Numerical NLO Calculations in QCD with Many Jets

or: how to perform $\int \mathrm{d}^4 k$ numerically

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LC 2013, Tue May 28

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Symmetry Breaking

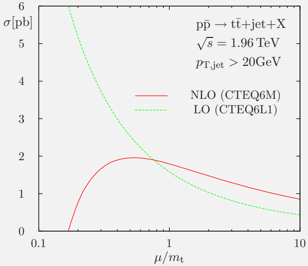
THEP

Motivation (I)

- E.g.: Discovery of new particles, requires large $\sqrt{s}$
- Large $\sqrt{s}$ means more events with many jets

$\Rightarrow$ Theoretical prediction (ME) is required

- LO suffers from scale uncertainties $\Rightarrow$ NLO
- NLO much more complicated than LO
- Many jets are computationally complex



An automated algorithm to calculate QCD matrix elements $|\mathcal{A}|^2$ at NLO precision for many jets with a good scaling behavior

[0] Picture taken from: S. Dittmaier, P. Uwer, and S. Weinzierl. "NLO QCD corrections to t anti-t + jet production at hadron colliders." In: *Physical Review Letters* 98 (2007), p. 262002.

Motivation (II)

Idea: Numerical Subtraction

- No symbolic integration of $\int d^D k$
- Instead perform loop together with phase space in a single MC integration $\rightarrow$ speedup; MC error independent of the dimension of the integrand
- Automated: Program takes the number of jets and computes the matrix element using recursion relations — no intermediate diagram generation

- Different from the usual approach

$\rightarrow$ How is regularization achieved, $D = 4 - 2\epsilon$!?

$\rightarrow$ How do the recursion relations look like?

$\rightarrow$ Is this really faster?

Outline

1. The Numerical Subtraction Method
   - Real Subtraction
   - Virtual Subtraction
2. Application to $e^+e^- \longrightarrow n$ jets
   - Jet-Rates
3. Summary

Reminder: Regularization in real corrections

NLO correction

$$\langle O\rangle^{\mathrm{NLO}} = \int_{n+1} O_{n+1}\mathrm{d}\sigma^{\mathrm{R}} + \int_{n} O_n \mathrm{d}\sigma^{\mathrm{V}} \tag{1}$$

- Two seperately divergent integrals, finite sum (KLN)
- Different integrals, unsuitable for numerical integration!
- Solution: Add zero in a "clever way", so integrals are separately finite:

$$\int_{n+1} \left(O_{n+1}\mathrm{d}\sigma^{\mathrm{R}} - O_n\mathrm{d}\sigma^{\mathrm{A}}\right) + \int_{n} \left(O_n\mathrm{d}\sigma^{\mathrm{V}} + O_n \int_{1} \mathrm{d}\sigma^{\mathrm{A}}\right) \tag{2}$$

- How to construct subtraction term $\mathrm{d}\sigma^{\mathrm{A}}$? → Use your favorite subtraction method

A Closer Look At the Virtual Correction

Virtual correction

$$\int_n \left(O_n \int_{\text{loop}} \mathrm{d}\sigma^{\tilde{\mathrm{V}}} - O_n \int_1 \mathrm{d}\sigma^{\mathrm{A}} \right) \tag{3}$$

$\rightarrow$ $\int_{\text{loop}} \mathrm{d}\sigma^{\tilde{\mathrm{V}}} = \mathrm{d}\sigma^{\mathrm{V}}$ is UV finite but IR divergent

- Loop integral in $D = 4 - 2\epsilon$ produces poles $\sim \frac{1}{\epsilon}, \frac{1}{\epsilon^2}$
- These are canceled by $\int_1 \mathrm{d}\sigma^{\mathrm{A}}$

$\Rightarrow$ Two divergent integrals, finite sum — Same situation as in the real case!

Numerical Subtraction

- Repeat the procedure for the virtual case, i.e.

