Kalman track fitting for nonuniform magnetic field

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Outline

Motivation

- It is important to get the precise momentum result at nonuniform magnetic field: small bias and reasonable error.
- A Kalman tracking algorithm, Kaltest, works well for uniform magnetic field.

We can try to update KalTest for nonuniform magnetic field.

KalTest

- KalTest is a Kalman filter tracking package written in C++.
- Assume the magnetic field is parallel with z axis of coordinate, helix can be plot as:

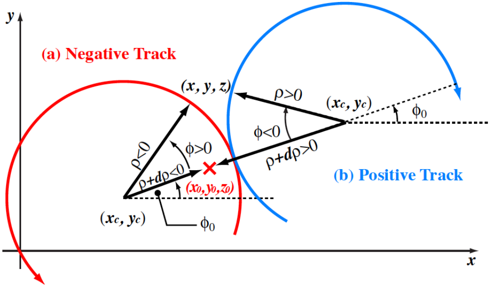


Figure 1 : Helix model(from KalTest mannul)

Parametrization of helix

- Then helix is parametrized by

$$\begin{cases} x &= x_0 + d_\rho \cos\phi_0 + \frac{\alpha}{\kappa}\left[\cos\phi_0 - \cos(\phi_0+\phi)\right] \\ y &= y_0 + d_\rho \sin\phi_0 + \frac{\alpha}{\kappa}\left[\sin\phi_0 - \sin(\phi_0+\phi)\right] \\ z &= z_0 + d_z - \frac{\alpha}{\kappa}\tan\lambda\cdot\phi \end{cases} \quad (1)$$

- Define the state as:

$$\boldsymbol{a}_k = \left(\ d_\rho, \phi_0, \kappa, d_z, \tan\lambda\ \right)^T \quad (2)$$

given a pivot and state vector, a helix is defined.

Kalman Filter

Kalman filter algorithm contains two steps at each site. The state vector is updated according to the current measurement and last state vector.
Prediction:

$$\boldsymbol{a}_k^{k-1} = \boldsymbol{f}_{k-1}(\boldsymbol{a}_{k-1}) \tag{3}$$

$$\boldsymbol{F}_{k-1} = \frac{\partial \boldsymbol{f}_{k-1}}{\partial \boldsymbol{a}_{k-1}} = \frac{\partial \boldsymbol{a}_k^{k-1}}{\partial \boldsymbol{a}_{k-1}} \tag{4}$$

$\boldsymbol{f}_k$ is propagation function, and $\boldsymbol{F}_k$ is propagation matrix.
Filtering:

$$\boldsymbol{a}_k = \boldsymbol{a}_k^{k-1} + \boldsymbol{K}_k \left(\boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^{k-1}) \right) \tag{5}$$

in which, $\boldsymbol{K}_k$ is the gain matrix, and $\boldsymbol{h}_k$ is the projection function.

Basic idea of updating propagation

- To use the existing track model in the nonuniform magnetic field situation, we have to transform the frame to make sure that z axis points to the direction of magnetic field.
- Assume that the magnetic field between two layers is uniform, the whole propagation procedure between two sites is:

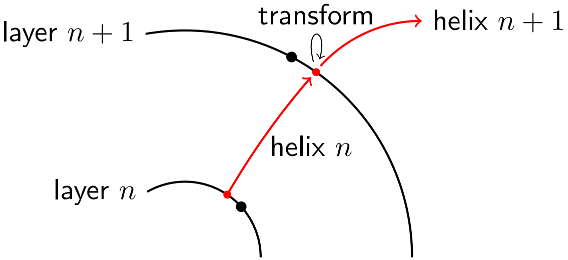


Figure 2 : Propagation

The formulas

- The propagation procedure can be represented by four equation:

$$\begin{cases} \boldsymbol{a}' &= \boldsymbol{f}_k(\boldsymbol{a}_k) \\ \boldsymbol{p} &= \boldsymbol{c}(\boldsymbol{a}') \\ \boldsymbol{p}' &= \boldsymbol{T}(\boldsymbol{p}) \\ \boldsymbol{a}'' &= \boldsymbol{c}^{-1}(\boldsymbol{p}') \end{cases} \tag{6}$$

- We have known $\boldsymbol{f}_k$ in the current KalTest; $\boldsymbol{c}$ is a function to calculate momentum from state vector, and $\boldsymbol{c}^{-1}$ is its inverse function; $\boldsymbol{T}$ is actually the rotation matrix.

