Three-dimensional effects in free-electron laser theory

Panagiotis Baxevanis

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Outline

- Introduction
- Transverse and longitudinal equations of motion
- Vlasov-Maxwell equations
- Eigenmode equation
- Parabolic model
- Variational solution and Ming Xie's formula

Introduction

- The previous analysis focused on 1D FEL theory. This resulted in a relatively simple and illuminating development which provides good insight into the physics of the FEL.

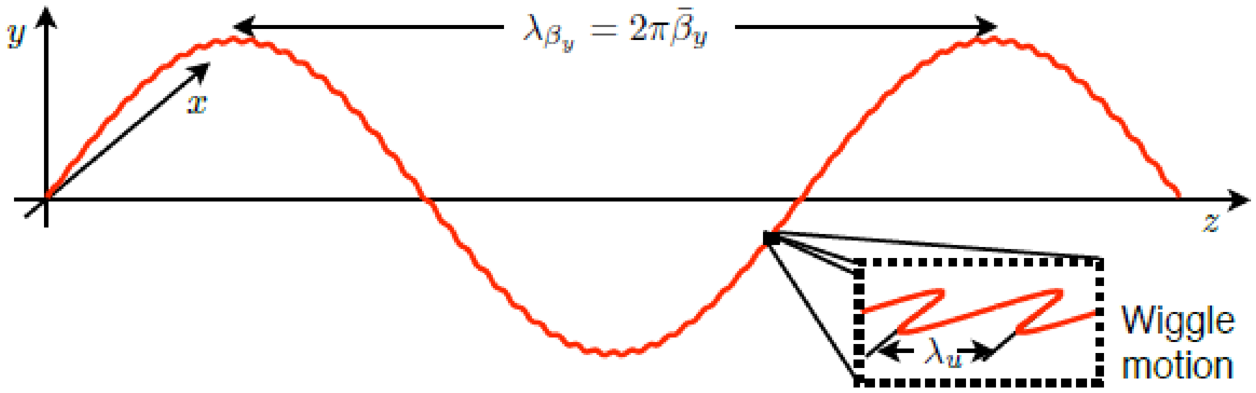
- However, the neglected three-dimensional (3D) effects due to radiation diffraction, e-beam emittance and undulator focusing can significantly affect the operation of the FEL, especially in the X-ray region.

- Here, we provide a discussion of 3D effects, with special emphasis on the high-gain regime of the interaction.

- Most of the material is drawn from the FEL notes of Zhirong Huang, Kwang-Je Kim and Ryan Lindberg (see USPAS-2013 course materials for more details).

Transverse equations of motion

- In the 3D picture, the averaged electron trajectories are no longer parallel to the undulator axis. In fact, the electrons execute a slow, large-amplitude transverse oscillatory motion (betatron oscillation) upon which the fast, small-amplitude wiggle motion is superimposed.

