s^2(\mu_R) \times \frac{\beta_0}{\pi}\left(\log\frac{\mu_R}{\mu_I} + 1 + \frac{K}{2\beta_0}\right) + \mathscr{O}(\alpha_s^3)\right]\right\}$$



$\mathscr{M} = 1.49 \pm 0.29$ (effect of the interplay between PT & PC)
[Dokshitzer, Lucenti, Marchesini, Salam '00]

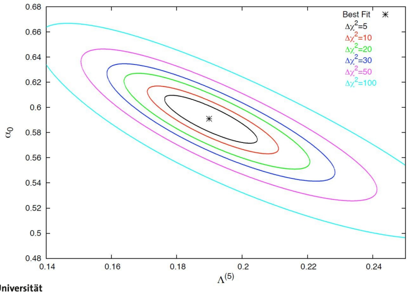
y	$1-T$	C	Y_3	ρ	B_T	B_W
a_y	2	3π	0	1	$\log\frac{1}{B_T} + F_{B_T}(B_T,\alpha_s)$	$\frac{1}{2}\log\frac{1}{B_W} + F_{B_W}(B_W,\alpha_s)$

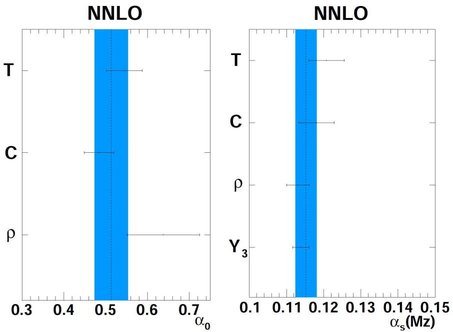
- T: NNLO+NLL+$\frac{1}{Q}$ global fit
  14 GeV $\leq$ Q $\leq$ 207 GeV
  [Davison, Webber '08]

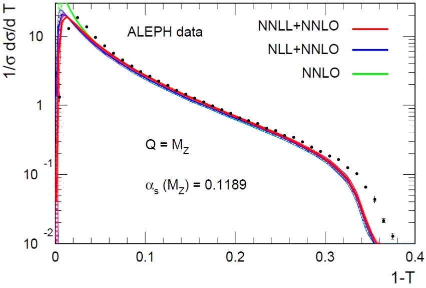
$$\alpha_s(M_Z) = 0.1164^{+0.0028}_{-0.0026}$$
$$\alpha_0(2\,\mathrm{GeV}) = 0.59 \pm 0.03$$

- NNLO+$\frac{1}{Q}$ JADE and OPAL event shape moments analysis (no Broadenings),
  [Gehrmann, Jaquier, Luisoni '10]

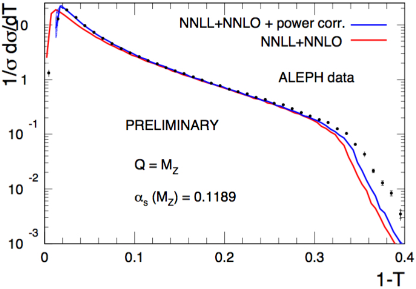
$$\alpha_s(M_Z) = 0.1153 \pm 0.0017_{\mathrm{exp}} \pm 0.0023_{\mathrm{th}}$$
$$\alpha_0(2\,\mathrm{GeV}) = 0.51 \pm 0.01_{\mathrm{exp}} \pm 0.04_{\mathrm{th}}$$



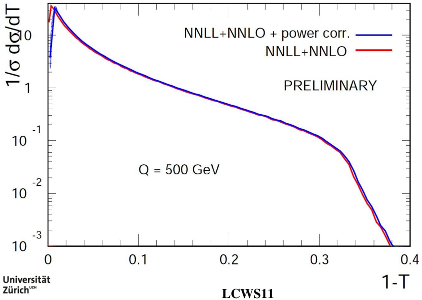




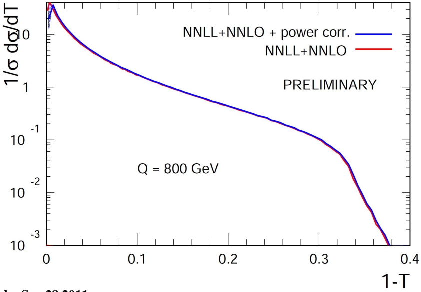
1/σ dσ/d T
ALEPH data
NNLL+NNLO
NLL+NNLO
NNLO
Q = M_Z
α_s (M_Z) = 0.1189
1-T



