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## REALIZATION AND CANONICAL FORMS

A linear time-invariant (LTI) system can be represented in many ways, including:

- differential equation (ODE)
- state variable (SV) form
- transfer function
- impulse response
- block diagram (BD) or flow graph

Each description can be converted to the others. In the lecture on [Mason's Formula](#) we saw how to represent systems in terms of block diagrams, and how to determine the transfer function of a block diagram system using Mason's Formula.

In this lecture we shall see how to make a block diagram from a given transfer problem. This is the inverse problem from the one Mason's Formula solves. Then, we shall see how to associate a state-space equation with any block diagram. The relation between SV systems and BD is very transparent since the output of each integrator is a state.

The problem of finding a SV or BD representation given a prescribed transfer function is called the *realization problem*.

## BD REALIZATION OF TRANSFER FUNCTIONS IN SERIES FORMS

A transfer function can be realized as a BD in series form or parallel form. Here we introduce two series forms that are very convenient for solving the BD realization problem for single-input/single-output (SISO) systems.

Consider the illustrative third-order transfer function

$$H(s) = \frac{b_2s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0}. \quad (1)$$

This is a *rational* function (e.g. a ratio of two polynomials in  $s$ ). For realization, it is important to ensure that the transfer function is *monic*, that is, the highest order term in

the denominator has a coefficient of 1. If not, divide through by this coefficient to put the transfer function in monic form.

The transfer function must also have relative degree of 1 or more. If the relative degree is zero (e.g. same power of  $s$  in the numerator as the denominator), then divide the denominator into the numerator in one step of long division to write  $H(s)$  as a constant term plus a term whose relative degree is at least one. The constant term is a direct feedthrough term, and the procedures below may be carried out to realize the remainder term.

A transfer function is said to be *proper* if its relative degree is greater than or equal to zero, and *strictly proper* if the relative degree is greater than or equal to one.

We use a third-order system to illustrate the approach, which works for any  $n$ -th order rational, monic, strictly proper transfer function.

To find a BD realization of  $H(s)$ , divide by the highest power of  $s$  to obtain

$$H(s) = \frac{b_2s^{-1} + b_1s^{-2} + b_0s^{-3}}{1 + a_2s^{-1} + a_1s^{-2} + a_0s^{-3}} = \frac{b_2s^{-1} + b_1s^{-2} + b_0s^{-3}}{1 - (-a_2s^{-1} - a_1s^{-2} - a_0s^{-3})}. \quad (2)$$

Now think of Mason's Formula. To draw a BD we can use three feedforward paths and three loops if we select the correct transmissions and loop structure.

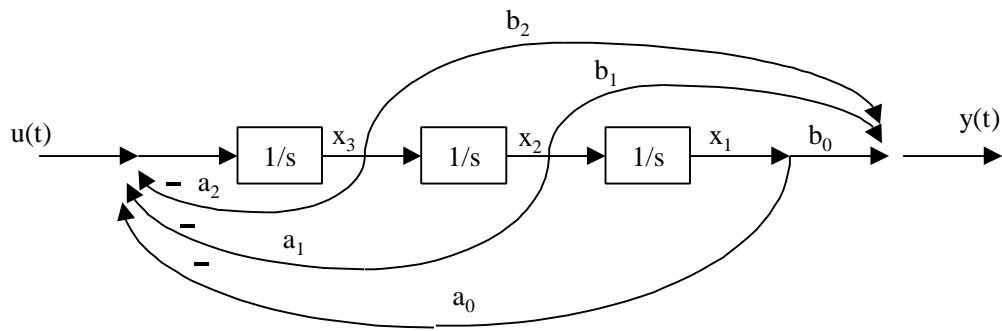
We give two series forms that have a convenient structure for realizing SISO systems. Note particularly that Mason's Formula is very easy to use if there are no disjoint loops, and all loops touch all feedforward paths. Then, the determinant  $\mathbf{D}(s)$  is simply 1 minus the sum of the loop gains, and all cofactors are equal to one.

## Reachable Canonical Form (RCF)

The rule used for RCF is

*feedback to the left*  
*feedforward to the right.*

This means that all feedback loops should join at a summer on the left, and all feedforward paths should join at a summer on the right. A BD satisfying this condition is drawn below.



### Reachable Canonical Form

Note that all loops and all feedforward paths have the left-hand integrator in common, so all cofactors are equal to 1 and the determinant has no higher-order terms. Applying Mason's Formula to this BD gives the transfer function (2). Determine the loop gains and path gains and make sure you believe this.