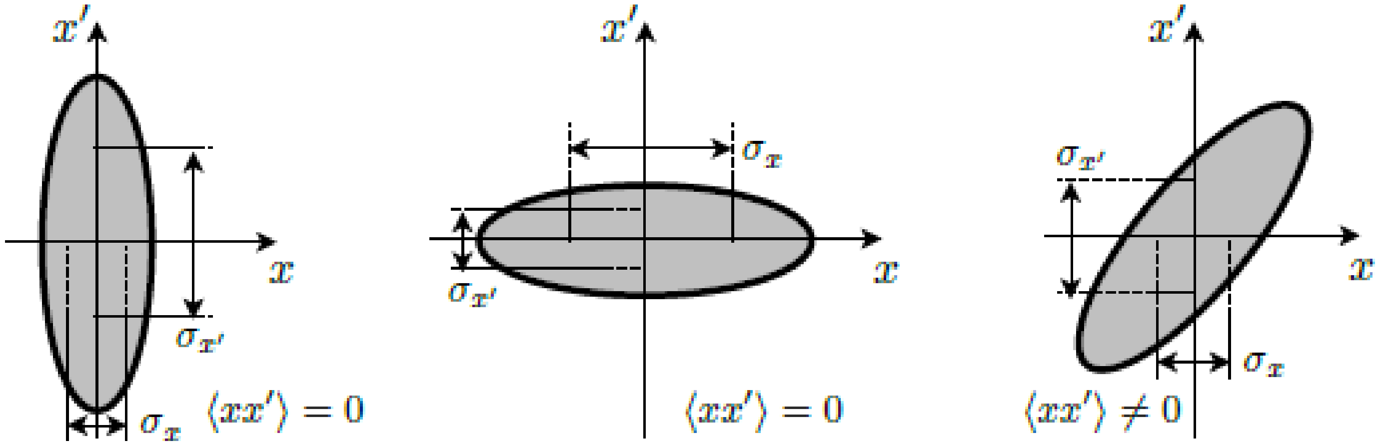


$$x = x_w + x_\beta$$

$$y = y_\beta$$

- As a result, the electron beam occupies a non-zero area in transverse phase space. A measure of this area is the transverse emittance, which (for uncoupled systems) is defined (say for the x-direction) as $\varepsilon_x \equiv \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$

- For linear focusing forces, emittance is an invariant of the motion (the shape of the phase space picture changes but not its area).



- The full magnetic field of a flat-pole undulator (i.e. the form that satisfies Maxwell's equations) has a longitudinal component as well as a transverse one. Both field components depend on y.

$$B(x;z) = -B_0 \cosh(k_u y)\sin(k_u z)\hat{y} - \underbrace{B_0 \sinh(k_u y)\cos(k_u z)\hat{z}}_{\text{disregarded in 1D theory}}.$$

$k_u = 2\pi/\lambda_u$

λ_u is the undulator period

- The equation of motion for an electron in the field of the undulator is given by

$$-e[v \times \boldsymbol{B}] = -e \begin{bmatrix} B_z \dfrac{dy}{dt} - B_y \dfrac{dz}{dt} \\ -B_z \dfrac{dx}{dt} \\ B_y \dfrac{dx}{dt} \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \gamma m \dfrac{dx}{dt} \\ \gamma m \dfrac{dy}{dt} \\ \gamma m \dfrac{dz}{dt} \end{bmatrix}.$$

- The horizontal (x) component can be integrated to give an expression for the wiggle velocity:

$$x' \equiv \frac{dx}{dz} = \frac{dx/dt}{dz/dt} \approx \frac{K}{\gamma} \cos(k_u z) \cosh(k_u y) + x'(0),$$

- Using the above, the vertical (y) component of the equation of motion becomes

$$\begin{aligned} y'' &\approx -\frac{K^2 k_u}{\gamma_r^2} \cos^2(k_u z) \sinh(k_u y) \cosh(k_u y) \\ &\approx -\left(\frac{K k_u}{\gamma_r}\right)^2 \cos^2(k_u z) y \end{aligned}$$

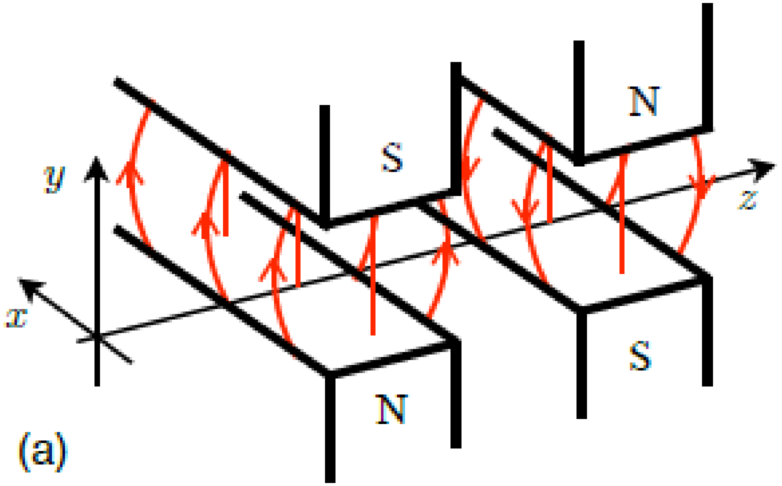
$$\boldsymbol{K = \frac{eB_0}{mck_u}}$$

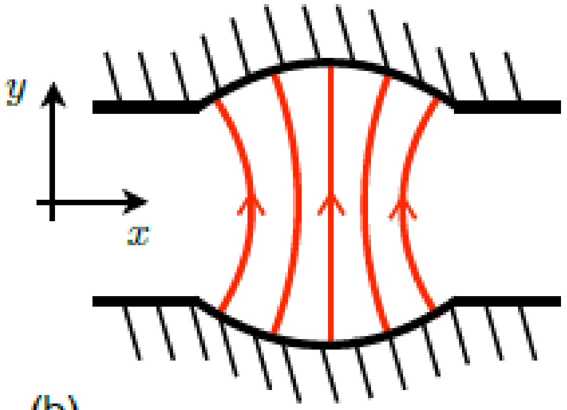
undulator parameter

- Averaging over the wiggle motion yields a harmonic oscillator equation for the vertical motion (in the horizontal direction, there is no natural focusing so the motion is simply a drift):