Transforming the frame

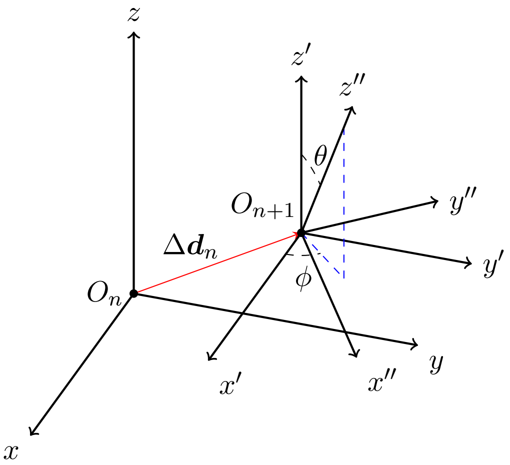


Figure 3 : Transformation

Rotation matrix

Define the transforming operation is to make the z axis rotate θ angle in the $z\prime Oz\prime\prime$ plane, then rotation matrix is equivalent to the product of three sequential rotations:

$$\boldsymbol{\Delta R} = \boldsymbol{\Delta R}_z(-\phi)\boldsymbol{\Delta R}_y(\theta)\boldsymbol{\Delta R}_z(\phi) \tag{7}$$

Since the rotation is passive:

$$\boldsymbol{\Delta R}_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{\Delta R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Transformation of vectors

- Concerning shift, a position vector in frame n $\boldsymbol{x}_n$ can be transformed to frame $n+1$ by

$$\boldsymbol{x_{n+1}} = \boldsymbol{\Delta R_n}(\boldsymbol{x_n} - \boldsymbol{\Delta d_n}) \tag{8}$$

- We may also want to transform from local to global and vice versa:

$$\boldsymbol{x_{n+1}} = \boldsymbol{R_n}(\tilde{\boldsymbol{x}} - \boldsymbol{d_n}) \tag{9}$$

In equation (8), $\tilde{\boldsymbol{x}}$ is defined in the global frame, $\boldsymbol{x}_n$ is defined in frame n, which can be transformed to this frame from global by:

$$\boldsymbol{x_n} = \boldsymbol{R_{n-1}}(\tilde{\boldsymbol{x}} - \boldsymbol{d_{n-1}}) \tag{10}$$

Transformation of vectors

Substitute it into equation (8)

$$\boldsymbol{x_{n+1}} = \boldsymbol{\Delta R_n}[\boldsymbol{R_{n-1}}(\tilde{\boldsymbol{x}} - \boldsymbol{d_{n-1}}) - \boldsymbol{\Delta d_n}]$$

that means

$$\begin{cases} \boldsymbol{R}_n &= \Delta \boldsymbol{R}_n \boldsymbol{R}_{n-1} \\ \boldsymbol{d}_n &= \boldsymbol{d}_{n-1} + \boldsymbol{R}_{n-1}^{-1} \Delta \boldsymbol{d}_n \end{cases} \quad (11)$$

- In local frame n, the magnetic field is

$$\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{R}_n \boldsymbol{B}(\tilde{\boldsymbol{x}}) \quad (12)$$

Momentum and state vector

We can calculate momentum from state and vice versa by Figure 4:

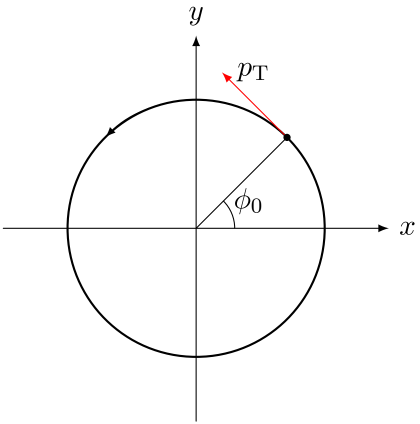


Figure 4 : Momentum and state vector

momentum vector

The momentum vector in last frame is:

$$\boldsymbol{p} = \begin{pmatrix} -\frac{1}{|\kappa|}\sin\phi_0 \\ \frac{1}{|\kappa|}\cos\phi_0 \\ \frac{1}{|\kappa|}\tan\lambda \end{pmatrix} \tag{13}$$

The rotated vector in the new frame is:

$$\boldsymbol{p}' = \Delta \boldsymbol{R}\boldsymbol{p} \tag{14}$$

We have taken particle charge $|Q| = 1$. We don't know the sign of charge before track fitting, we can get it from the sign of κ.

Calculate state vector from momentum

From the rotated momentum, the new state vector is:

$$\boldsymbol{a}'' = \begin{pmatrix} d_\rho \\ \mathrm{atan2}(-p'_x, p'_y) \\ \frac{s_\kappa}{\left(p'^2_x + p'^2_y\right)^{\frac{1}{2}}} \\ d_z \\ \frac{p'_z}{\left(p'^2_x + p'^2_y\right)^{\frac{1}{2}}} \end{pmatrix} \tag{15}$$

Obviously the components d_ρ and d_z are 0. In the equation, $s_\kappa \equiv \mathrm{sgn}(\kappa)$, and we keep the sign of κ not changed, this is true if magnetic field changes moderately.