Each integrator output is labeled as a state. The rule used in this course for labeling states will be:

*Label the states from right to left, from top to bottom.*

We will see some examples of this to clarify it.

With the states labeled as shown, one may write down directly the state equations

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u = Ax + Bu \quad (3)$$

$$y = [b_0 \quad b_1 \quad b_2] x = Cx$$

As an exercise, one may find the transfer function

$$H(s) = C(sI - A)^{-1}B + D$$

and verify that it is the same as the one we started with.

This development gives a very easy way to realize SISO transfer functions in SV form. One notes that it is easy to write down (3) directly from (1) without having to draw the BD. In fact, simply take the denominator of  $H(s)$ , turn the coefficients backwards, make them negative, and place them into the bottom row of the  $A$  matrix. Take the coefficients of the numerator, turn them backwards, and place them into the  $C$  matrix.

The  $A$  matrix in (3) is known as a *bottom companion matrix* for the characteristic polynomial

$$\Delta(s) = s^3 + a_2s^2 + a_1s + a_0.$$

The superdiagonal 1's in  $A$  and the lower 1 in  $B$  mean simply that the three integrators are connected in series.

### Example 1. Realize Transfer Function as RCF SV System

Let there be prescribed

$$H(s) = \frac{s^2 + 2s - 1}{s^3 + 2s^2 + 3s + 4}.$$

The SV equations are directly written down as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u = Ax + Bu$$

$$y = [-1 \quad 2 \quad 1]x = Cx$$

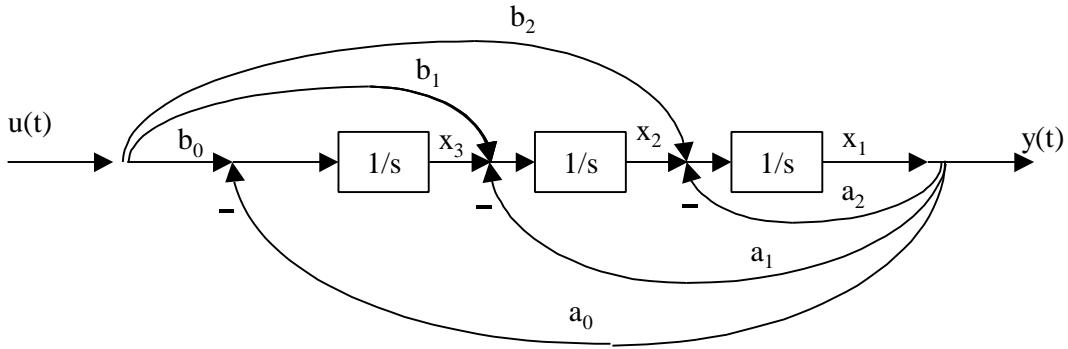
Now one may analyze the system including simulation, finding output given an input and ICs, etc.

### Observable Canonical Form (OCF)

The rule used for OCF is

*feedback from the right*  
*feedforward from the left*

A BD satisfying this condition is drawn below.



### Observable Canonical Form

Note that all loops and all feedforward paths have the right-hand integrator in common, so all cofactors are equal to 1 and the determinant has no higher-order terms. Applying Mason's Formula to this BD gives the transfer function (2). Determine the loop gains and path gains and make sure you believe this.

With the states labeled from right to left as shown, one may write down directly the state equations

$$\dot{x} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u = Ax + Bu \tag{4}$$

$$y = [1 \ 0 \ 0]x = Cx$$

As an exercise, one may find the transfer function

$$H(s) = C(sI - A)^{-1}B + D$$

and verify that it is the same as the one we started with.

This development gives a very easy way to realize SISO transfer functions in SV form. One notes that it is easy to write down (4) directly from (1) without having to draw the BD. In fact, simply take the denominator of  $H(s)$ , stack the coefficients on end, make them negative, and place them into the first column of the  $A$  matrix. Take the coefficients of the numerator, stack them on end, and place them into the  $B$  matrix.

Note that this OCF state-space form is not the same as RCF, though both have the same transfer function. In fact, RCF and OCF are related by a *state-space transformation*, which we shall not discuss in this course (it is discussed in EE 5307, Linear Systems).

The  $A$  matrix in (4) is known as a *left companion matrix* for the characteristic polynomial

$$\Delta(s) = s^3 + a_2s^2 + a_1s + a_0.$$

The superdiagonal 1's in  $A$  and the left-hand 1 in  $C$  mean simply that the three integrators are connected in series.