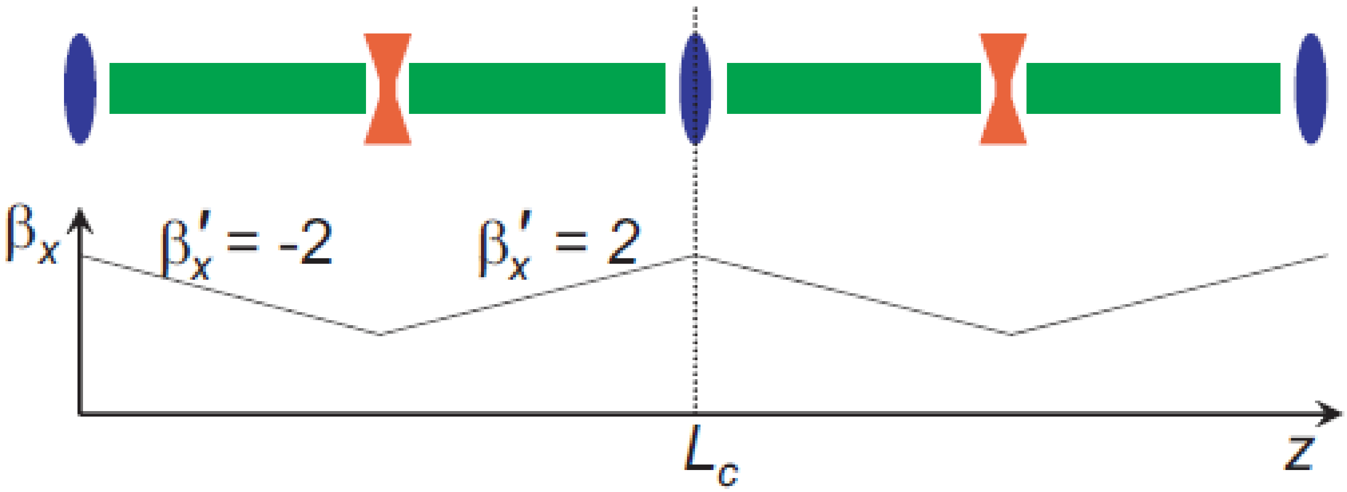
$$y'' = -k_{n0}^2 y, \quad \text{with} \quad k_{n0} = \frac{K k_u}{\sqrt{2}\gamma} \equiv \frac{1}{\beta_n}.$$

- Using an undulator with a parabolically-shaped pole face introduces focusing in the horizontal (x) direction as well (see homework problems).





- Typically, natural focusing (~1/γ) is not sufficient in an XFEL and is supplemented by external focusing. The latter is usually implemented by means of a FODO lattice, with quadrupoles placed in between the undulator segments.



- In general, this results in z-dependent focusing forces.

- In the case of small phase advance per cell, a smooth focusing approximation is applicable. This results in a symmetric, constant focusing strength.

$$\frac{dx}{dz} = p, \qquad \frac{dp}{dz} = -k_\beta^2 x,$$

$\boldsymbol{x} = (x, y)$ is the transverse position vector

Longitudinal equations of motion

- Another major departure from the 1D picture is the inclusion of the radiation diffraction. For linearly polarized radiation (along the x direction), the electric field is

$$E_x = \tilde{E}(x,t;z)e^{ik_1(z-ct)} + c.c.$$
$$= \int d\nu\, e^{i\nu k_1(z-ct)} \underbrace{E_\nu(x;z)}_{} e^{i\Delta\nu k_u z} + c.c.$$

Slowly-varying Fourier amplitude
(note the transverse dependence) $|\partial E_\nu/\partial z| \ll k_1 |E_\nu|$

$\nu = \omega/\omega_1 \quad \Delta\nu \equiv (\nu - 1)$

- Scaled frequency/detuning variables
- The FEL effect occurs near the resonant frequency ω_1 so $\nu \sim 1$