$\rightarrow$ Need subtraction terms rendering each integral separately finite

$\rightarrow$ Perform loop integral with Monte Carlo

REGULARIZATION OF THE VIRTUAL CORRECTION (I)

- Disassemble the virtual correction:

$$\mathrm{d}\sigma^{\mathrm{V}} = 2\mathfrak{Re}\left(\mathcal{A}^{(0)^*}\mathcal{A}^{(1)}\right)\mathrm{d}\phi_n \qquad \mathcal{A}^{(1)} = \mathcal{A}^{(1)}_{\mathrm{bare}} + \mathcal{A}^{(1)}_{\mathrm{CT}} \tag{4}$$

- Define loop integrands, e.g.

$$\mathcal{A}^{(1)}_{\mathrm{bare}} = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\mathcal{G}^{(1)}_{\mathrm{bare}} \tag{5}$$

- Using subtraction terms $\mathcal{A}^{(1)}_{\mathrm{X}}$ rewrite loop integral[1]

$$\begin{aligned}\mathcal{A}^{(1)}_{\mathrm{bare}} + \mathcal{A}^{(1)}_{\mathrm{CT}} = \quad & \left(\mathcal{A}^{(1)}_{\mathrm{bare}} - \mathcal{A}^{(1)}_{\mathrm{soft}} - \mathcal{A}^{(1)}_{\mathrm{coll}} - \mathcal{A}^{(1)}_{\mathrm{UV}}\right) + \\ & \left(\mathcal{A}^{(1)}_{\mathrm{CT}} + \mathcal{A}^{(1)}_{\mathrm{soft}} + \mathcal{A}^{(1)}_{\mathrm{coll}} + \mathcal{A}^{(1)}_{\mathrm{UV}}\right)\end{aligned} \tag{6}$$

[1] S. Becker, C. Reuschle, and S. Weinzierl. "Numerical NLO QCD calculations." In: *Journal of High Energy Physics* 2010 (12 2010), pp. 1–61.

Regularization of the Virtual Correction (I)

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- Using subtraction terms $\mathcal{A}^{(1)}_{\mathrm{X}}$ rewrite loop integral[1]

$$\mathcal{A}^{(1)}_{\mathrm{bare}} + \mathcal{A}^{(1)}_{\mathrm{CT}} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\left(\mathcal{G}^{(1)}_{\mathrm{bare}} - \mathcal{G}^{(1)}_{\mathrm{soft}} - \mathcal{G}^{(1)}_{\mathrm{coll}} - \mathcal{G}^{(1)}_{\mathrm{UV}}\right) + \left(\mathcal{A}^{(1)}_{\mathrm{CT}} + \mathcal{A}^{(1)}_{\mathrm{soft}} + \mathcal{A}^{(1)}_{\mathrm{coll}} + \mathcal{A}^{(1)}_{\mathrm{UV}}\right) \tag{6}$$

rendering the first brace IR and UV finite in $D = 4$!

[1] S. Becker, C. Reuschle, and S. Weinzierl. "Numerical NLO QCD calculations." In: *Journal of High Energy Physics* 2010 (12 2010), pp. 1–61.

Regularization of the Virtual correction (II)

The remaining terms define the insertion term

$$\mathbf{L} \otimes \mathrm{d}\sigma^{\mathrm{B}} = 2\mathfrak{Re}\left[\mathcal{A}^{(0)^*}\left(\mathcal{A}^{(1)}_{\mathrm{CT}} + \mathcal{A}^{(1)}_{\mathrm{soft}} + \mathcal{A}^{(1)}_{\mathrm{coll}} + \mathcal{A}^{(1)}_{\mathrm{UV}}\right)\right]\mathrm{d}\phi_n \quad (7)$$

which together with the insertion term from the real correction

$$\mathbf{I} \otimes \mathrm{d}\sigma^{\mathrm{B}} = \int_1 \mathrm{d}\sigma^{\mathrm{A}} \quad (8)$$

is finite in $D = 4$ and can be integrated once and for all:

$$\mathbf{L} + \mathbf{I} = \frac{\alpha_s}{2\pi}\mathfrak{Re}\left[\sum_i \sum_{j \neq i} \mathbf{T}_i \mathbf{T}_j \left(\frac{\gamma_i}{\mathbf{T}_i^2} \ln \frac{|2p_i \cdot p_j|}{\mu_{\mathrm{UV}}^2} - \frac{\pi^2}{2}\theta(2p_i \cdot p_j)\right) + \sum_i \left(\gamma_i + K_i - \frac{\pi^2}{3}\mathbf{T}_i^2\right) - \frac{n-2}{2}\beta_0 \ln \frac{\mu_{\mathrm{UV}}^2}{\mu^2}\right] + \mathcal{O}(\epsilon) \quad (9)$$