The propagator matrix

The modified propagator matrix is

$$\boldsymbol{F}^m_{k-1} = \frac{\partial \boldsymbol{a}''}{\partial \boldsymbol{a}} \tag{16}$$

$$\boldsymbol{F}^m_{k-1} = \frac{\partial \boldsymbol{a}''}{\partial \boldsymbol{p}'}\frac{\partial \boldsymbol{p}'}{\partial \boldsymbol{p}}\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{a}'}\frac{\partial \boldsymbol{a}'}{\partial \boldsymbol{a}} = \boldsymbol{F}^r_{k-1}\boldsymbol{F}_{k-1} \tag{17}$$

The matrix $\boldsymbol{F}_{k-1}$ is known. We just use non-zero components to calculate the propagation matrix, i.e, $\boldsymbol{F}^r_{k-1}$ can be simplified to a 3×3 matrix. And then we it can be extend it to be a 5×5 matrix.

The covariance matrix

- The original equation of covariance prediction is:

$$\boldsymbol{C}_k^{k-1} = \boldsymbol{F}_{k-1}\boldsymbol{C}_{k-1}\boldsymbol{F}_{k-1}^T + \boldsymbol{Q}_{k-1} \tag{18}$$

  in which $\boldsymbol{Q}$ is covariance matrix from multiple scattering, and

$$\boldsymbol{Q}_{k-1} = \boldsymbol{F}_{k-1}\boldsymbol{Q}_m\boldsymbol{F}_{k-1}^T \tag{19}$$

- We believe there is no noise when transforming frame, so the modified covariance matrix should be:

$$\boldsymbol{C}_k^{k-1,m} = \boldsymbol{F}_{k-1}^r\boldsymbol{C}_k^{k-1}\boldsymbol{F}_{k-1}^{rT} \tag{20}$$

  Then

$$\boldsymbol{C}_k^{k-1,m} = \boldsymbol{F}_{k-1}^m\boldsymbol{C}_k\boldsymbol{F}_{k-1}^{mT} + \boldsymbol{F}_{k-1}^m\boldsymbol{Q}_m\boldsymbol{F}_{k-1}^{mT} \tag{21}$$

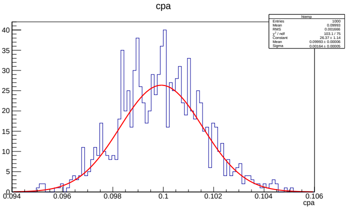
  for one step.

Implementation

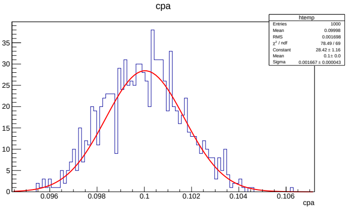
- A frame class, which contains rotation matrix and shift vector. Every site has a frame.
- Modified helix class: all the frame changing related code is kept in this class.
- Generator: calculating the crossing point of helix and measurement layer and smearing.
  - Newtonian method.
  - Runge-Kutta method and bisection method.
- The magnetic field can be easily accessed.

Test

- Firstly, compare the momentum distribution at uniform magnetic from two KalTest versions, with codition: 10GeV, N=50, B=3.5, $\sigma_{r\phi} = 0.01\text{mm}$



(a) Original version



(b) Updated version

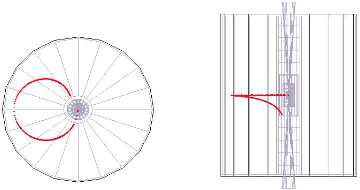
- The reconstructed results(mean and sigma) are the same as expected.

Event display

Suppose the nonuniform magnetic field is

$$\begin{cases} B_x &= B_0 kxz \\ B_y &= B_0 kyz \\ B_z &= B_0(1-kz^2) \end{cases}$$

in which $k = \frac{k_0}{z_m r_m}$. If $B_0 = 3\mathrm{T}$, $k_0 = 30$, $z_m = r_m = 300\mathrm{cm}$, the simulated track $(p = 0.8\mathrm{GeV}, \tan\lambda = 0)$ is



