Analytic Formula of Spatial Resolution

(from RY Ph.D. thesis)

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Overview of Basic Processes

1. Charged particles ionize gas molecules. The electrons generated in this process are called seed electrons.

- **Primary Ionization** $P_{PI}(N)$ **: collision between incident particle and gas molecules**
- **Secondary Ionization** $P_{SI}(M)$ **: further ionization by primary ionized electrons**

N : # of primary clusters M : cluster size

2. The seed electrons drift toward the readout plane while diffusing.

$$P(\Delta x;\sigma_d)\frac{1}{\sqrt{2\pi}\sigma_d}\exp\left[-\frac{1}{2}\left(\frac{\Delta x}{\sigma_d}\right)^2\right]$$

$$\sigma_d^2 = C_d^2\, z$$

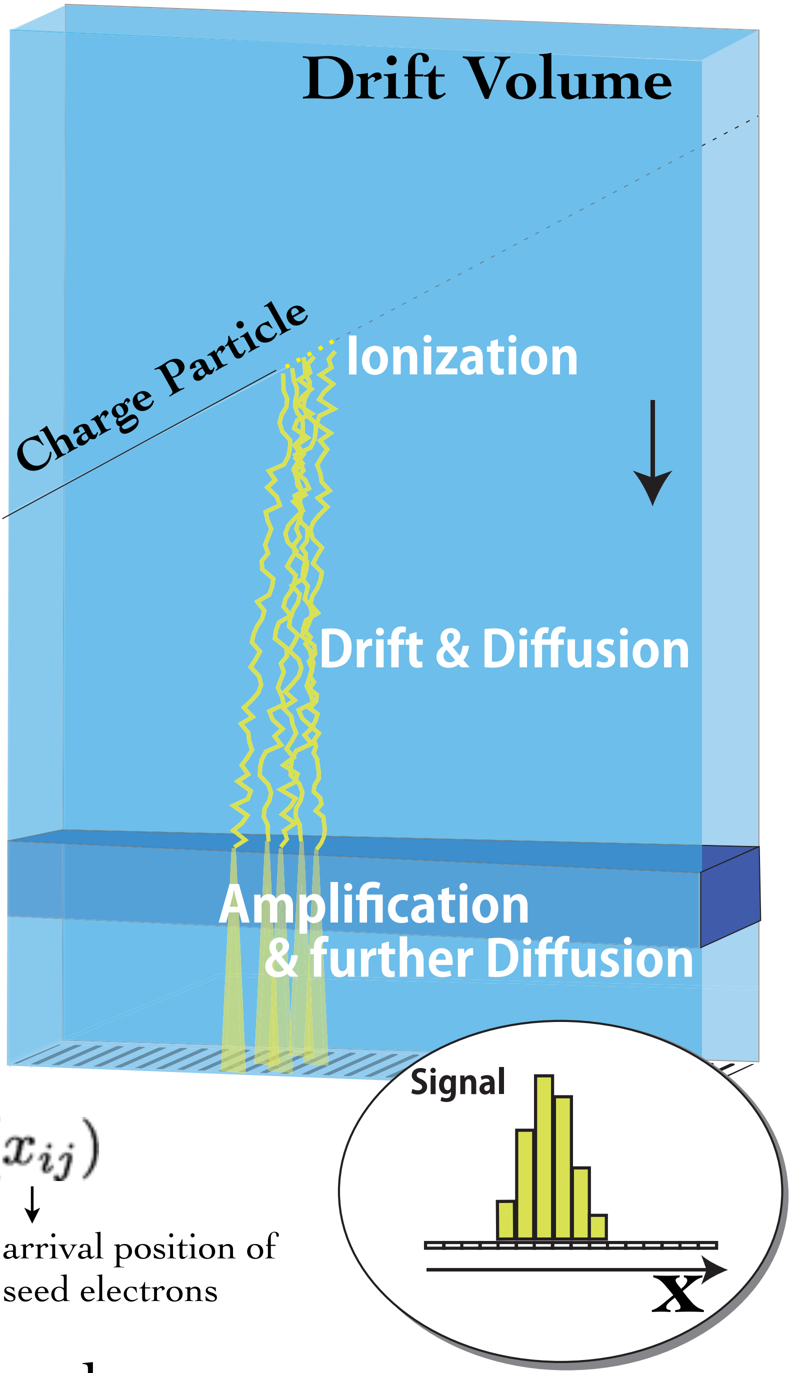
3. The Seed electrons are multiplied by a gas amplification device.

Polya distribution

$$P_G(G/\bar{G};\theta) = \frac{(\theta+1)^{\theta+1}}{\Gamma(\theta+1)}\left(\frac{G}{\bar{G}}\right)^{\theta}\exp\left(-(\theta+1)\left(\frac{G}{\bar{G}}\right)\right)$$

4. There may be further charge spread after gas amplification, and the charge spread is expressed by pad response function $F_a(x_{ij})$ and its width is specified by σ_{PRF}. This process is detector-specific.

$\downarrow$ arrival position of seed electrons



5. Finally the gas-amplified signals are readout with finite-width pads. We measure the coordinate of seed electrons with the charge centroid method.