SUMMARY: NUMERICAL SUBTRACTION

NLO Correction

$$\langle O\rangle^{\text{NLO}} = \langle O\rangle^{\text{NLO}}_{\text{real}} + \langle O\rangle^{\text{NLO}}_{\text{virtual}} + \langle O\rangle^{\text{NLO}}_{\text{insertion}} \tag{10}$$

- Real correction:

$$\langle O\rangle^{\text{NLO}}_{\text{real}} = \int\limits_{n+1} \left(O_{n+1}\mathrm{d}\sigma^{\mathrm{R}} + O_n \mathrm{d}\sigma^{\mathrm{A}}\right) \tag{11}$$

- Insertion:

$$\langle O\rangle^{\text{NLO}}_{\text{insertion}} = \int\limits_{n} (\mathbf{I} + \mathbf{L}) \otimes \mathrm{d}\sigma^{\mathrm{B}} \tag{12}$$

- Virtual correction:

$$\langle O\rangle^{\text{NLO}}_{\text{virtual}} = \underbrace{\int\limits_{n}\int\limits_{\text{loop}}}_{\text{MC}} \mathcal{O}_n 2\mathfrak{Re}\left[\mathcal{A}^{(0)^*}\left(\mathcal{G}^{(1)}_{\text{bare}} - \mathcal{G}^{(1)}_{\text{soft}} - \mathcal{G}^{(1)}_{\text{coll}} - \mathcal{G}^{(1)}_{\text{UV}}\right)\right] \tag{13}$$

Recursion Relations 1O1

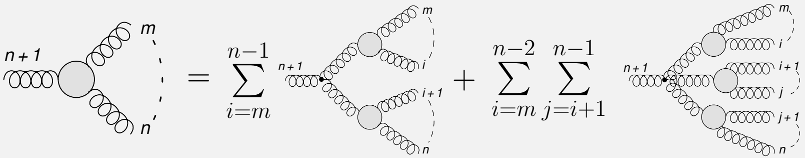
$$\langle O\rangle^{\text{NLO}}_{\text{virtual}} = \int_n \mathrm{d}\phi_n \mathcal{O}_n 2\mathfrak{Re}\left[A^{(0)*} \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(G^{(1)}_{\text{bare}} - G^{(1)}_{\text{soft}} - G^{(1)}_{\text{coll}} - G^{(1)}_{\text{UV}}\right)\right]$$

We make use of the color decomposition:

$$\mathcal{A}^{(0)} = \sum_i C_i A^{(0)}_i\left(g_{\sigma_i(1)}, g_{\sigma_i(2)}, \ldots, g_{\sigma_i(n)}\right) \tag{14}$$

with

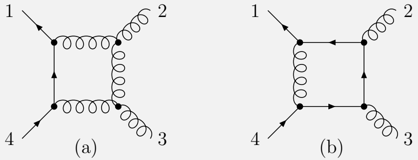
- $\sum_i$ the sum running over permutations σ_i of the outgoing particles,
- C_i color factors (string of Kronecker-deltas), and
- $A^{(0)}_i$ the *partial amplitudes* with a fixed permutation of the external legs:



BARE AMPLITUDE RECURSION: G_{BARE} (I)

$$\langle O \rangle^{\text{NLO}}_{\text{virtual}} = \int_n \mathrm{d}\phi_n \mathcal{O}_n 2\mathfrak{Re}\left[A^{(0)*} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left(G^{(1)}_{\text{bare}} - G^{(1)}_{\text{soft}} - G^{(1)}_{\text{coll}} - G^{(1)}_{\text{UV}} \right) \right]$$