1/σ dσ/dT
NNLL+NNLO + power corr.
NNLL+NNLO
ALEPH data
PRELIMINARY
Q = M_Z
α_s (M_Z) = 0.1189
1-T



1/σ dσ/dT
NNLL+NNLO + power corr.
NNLL+NNLO
PRELIMINARY
Q = 500 GeV
1-T



1/σ dσ/dT
NNLL+NNLO + power corr.
NNLL+NNLO
PRELIMINARY
Q = 800 GeV
1-T

NNLO/NNLO+NLL & MC hadronisation
(All Event Shapes)

ALEPH data, [Dissertori et al. '07]

$\alpha_s(M_Z) = 0.1240 \pm 0.0012 \pm 0.0031$

$\alpha_s(M_Z) = 0.1224 \pm 0.0012 \pm 0.0037$

JADE data, [Bethke et al. '08]

$\alpha_s(M_Z) = 0.1210 \pm 0.0022 \pm 0.0056$

$\alpha_s(M_Z) = 0.1172 \pm 0.0020 \pm 0.0046$

OPAL data, [OPAL coll. '11]

$\alpha_s(M_Z) = 0.1201 \pm 0.0015 \pm 0.0026$

$\alpha_s(M_Z) = 0.1189 \pm 0.0017 \pm 0.0037$

NNLO+NN(N)LL+MC hadronisation

T: ALEPH & OPAL data, [Becher, Schwartz '08]

$\alpha_s(M_Z) = 0.1172 \pm 0.0012 \pm 0.0016$

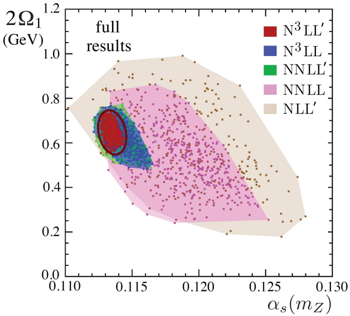
NNLO+NN(N)LL+Shape Function

ρ: ALEPH data, [Chien, Schwartz '08]

$\alpha_s(M_Z) = 0.1220 \pm 0.0019 \pm 0.0023$

T: Global fit, [Abbate et al. '10]

$\alpha_s(M_Z) = 0.1135 \pm 0.0002 \pm 0.001$



Conclusions

- Computation of the two loop soft corrections to the Thrust distribution
  - Analytic calculation of the non-logarithmic part
  - Determination of the constant term C_2 and the logarithmic coefficient $G_{3,1}$
- N^2LL resummation using RG evolution and matching to NNLO
  - Logarithmic agreement with SCET-based results up to subleading terms
  - New implementation of the R-matching scheme and numerical fit of the $\mathcal{O}(\alpha_s^3)$ constant term C_3

Perspectives

- Study of hadronisation corrections and fit of the strong coupling
- Study of general features of final state resummation at NNLL and extension to other jet observables

$$f_1(\lambda) = -\frac{A^{(1)}}{\beta_0 \lambda}\left[(1-2\lambda)\log(1-2\lambda) - 2(1-\lambda)\log(1-\lambda)\right]$$

$$f_2(\lambda) = -\frac{A^{(2)}}{\beta_0^2}\left[2\log(1-\lambda) - \log(1-2\lambda)\right] + 2\frac{B^{(1)}}{\beta_0}\log(1-\lambda)$$
$$-\frac{A^{(1)}\beta_1}{\beta_0^3}\left[\log(1-2\lambda) + \frac{1}{2}\log^2(1-2\lambda) - \log(1-\lambda)(2+\log(1-\lambda)\right]$$
$$-2\frac{A^{(1)}\gamma_E}{\beta_0}\log\frac{1-\lambda}{1-2\lambda}$$