Resolution Formula

Definition of spatial resolution

$$\sigma_{\bar{x}}^2 \equiv \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \int d\bar{x}\, P(\bar{x};\tilde{x})\,(\bar{x}-\tilde{x})^2$$

Probability Distribution Function (PDF)

$$\begin{aligned} P(\bar{x};\tilde{x}) = {} & \sum_{N=1}^{\infty} \underbrace{P_{PI}(N;\overset{\substack{\text{averaged number of primary clusters}\\ \uparrow}}{\bar{n}}\Delta Y)}_{\text{Primary ionization}} \prod_{i=1}^{N} \left[\int_{-\Delta Y/2}^{+\Delta Y/2} \frac{dy_i}{\Delta Y} \underbrace{\sum_{M_i=1}^{\infty} P_{SI}(M_i)}_{\text{Secondary ionization}} \right. \\ & \left. \prod_{j=1}^{M_i} \left(\underbrace{\int_{-\infty}^{+\infty} d\Delta y_{ij} P_D(\Delta y_{ij};\sigma_d)}_{\text{Diffusion (y direction)}} \underbrace{\int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij};\sigma_d)}_{\text{Diffusion (x direction)}} \underbrace{\int d\frac{G_{ij}}{\bar{G}} P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right)}_{\text{gas gain}} \right) \right] \\ & \times \int d\Delta Q_a \underbrace{P_E(\Delta Q_a;\sigma_E)}_{\text{electronic noise}} \delta\left(Q_a - \left(\underbrace{\sum\sum G_{ij}F_a(x_{ij})R(y_{ij}) + \Delta Q_a}_{\text{Charge on a-th pad}}\right)\right) \delta\left(\bar{x} - \underbrace{\frac{\sum_a Q_a(aw)}{\sum_a Q_a}}_{\text{Charge centroid}}\right) \end{aligned}$$

$R(y_{ij})$: Pad response function in pad-row direction.
This factor represents the efficiency for seed electrons to arrive at the pad-row in question.

ΔQ_a : Electronic noise charge on a-th pad.

Resolution Formula

General Expression

$$\begin{aligned}
\sigma_{\bar{x}}^2 \;\simeq\; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{n}\Delta Y) \prod_{i=1}^{N} \Bigg[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \\
& \prod_{j=1}^{M_i} \left(\int d\left(\frac{G_{ij}}{\bar{G}}\right) P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right) \int_{-\infty}^{+\infty} dy_{ij}\, P_D(y_{ij}-y_i;\sigma_d) \right) \Bigg] \\
& \qquad \times \left\{ \frac{\sum_{a,b}(abw^2)\, \sum_{i=1}^{N} \left[\langle F_a F_b\rangle_{\Delta x}^{y_i} - \langle F_a\rangle_{\Delta x}^{y_i} \langle F_b\rangle_{\Delta x}^{y_i}\right] \sum_{j=1}^{M_i} (G_{ij}\, R(y_{ij}))^2}{\left(\sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})\right)^2} \right. \\
& \qquad\qquad \left. + \left(\frac{\sum_a (aw)\, \sum_{i=1}^{N} \langle F_a\rangle_{\Delta x}^{y_i} \sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})}{\sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})} - \tilde{x} \right)^2 \right\}
\end{aligned}$$

Notation:

$$\langle g(\Delta x)\rangle_{\Delta x} := \int_{\infty}^{\infty} \frac{d\Delta x}{\sqrt{2\pi}\sigma_d} \exp\left[-\frac{1}{2}\left(\frac{\Delta x}{\sigma_d}\right)^2\right] g(\Delta x)$$

In the Case of Perpendicular Tracks (ϕ=0)

$$\boxed{\sigma_x^2(z;w,C_d,N_{eff},[f]) = [A] + \frac{1}{N_{eff}}[B] + [C]}$$

Notation:

$$\langle g(\Delta x)\rangle_{\Delta x} := \int_{\infty}^{\infty} \frac{d\Delta x}{\sqrt{2\pi}\sigma_d} \exp\left[-\frac{1}{2}\left(\frac{\Delta x}{\sigma_d}\right)^2\right] g(\Delta x)$$

Pad response function

True position

$$[A] := \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left(\underbrace{\underbrace{\sum_a (aw)\left\langle \overset{\downarrow}{F_a}(\tilde{x}+\Delta x)\right\rangle_{\Delta x}}_{\text{diffusion-averaged charge centroid}} - \overset{\downarrow}{\tilde{x}}}_{\text{systematics}}\right)^2$$

→ Visible only in short drift region.
systematic term[S-shape, hodoscope]

$$[B] := \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\underbrace{\sum_a (aw) F_a(\tilde{x}+\Delta x) - \underbrace{\sum_a (aw)\left\langle F_a(\tilde{x}+\Delta x)\right\rangle_{\Delta x}}_{\text{diffusion-averaged charge centroid}}}_{\text{displacement due to diffusion for a single electron}}\right)^2 \right\rangle_{\Delta x}$$

$$\approx [A]_{z=0} + \sigma_d^2$$

→ diffusion term

[B] represents resolution per seed electron.

asymptotic formula

$$\boxed{\sigma_x^2 = \sigma_0^2 + \frac{C_d^2}{N_{eff}} z}$$

$$\sigma_0^2 = \frac{[A]_{z=0}}{N_{eff}} \quad \left([A]_{z=0} \to \frac{w^2}{12} \text{ for } \sigma_{PRF} \ll w\right)$$