Loop integrands can be computed similarly to Born-level ME; but …



…loop propagators in partial amplitudes are not uniquely determined. We can further decompose the amplitudes, using *primitive amplitudes*:

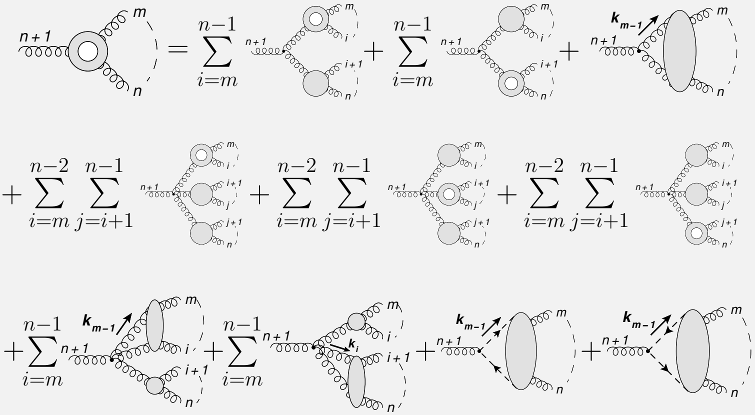
$$\mathcal{A}^{(1)} = \sum_{i,j} C_j A^{(1)}_{\text{bare},i,j} = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \sum_{i,j} C_j G^{(1)}_{\text{bare},i,j} \quad (15)$$

In $G^{(1)}_{\text{bare},j,j}$ propagators are uniquely defined!

BARE AMPLITUDE RECURSION: G_{BARE} (II)

$$\langle O\rangle^{\text{NLO}}_{\text{virtual}} = \int_n \mathrm{d}\phi_n \mathcal{O}_n 2\mathfrak{Re}\left[A^{(0)^*}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(G^{(1)}_{\text{bare}} - G^{(1)}_{\text{soft}} - G^{(1)}_{\text{coll}} - G^{(1)}_{\text{UV}}\right)\right]$$

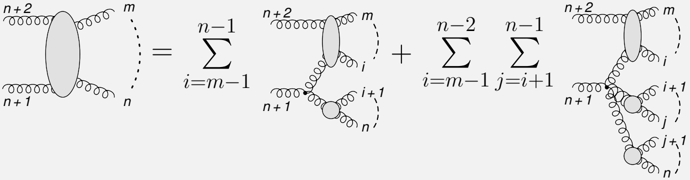
Recursion relation for gluons:



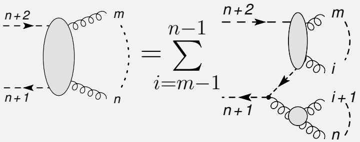
BARE AMPLITUDE RECURSION: G_{BARE} (III)

$$\langle O\rangle^{\text{NLO}}_{\text{virtual}} = \int_n \mathrm{d}\phi_n \mathcal{O}_n 2\mathfrak{Re}\left[A^{(0)^*}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(G^{(1)}_{\text{bare}} - G^{(1)}_{\text{soft}} - G^{(1)}_{\text{coll}} - G^{(1)}_{\text{UV}}\right)\right]$$

Tensor current for gluons:



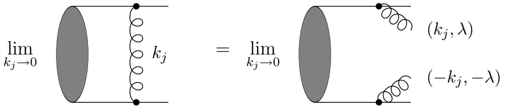
Tensor current for ghosts:



SUBTRACTION TERMS (I): $G^{(1)}_{\mathrm{SOFT}}$

$$\langle O\rangle^{\mathrm{NLO}}_{\mathrm{virtual}} = \int_n \mathrm{d}\phi_n \mathcal{O}_n 2\Re\mathfrak{e}\left[A^{(0)^*}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(G^{(1)}_{\mathrm{bare}} - G^{(1)}_{\mathrm{soft}} - G^{(1)}_{\mathrm{coll}} - G^{(1)}_{\mathrm{UV}}\right)\right]$$