(a) $r\phi$ (b) rz

Figure 5 : Track in nonuniform b field

Comparison of reconstructed momentum

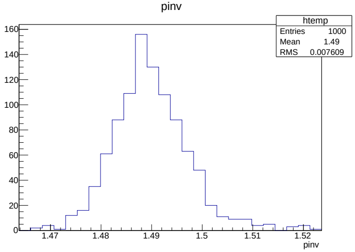
Tested with $\boldsymbol{B} = (kx, ky, (3.5^2 - (kx)^2 - (ky)^2)^{\frac{1}{2}})$

Table 1 : Results at different conditions at 5GeV(G - generator, R - reconstruction, U/N - uniform/nonuniform magnetic field)

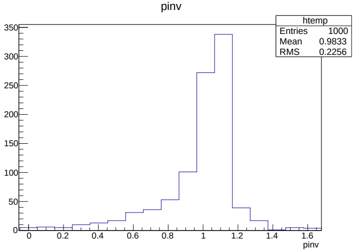
$k, \tan\lambda$	G.R.	$1/p$	$1/p_t$
$10^{-3}, 0$	UU	0.2 ± 0.00018	0.2 ± 0.00018
	NU	0.1973 ± 0.00018	0.1973 ± 0.00018
	NN	0.2 ± 0.00018	0.2095 ± 0.00019
$2 \times 10^{-3}, 0$	UU	0.2 ± 0.00018	0.2 ± 0.00018
	NU	0.1888 ± 0.00018	0.1973 ± 0.00017
	NN	0.2 ± 0.00019	0.2488 ± 0.00024
$10^{-3}, 0.1$	UU	0.199 ± 0.00018	0.2 ± 0.00018
	NU	0.1932 ± 0.00018	0.1942 ± 0.00018
	NN	0.199 ± 0.00018	0.2162 ± 0.00020

Smoothing

P=1GeV, and in order to see to effect of smoothing, the density of material is set to very large.



(a) without smoothing



(b) with smoothing

Figure 6 : Results with M.S. and dE/dx switched on

Test with RK track generator

- As for uniform magnetic field, the updated KalTest also can gives expected momentum results which is the same with original KalTest;
- Test in nonuniform magnetic field:
  - Field: $B_r = 3.5(r/1500)^2, B_\phi = 0, B_z = 3.5\sqrt{1-(r/1500)^4}$
  - Detector: 200 measurement layers; step between two nearby layer is 6 mm; $r\phi$ resolution is 100 $\mu\mathrm{m}$.
  - Results:

Table 2 : Results calculated by the updated and original KalTest

Momentum	Ver.	$1/p$	$\delta(1/p)$
5GeV	original	$1.937e-01$	$1.138e-04$
	updated	$1.999e-01$	$1.147e-04$
10GeV	original	$9.687e-02$	$1.141e-04$
	updated	$9.995e-02$	$1.158e-04$

Summary

- We have updated KalTest for nonuniform magnetic field, using the existing helix track model.
- According to the test, the updated algorithm can get improved momentum results.

Backup slides

In equation (17),

$$\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{a}'} = \begin{pmatrix} -\frac{1}{|\kappa|}\cos\phi_0 & \frac{s_\kappa}{\kappa^2}\sin\phi_0 & 0 \\ -\frac{1}{|\kappa|}\sin\phi_0 & -\frac{s_\kappa}{\kappa^2}\cos\phi_0 & 0 \\ 0 & -\frac{s_\kappa}{\kappa^2}\tan\lambda & \frac{1}{|\kappa|} \end{pmatrix} \tag{22}$$

$$\frac{\partial \boldsymbol{p}'}{\partial \boldsymbol{p}} = \Delta \boldsymbol{R} \tag{23}$$

$$\frac{\partial \boldsymbol{a}''}{\partial \boldsymbol{p}'} = \begin{pmatrix} -\frac{p_y}{p_\mathrm{T}^2} & \frac{p_x}{p_\mathrm{T}^2} & 0 \\ -\frac{s_\kappa p_x}{p_\mathrm{T}^3} & -\frac{s_\kappa p_y}{p_\mathrm{T}^3} & 0 \\ -\frac{p_x p_z}{p_\mathrm{T}^3} & -\frac{p_y p_z}{p_\mathrm{T}^3} & \frac{1}{p_\mathrm{T}^3} \end{pmatrix} \tag{24}$$

Let

$$\boldsymbol{M} = \frac{\partial \boldsymbol{a}''}{\partial \boldsymbol{p}'} \frac{\partial \boldsymbol{p}'}{\partial \boldsymbol{p}} \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{a}'}$$

then

$$\boldsymbol{F}_r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & M_{00} & M_{01} & 0 & M_{02} \\ 0 & M_{10} & M_{11} & 0 & M_{12} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & M_{20} & M_{21} & 0 & M_{22} \end{pmatrix} \tag{25}$$