= hodoscope effect

$$N_{eff} := \left[\left\langle \frac{1}{N} \right\rangle_N \left\langle \left(\frac{G}{\bar{G}}\right)^2 \right\rangle_G\right]^{-1}$$

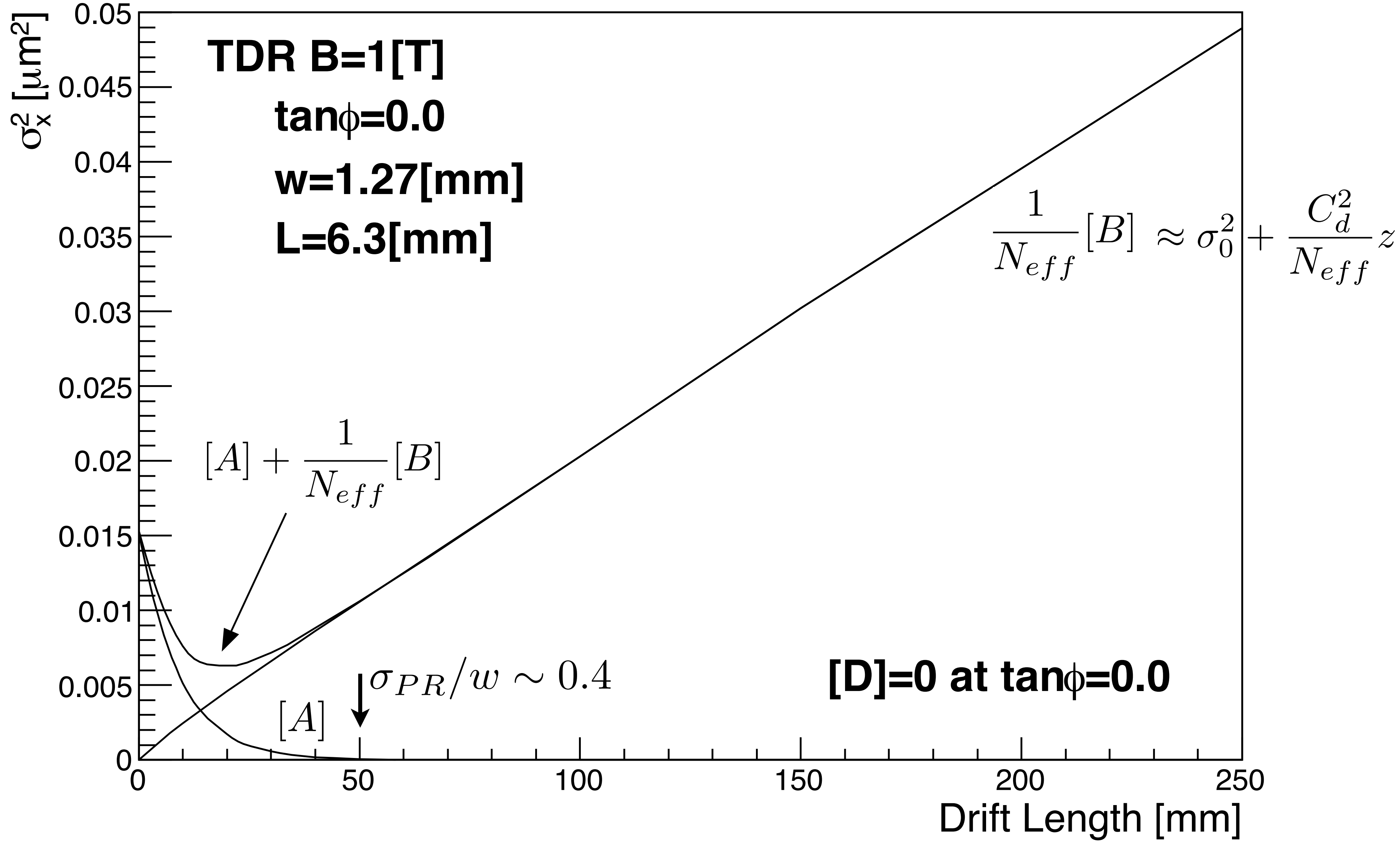
→ Effective number of electrons.
N_{eff} depends on ionization statistics and gas gain fluctuation.
This expression tells us why $N_{eff} \ll \langle N \rangle_N$.
Experimentally we can obtain N_{eff} from the spatial resolution plot as a function of drift length.

$$[C] := \left(\frac{\sigma_G}{\bar{G}}\right)^2 \left\langle \frac{1}{N^2} \right\rangle_N \sum_a (aw)^2$$

→ Electronics noise effect.

Sample Calculation I

Perpendicular track



[A] vanishes above $\sigma_{PR}/w \sim 0.4$ (corresponding to z~50mm in the figure).
[B] is almost linear -->$[B] \approx [A]_{z=0} + \sigma_d^2$ approximation is valid almost all over the drift length.