Soft case[2]: $m_i = 0, \quad p_j^2 = m_{j-1}^2, \quad p_{j+1}^2 = m_{j+1}^2$



$$G^{(1)}_{\mathrm{soft}} = 4i \sum_{j\in I_g} \frac{p_j \cdot p_{j+1}}{\left(k_{j-1}^2 - m_{j-1}^2\right) k_j^2 \left(k_{j+1}^2 - m_{j+1}^2\right)} A_j^{(0)} \tag{16}$$

I_g Set of indices j where j corresponds to a gluon propagator

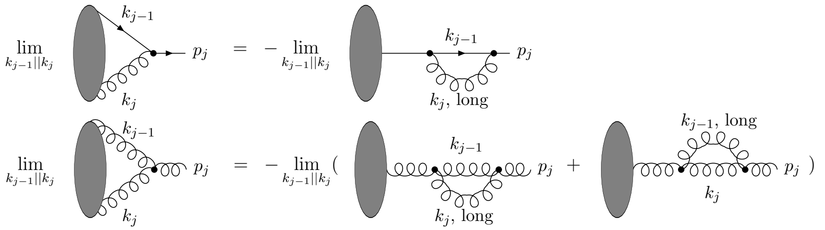
$A_j^{(0)}$ Amplitudes without propagator j (tree-level)

[2] M. Assadsolimani, S. Becker, and S. Weinzierl. "A Simple formula for the infrared singular part of the integrand of one-loop QCD amplitudes." In: *Physical Review D* 81 (2010), p. 094002.

Subtraction Terms (II): $G^{(1)}_{\text{COLL}}$

$$\langle O \rangle^{\text{NLO}}_{\text{virtual}} = \int_n d\phi_n \mathcal{O}_n 2\mathfrak{Re}\left[A^{(0)^*} \int \frac{d^4k}{(2\pi)^4}\left(G^{(1)}_{\text{bare}} - G^{(1)}_{\text{soft}} - G^{(1)}_{\text{coll}} - G^{(1)}_{\text{UV}}\right)\right]$$

Collinear case[3]: $p_j^2 = 0, \quad m_{i-1} = 0, \quad m_i = 0$

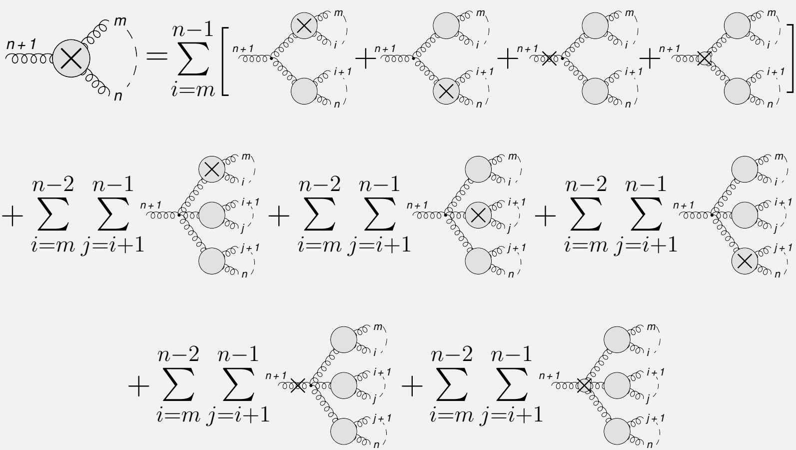


$$G^{(1)}_{\text{coll}} = -2i \sum_{j \in I_g}\left[\frac{S_j g_{\text{UV}}\left(k^2_{j-1}, k^2_j\right)}{k^2_{j-1}k^2_j} + \frac{S_{j+1} g_{\text{UV}}\left(k^2_j, k^2_{j+1}\right)}{k^2_j k^2_{j+1}}\right] A^{(0)}_j \qquad (17)$$

[3] M. Assadsolimani, S. Becker, and S. Weinzierl. "A Simple formula for the infrared singular part of the integrand of one-loop QCD amplitudes." In: *Physical Review D* 81 (2010), p. 094002.