### Example 2. Realize Transfer Function as OCF SV System

Let there be prescribed

$$H(s) = \frac{s^2 + 2s - 1}{s^3 + 2s^2 + 3s + 4}.$$

The SV equations are directly written down as

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} u = Ax + Bu$$

$$y = [1 \ 0 \ 0]x = Cx$$

Now one may analyze the system including simulation, finding output given an input and ICs, etc.

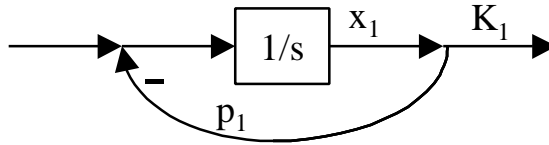
## BD REALIZATION OF TRANSFER FUNCTIONS IN PARALLEL FORM

To realize a system in parallel form, one performs a PFE on the transfer function to obtain

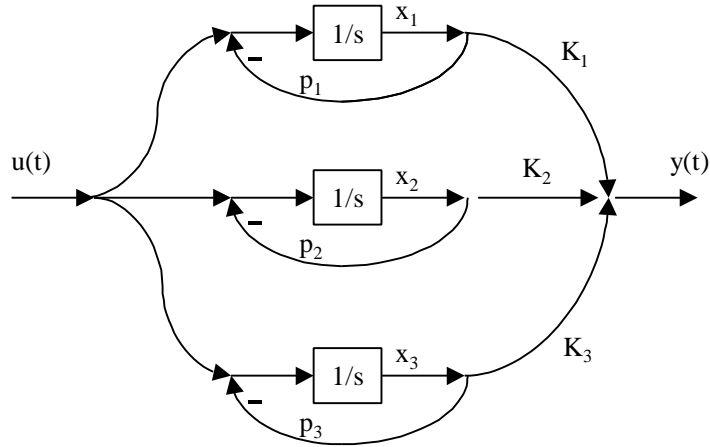
$$H(s) = \frac{b_2s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0} = \frac{K_1}{s + p_1} + \frac{K_2}{s + p_2} + \frac{K_3}{s + p_3} \quad (5)$$

where the poles are at  $s = -p_1, -p_2, -p_3$  and the residues are  $K_1, K_2, K_3$ .

Now note that a single term of this form can be realized using the simple BD shown. Use Mason's Formula to determine the transfer function to make sure you believe this.



The complete transfer function with three parallel paths can be realized as shown.



This realization is known as parallel form. If there are repeated poles, then the transfer function has higher-order poles in the PFE. In this event, some parallel paths will contain multiple integrators. We do not need to know the details for this course.

A system which has a PFE with no higher-order poles is called *simple*.

The parallel form is known as *Jordan Normal Form* in mathematics. The case of higher-order pole factors, corresponding to multiple integrators in some paths, corresponds to what is known as *eigenvector chains* in those paths. The details, though fascinating, are not needed in this course.

With the states labeled from top to bottom as shown, one may write down directly the state equations

$$\dot{x} = \begin{bmatrix} -p_1 & 0 & 0 \\ 0 & -p_2 & 0 \\ 0 & 0 & -p_3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u = Ax + Bu$$

(6)

$$y = [K_1 \quad K_2 \quad K_3]x = Cx$$

As an exercise, one may find the transfer function

$$H(s) = C(sI - A)^{-1}B + D$$

and verify that it is the same as the one we started with.

Note that for a scalar system (e.g.  $n=1$ ) one may write

$$H(s) = \frac{cb}{s-a} + d.$$

This shows that the residues can be placed on the input paths in the figure above. In fact, as long as  $cb = K$  in each path, one can split the residues between input and output paths.

This development gives a very easy way to realize SISO transfer functions in SV form. One notes that it is easy to write down (6) directly from (5) without having to draw the BD.

Note that this parallel state-space form is not the same as RCF or OCF, though all three have the same transfer function. In fact, RCF, OCF, and the Jordan form are related by *state-space transformations*, which we shall not discuss in this course.

The  $A$  matrix in (6) is known as a *parallel form matrix* for the characteristic polynomial

$$\Delta(s) = s^3 + a_2s^2 + a_1s + a_0 = (s + p_1)(s + p_2)(s + p_3).$$

If the  $A$  matrix is diagonal with the poles appearing on the diagonal it is called *simple*. If the system is not simple, then there will be some off diagonal 1's in  $A$  corresponding to Jordan eigenvector chains of length greater than one. You will hear more about this in another course (EE 5307).