Generalization to Inclined Tracks

$$\boxed{\sigma_x^2(z; w, L\tan\phi, C_d, N_{eff}, \hat{N}_{eff}, [f]) = [A] + \frac{1}{N_{eff}}[B] + [C] + \frac{1}{\hat{N}_{eff}}[D]}$$

Notation:

$$\langle g(y)\rangle_y := \int_{-\Delta Y/2}^{+\Delta Y/2} \frac{dy}{\Delta Y} \bar{\bar{P}}_{SI}(k, y)\ g(y)$$

Effective secondary ionization probability

$$[A] := \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left(\underbrace{\underbrace{\sum_a (aw)\left\langle \langle F_a(\tilde{x} + y\tan\phi + \Delta x)\rangle_{\Delta x}\right\rangle_y}_{\text{diffusion-averaged \& cluster position average charge centroid}} - \tilde{x}}_{\text{systematics}} \right)^2$$

→ Generalization of [A] for perpendicular tracks

$$[B] := \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\underbrace{\sum_a (aw) F_a(\tilde{x} + \Delta x) - \underbrace{\sum_a (aw) \langle F_a(\tilde{x} + \Delta x)\rangle_{\Delta x}}_{\text{diffusion-averaged charge centroid}}}_{\text{displacement due to diffusion for a single electron}} \right)^2 \right\rangle_{\Delta x}$$

$$\approx [A]_{z=0} + \sigma_d^2$$

→ Exactly same as with the [B] for perpendicular tracks.

$$[C] := \left(\frac{\sigma_G}{\bar{G}}\right)^2 \left\langle \frac{1}{N^2} \right\rangle_N \sum_a (aw)^2$$

→ Exactly same as with the [C] for perpendicular tracks.

$$[D] := \frac{L^2 \tan^2\phi}{12}$$

variance projected to x-axis for a primary cluster

→ Visible only when ϕ ≠ 0.

$$N_{eff} := \left[\left\langle \sum_{i=1}^{N} \sum_{j=1}^{k_i} \left\langle \left(\underbrace{\frac{G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}}}_{\text{Normalized gain for a seed electron}} \right)^2 \right\rangle_G \right\rangle_{N,k} \right]^{-1}$$

→ Generalized effective number of electrons.

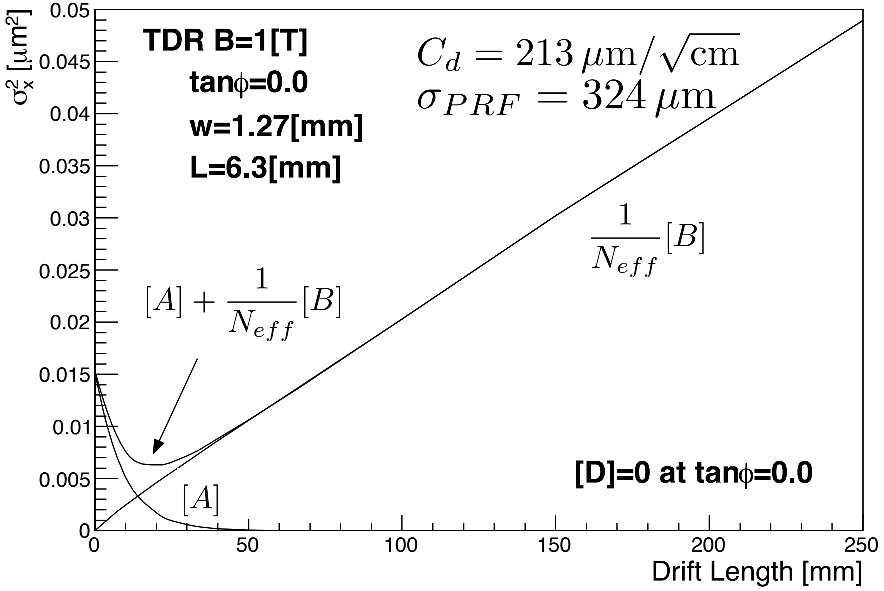
$$\hat{N}_{eff} := \left[\int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \sum_{i=1}^{N} \left\langle \left(\sum_a (aw) \langle F_a\rangle_{\Delta x} - \sum_a (aw) \left\langle \langle F_a\rangle_{\Delta x}\right\rangle_y \right)^2 \right\rangle_y \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i}} \right)^2 \right\rangle_G \right\rangle_{N,k} \right]^{-1} \times \tan^2\phi \frac{L^2}{12}$$

$$\approx \left[\left\langle \sum_{i=1}^{N} \left\langle \left(\underbrace{\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}}}_{\text{Normalized gain for a single primary cluster}} \right)^2 \right\rangle_G \right\rangle_{N,k} \right]^{-1}$$

→ Effective number of primary clusters.