SUBTRACTION TERMS (IV): $G^{(1)}_{\mathrm{UV}}$

$$\langle O\rangle^{\mathrm{NLO}}_{\mathrm{virtual}} = \int_n \mathrm{d}\phi_n \mathcal{O}_n 2\mathfrak{Re}\left[A^{(0)^*}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(G^{(1)}_{\mathrm{bare}} - G^{(1)}_{\mathrm{soft}} - G^{(1)}_{\mathrm{coll}} - G^{(1)}_{\mathrm{UV}}\right)\right]$$



Contour Deformation: Integration in $\mathbb{C}^4$

$$\langle O\rangle^{\text{NLO}}_{\text{virtual}} = \int_n \mathrm{d}\phi_n \mathcal{O}_n 2\mathfrak{Re}\left[A^{(0)^*}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(G^{(1)}_{\text{bare}} - G^{(1)}_{\text{soft}} - G^{(1)}_{\text{coll}} - G^{(1)}_{\text{UV}}\right)\right]$$

→ Loop-Integral is finite, but loop propagators can still go on-shell:

$$I = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{R(k)}{\prod_{j=0}^{n-2}\left(k_j^2 - m_j^2 + i\delta\right)} \tag{18}$$

- "Classic solution" are Feynman/Schwinger parameters → Reduce problem to effectively one propagator → Wick-Rotation → MI
- Here: *Direct* Contour Deformation[4] with $k = \tilde{k} + i\kappa\left(\tilde{k}\right)$:

$$I = \int \frac{\mathrm{d}^4\tilde{k}}{(2\pi)^4}\left|\frac{\partial k^\mu}{\partial \tilde{k}^\nu}\right| \frac{R\left(k\left(\tilde{k}\right)\right)}{\prod_{j=0}^{n-2}\left(\tilde{k}_j^2 - m_j^2 - \kappa^2 + 2i\tilde{k}_j\cdot\kappa\right)} \tag{19}$$

- Match $i\delta$ prescription by constructing κ such that:

$$\tilde{k}_j^2 - m_j^2 = 0 \quad \rightarrow \quad \tilde{k}_j\cdot\kappa \geq 0 \tag{20}$$

[4] W. Gong, Z. Nagy, and D. E. Soper. "Direct numerical integration of one-loop Feynman diagrams for N-photon amplitudes." In: *Physical Review D* 79 (2009), p. 033005; S. Becker, C. Reuschle, and S. Weinzierl. "Efficiency Improvements for the Numerical Computation of NLO Corrections." In: *Journal of High Energy Physics* 1207 (2012), p. 090.

APPLICATION FOR e^+e^- ANNIHILATION

We computed[5] the jet ratios for $e^+e^- \to n$ jets in the large N_c limit (LC) with NLO acccuracy.

- Jet resolution variable for the Durham algorithm:

$$y_{ij} = \frac{2\min\left(E_i, E_j\right)(1-\cos\theta_{ij})}{Q^2} \tag{21}$$

- Jet-rate:

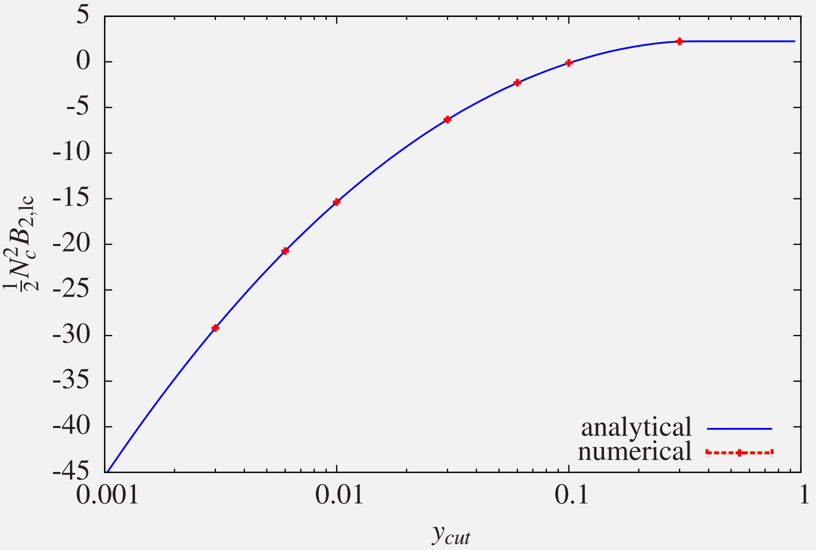
$$R_n(\mu) = \frac{\sigma_{\text{n jets}}(\mu)}{\sigma_{\text{tot}}(\mu)} \tag{22}$$