$$f_3(\lambda) = \frac{2c_s^{(1)}}{\beta_0}\frac{\lambda}{1-2\lambda} + \frac{2c_j^{(1)}}{\beta_0}\frac{\lambda}{1-\lambda} - \frac{2B^{(2)}}{\beta_0^2}\frac{\lambda}{1-\lambda} - \frac{A^{(3)}}{\beta_0^3}\frac{\lambda^2}{(1-\lambda)(1-2\lambda)}$$
$$-\frac{2A^{(2)}\gamma_E}{\beta_0^2}\frac{\lambda}{(1-\lambda)(1-2\lambda)} + \frac{A^{(2)}\beta_1}{\beta_0^4}\frac{3\lambda^2 + (1-\lambda)\log(1-2\lambda) - 2(1-2\lambda)\log(1-\lambda)}{(1-\lambda)(1-2\lambda)}$$
$$-2\frac{B^{(1)}}{\beta_0}\gamma_E\frac{\lambda}{1-\lambda} + \frac{2B^{(1)}\beta_1}{\beta_0^3}\frac{\lambda+\log(1-\lambda)}{1-\lambda}$$
$$+\frac{A^{(1)}}{\beta_0}\frac{1}{(1-\lambda)(1-2\lambda)}\Big[-\gamma_E^2\lambda(3-2\lambda) + \frac{2\gamma_E\beta_1}{\beta_0^2}\left[\lambda + (1-\lambda)\log(1-2\lambda)\right.$$
$$\left.-(1-2\lambda)\log(1-\lambda)\right] + \frac{\beta_2}{\beta_0^3}\left[-\lambda^2 + (1-3\lambda+2\lambda^2)(2\log(1-\lambda) - \log(1-2\lambda))\right]\Big]$$
$$-\frac{A^{(1)}\beta_1^2}{\beta_0^5}\Big[\frac{1-\lambda}{2(1-\lambda)(1-2\lambda)}\log(1-2\lambda)\left[4\lambda + \log(1-2\lambda)\right]$$
$$-\frac{2}{2(1-\lambda)(1-2\lambda)}\left[\lambda^2 - (1-2\lambda)\log(1-\lambda)(2\lambda + \log(1-\lambda))\right]\Big]$$

- *Single* cut: single soft emission

$$\mathscr{J}_{cut}(\tau) \rightarrow \delta^{(+)}(q^2)(\delta(\tau Q - q\cdot n)\Theta(q\cdot\bar{n} - q\cdot n) + \delta(\tau Q - q\cdot\bar{n})\Theta(q\cdot n - q\cdot\bar{n}))$$

- *Double* cut: double soft emission

$$\begin{aligned}
&\mathscr{J}_{cut}(\tau) \rightarrow \delta^{(+)}(q^2)\delta^{(+)}(k^2)\\
&\times(\delta(\tau Q - q\cdot n - k\cdot n)\Theta(q\cdot\bar{n} - q\cdot n)\Theta(k\cdot\bar{n} - k\cdot n)\\
&+\delta(\tau Q - q\cdot\bar{n} - k\cdot\bar{n})\Theta(q\cdot n - q\cdot\bar{n})\Theta(k\cdot n - k\cdot\bar{n})\\
&+\delta(\tau Q - q\cdot n - k\cdot\bar{n})\Theta(q\cdot\bar{n} - q\cdot n)\Theta(k\cdot n - k\cdot\bar{n})\\
&+\delta(\tau Q - k\cdot n - q\cdot\bar{n})\Theta(k\cdot\bar{n} - k\cdot n)\Theta(q\cdot n - q\cdot\bar{n}))
\end{aligned}$$

- All phase-space integrals evaluated with analytic techniques
  - numerical cross-check using sector decomposition: SecDec+BASES/VEGAS [Carter, Heinrich '11]