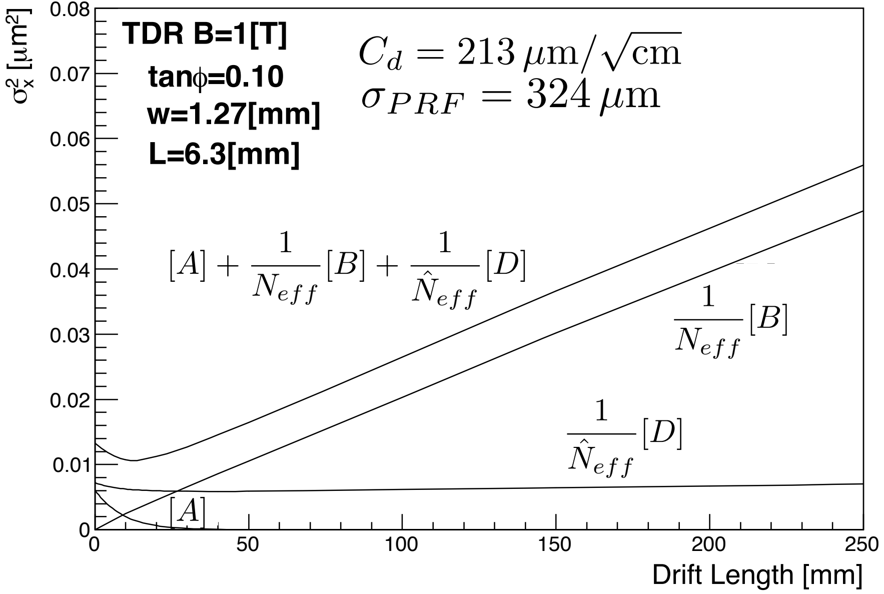
Sample calculations II

Perpendicular track



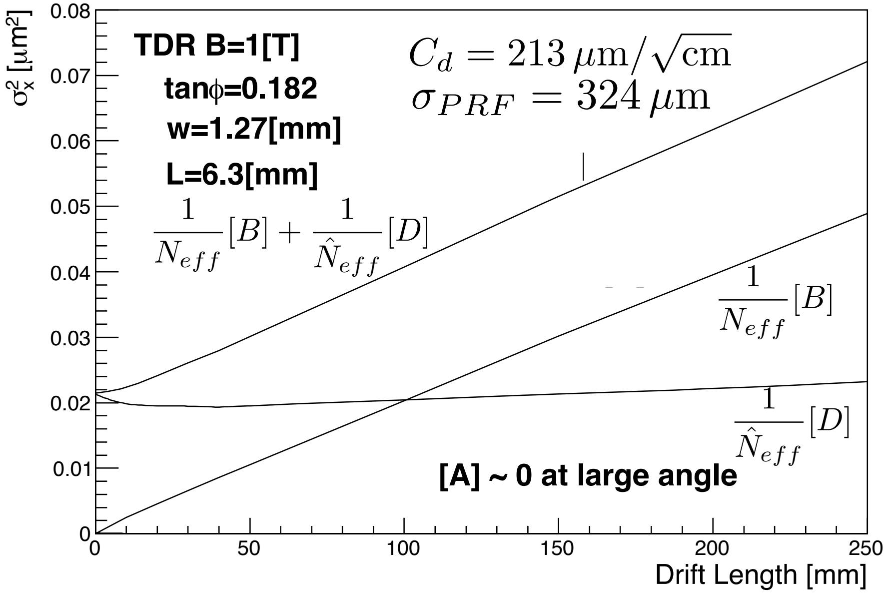
[D] is invisible.

Inclined track (tanφ=0.1)



[A] becomes smaller.
[D] becomes visible.

Inclined track (tanφ=0.18)



[A] becomes invisible.
[D] becomes sizable.

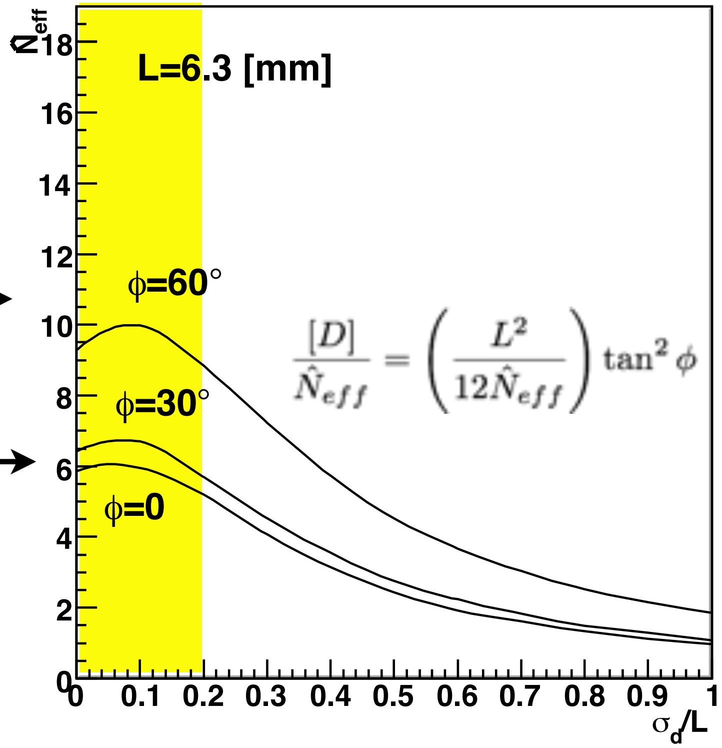
The σ_d/L will not exceed 0.2 for the LC-TPC.

$\hat{N}_{eff}$ is roughly constant in this range. →

[D] term contribution is determined
not by N_{eff} but by $\hat{N}_{eff}$.