Using the approximation $\sigma_{\text{tot}} \approx \sigma_{\text{2 jets}}$ @ LO we obtain:

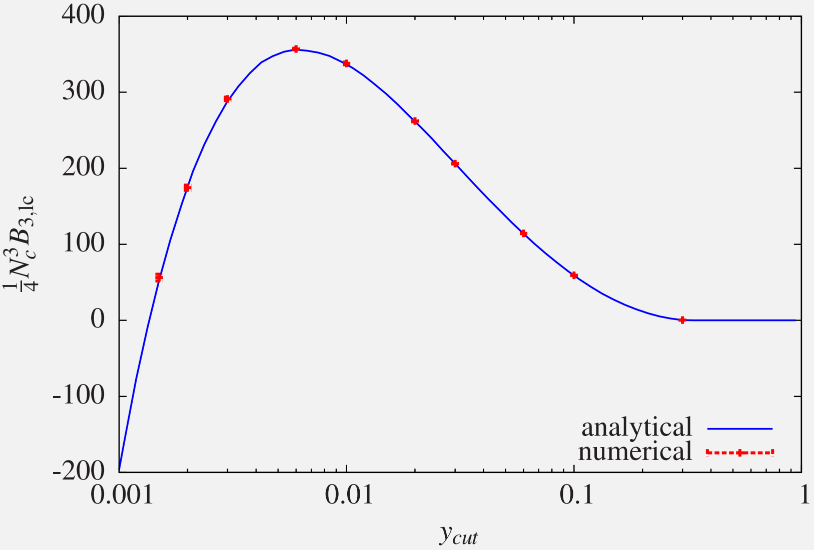
$$R_n(\mu) \approx \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-2} \underbrace{A_n(\mu)}_{\text{LO}} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-1} \underbrace{B_n(\mu)}_{\text{NLO}} \tag{23}$$

[5] S. Becker, D. Goetz, C. Reuschle, C. Schwan, and S. Weinzierl. “NLO results for five, six and seven jets in electron-positron annihilation.” In: *Physical Review Letters* 108 (2012), p. 032005.

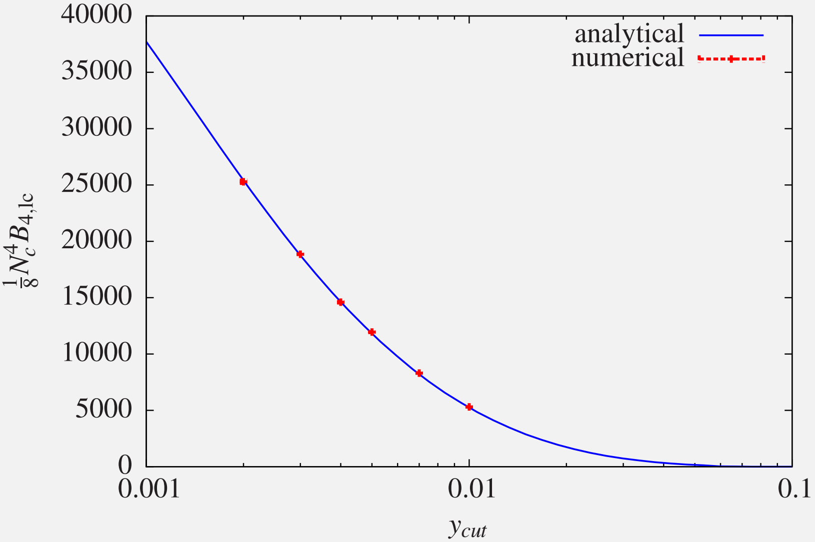
Two Jet-Rate: $e^+e^- \longrightarrow 2$ jets



Three Jet-Rate: $e^+e^- \longrightarrow 3$ jets



FOUR JET-RATE: $e^+e^- \longrightarrow 4$ JETS



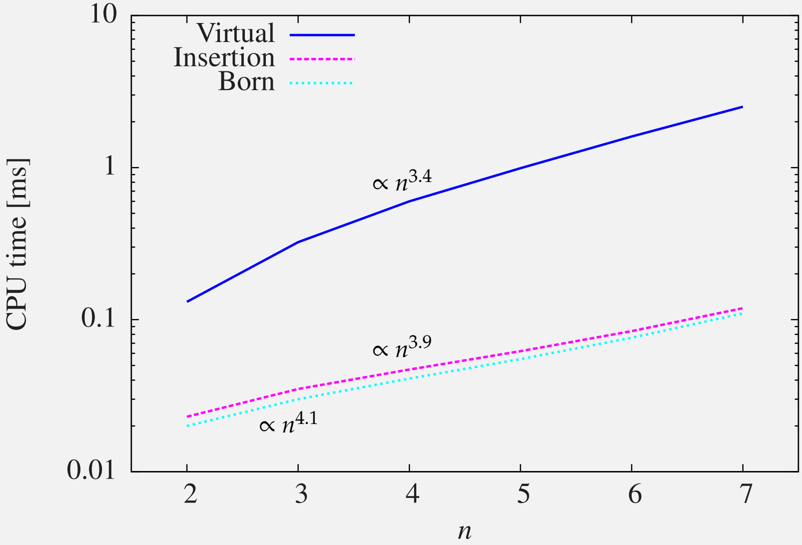
More Jets: $e^+e^- \longrightarrow 5,6,7$ jets

For the following results no analytic calculations were available, results for $n = 6, 7$ are calculated with this method for the first time:

n	y_{cut}	$\frac{N_c^n}{2^{n-1}} B_{n,\text{lc}}$
5	0.001	$(4.275 \pm 0.006) \times 10^5$
	0.002	$(1.050 \pm 0.026) \times 10^6$
	0.0006	$(1.84 \pm 0.15) \times 10^6$
6	0.001	$(1.46 \pm 0.04) \times 10^7$
	0.0006	$(3.88 \pm 0.18) \times 10^7$
7	0.0006	$(5.4 \pm 0.3) \times 10^8$

Time for 7 jet rate: $\sim$ 5 days @ 200 cores

Scaling behavior: $e^+e^- \longrightarrow n$ jets



$\Rightarrow$ Practical limitation arises from MC statistics only!

SUMMARY

- Extension of numerical subtraction to virtual part
- Needed subtraction terms have been derived and optimized
- Virtual part is computed using Berends-Giele-like recursion relations; no diagrams needed!
- (Direct) contour deformation is needed
- Methods has been shown to work and is capable of efficiently computing cross sections for many jets

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Thanks!

- Assadsolimani, M., S. Becker, and S. Weinzierl. "A Simple formula for the infrared singular part of the integrand of one-loop QCD amplitudes." In: *Physical Review D* 81 (2010), p. 094002.
- Becker, S., C. Reuschle, and S. Weinzierl. "Efficiency Improvements for the Numerical Computation of NLO Corrections." In: *Journal of High Energy Physics* 1207 (2012), p. 090.
- – ."Numerical NLO QCD calculations." In: *Journal of High Energy Physics* 2010 (12 2010), pp. 1–61.
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- Dittmaier, S., P. Uwer, and S. Weinzierl. "NLO QCD corrections to t anti-t + jet production at hadron colliders." In: *Physical Review Letters* 98 (2007), p. 262002.
- Gong, W., Z. Nagy, and D. E. Soper. "Direct numerical integration of one-loop Feynman diagrams for N-photon amplitudes." In: *Physical Review D* 79 (2009), p. 033005.