$\hat{N}_{eff} \ll N_{eff}$ → **σx quickly deteriorates with angle.**

tanφ=0.18→
-->φ=10°



Obtained Knowledge

- Spatial resolution consists of 4 components.
- [A] : systematics due to finite pad readout.

  disappears if $\sigma_{PR}/w \gtrsim 0.4$
  (long drift length or inclined tracks)
- [B] : diffusion effect

  - Gas property
  - We found that σ_0^2 in the asymptotic formula

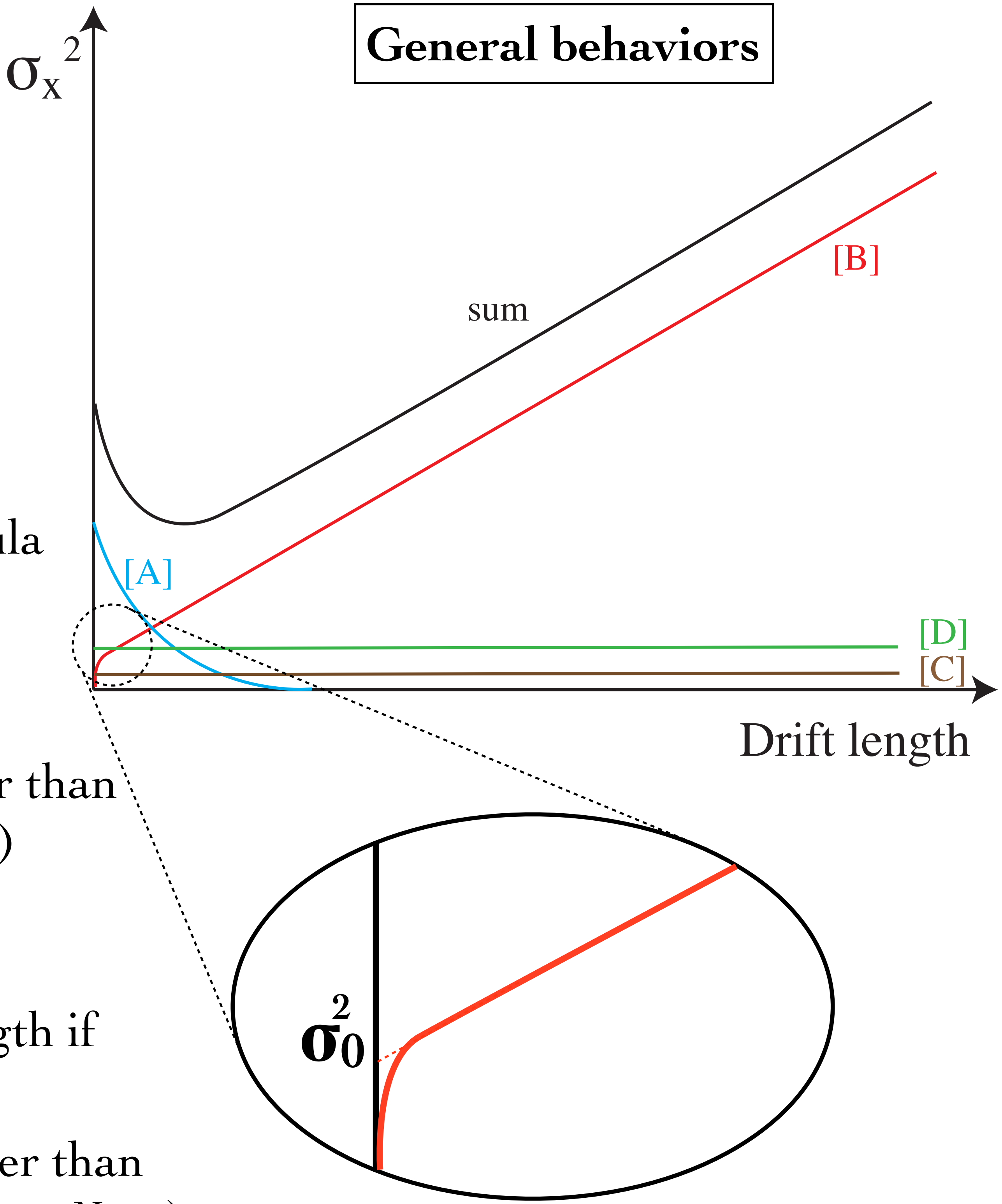
  $$\sigma_x^2 = \sigma_0^2 + \frac{C_d^2}{N_{eff}} z$$

  can be written as $\sigma_0^2 = [A]_{z=0}/N_{eff}$.

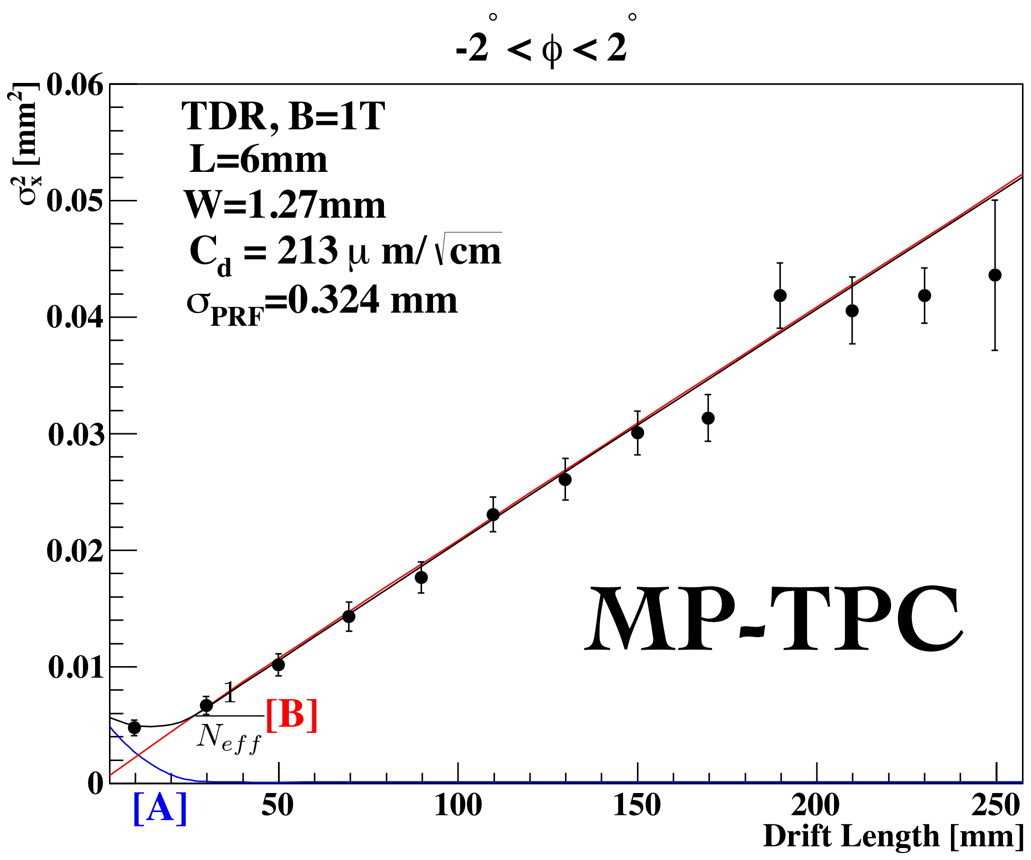
  - We understood why Neff is much smaller than average of seed electrons. ($N_{eff} \ll \langle N \rangle_N$)
- [C] : electronic noise effect
- [D] : primary cluster fluctuation

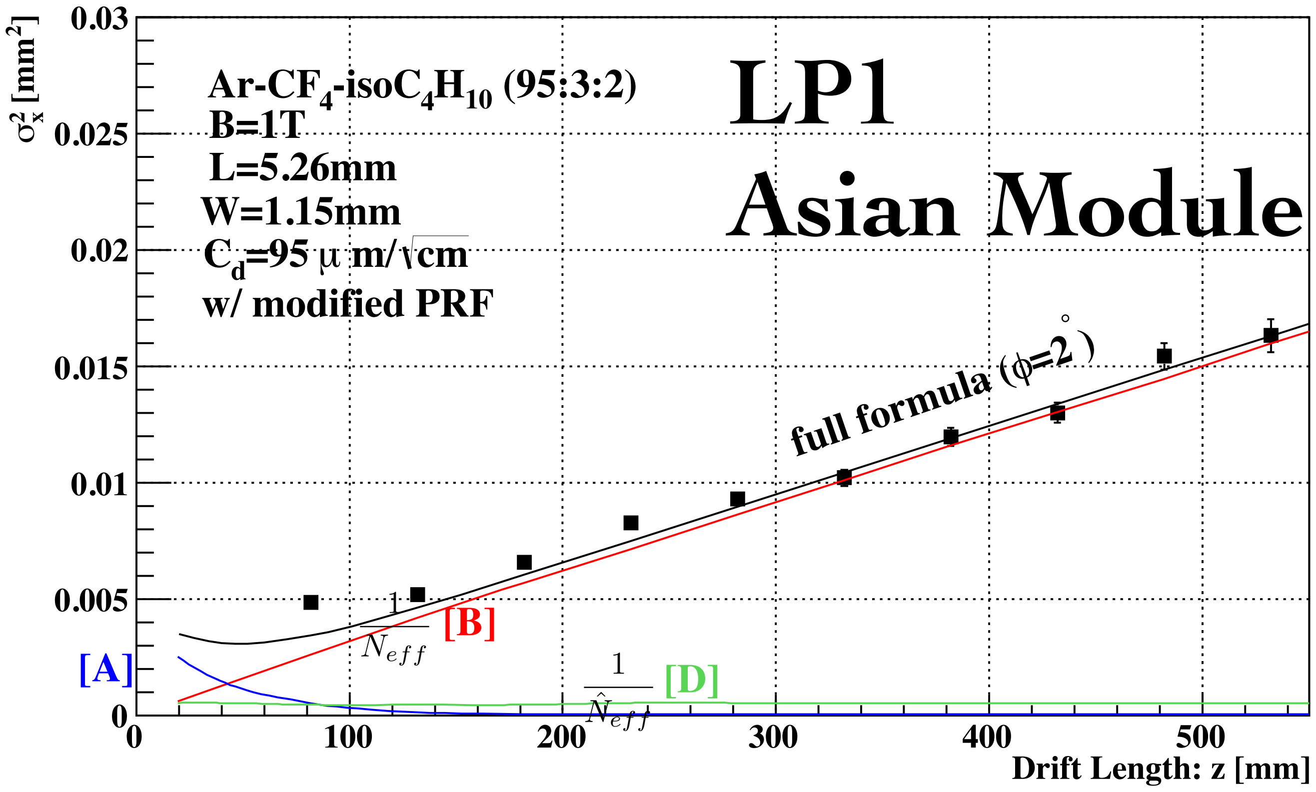
  almost constant as a function of drift length if ϕ is fixed. It vanished for ϕ=0.

  - We understood why $\hat{N}_{eff}$ is much smaller than effective number of seed electrons. ($\hat{N}_{eff} \ll N_{eff}$)



Comparison with Data

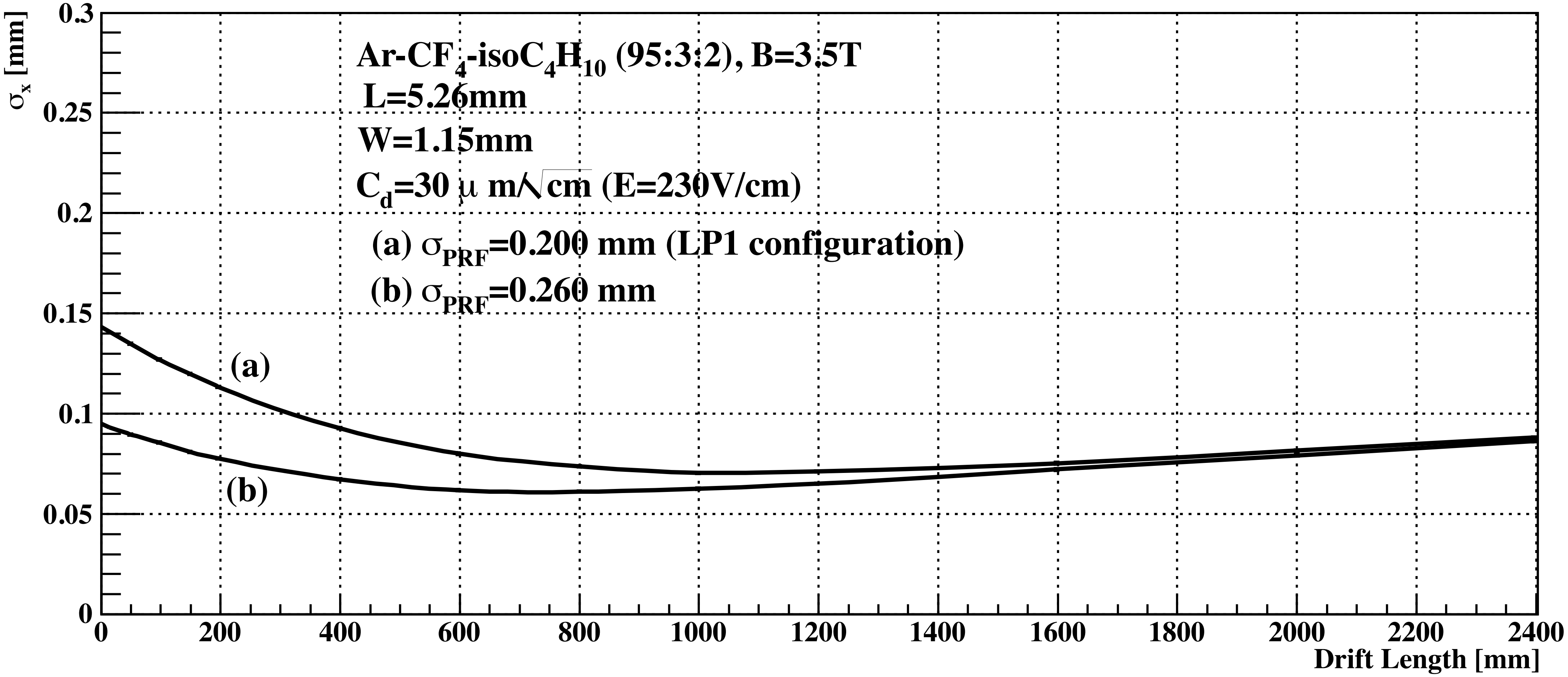




Noise term is negligible.
[B] is dominant in these plots.
For further improvement, we need smaller diffusion constant, smaller gas gain fluctuation, or better ionization statistics.

Extrapolation

Drift length 600mm --> 2400mm
B 1 T --> 3.5 T



Hodoscope effect at short drift lengths is sizable for (a).
(b) is a virtual case where we assumed wider σ_{PRF}. (b) can be realized if we optimize distance and electric field between amplification region.

Conclusion

- Our motivation was to demonstrate feasibility to achieve physics performance goal with TPC.
- In order to answer the question, we derived an analytic formula for spatial resolution.
- We clarified physical meanings of the 4 components that decide the spatial resolution.
- The formula is applicable to any MPGD TPC if we introduce a proper pad response function.
- Practically, we can find dominant component and thus we can efficiently find ways to improve spatial resolution. The analytic formula plays an important role to extrapolate results from small prototype TPC to a real-sized TPC.
- We will prepare a small paper for this topic.