$k_1 = 2\pi/\lambda_1$

- As in 1D theory, the ponderomotive phase variable is defined as the sum of the undulator and the radiation phases:

$$\theta_j(z) \equiv (k_u + k_1)z - ck_1\bar{t}_j(z)$$

arrival time averaged over the wiggle motion

$$\equiv (k_u + k_1)z - ck_1\left[t_j(z) - \frac{K^2}{ck_1(4+2K^2)}\sin(2k_u z)\right],$$

- As far as the phase equation is concerned:

(we use the relations
$1-(v_x^2+v_y^2+v_z^2)/c^2 = 1/\gamma^2$
$v_x \approx c dx/dz$ etc)

- the phase derivative is $\frac{d\theta}{dz} = (k_u + k_1) - k_1 \frac{c}{\bar{v}_z}$,

- the average z-velocity is given by $\frac{c}{\bar{v}_z} \approx 1 + \frac{1}{2\gamma^2} + \frac{\overline{x'^2 + y'^2}}{2}$

- define the energy deviation $\eta = (\gamma - \gamma_r)/\gamma_r$ and use the FEL resonance condition $\lambda_1 = \lambda_u(1+K^2/2)/2\gamma_r^2$ as well as the expressions for transverse slopes dx/dz and dy/dz

- The final result is the relation $\frac{d\theta}{dz} = 2k_u\eta - \frac{k_1}{2}\underbrace{\left(p^2 + k_\beta^2 x^2\right)}_{\text{emittance term, introduced by 3D effects}}$

- We also need to consider the energy exchange equation:

$$mc^2\frac{d\gamma}{dz} = -e\frac{dx}{dz}E_x$$
$$= \frac{eK}{\gamma}\cos(k_u z)\left[\underbrace{\int d\nu\, E_\nu(x;z)e^{i\nu(k_1 z-\omega_1 t)}}_{\text{extract slowly varying part}} e^{i\Delta\nu k_u z} + c.c.\right].$$

- To average over the wiggle motion, we use:

- The definition of the phase θ [$\theta = (k_u + k_1)z - ck_1 t + Q sin(2k_u z)$ with $Q = K^2/(4+2K^2)$] in order to eliminate t

- The Jacobi Anger identity [$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\theta}$]

- The end result is

$$\frac{d\eta}{dz} = \chi_1 \int d\nu\, E_\nu(x;z)e^{i\nu\theta} + c.c.$$

JJ factor

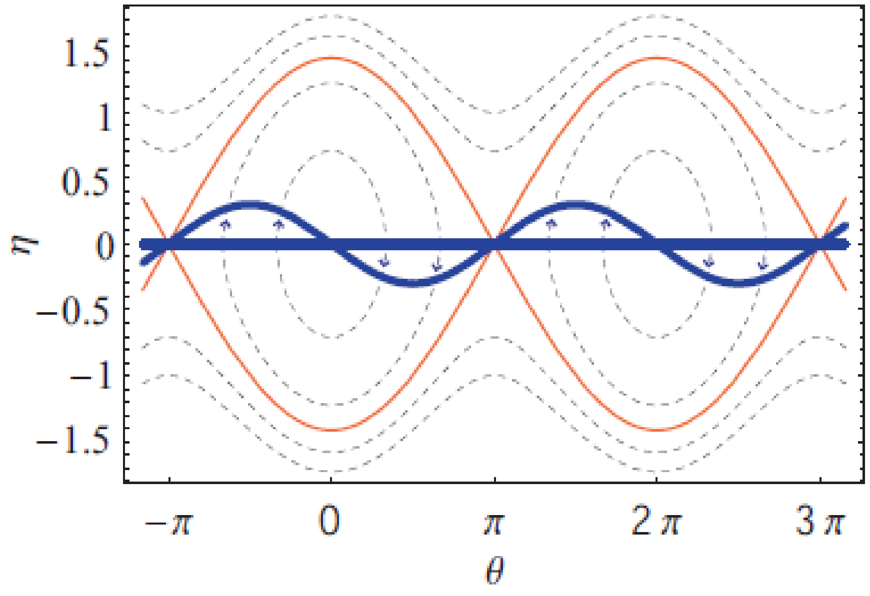
$$\chi_1 \equiv eK[\mathrm{JJ}]/(2\gamma_r^2 mc^2) \qquad [\mathrm{JJ}] \equiv J_0\left(\frac{K^2}{4+2K^2}\right) - J_1\left(\frac{K^2}{4+2K^2}\right)$$

Summary of the 3D averaged equations of motion

$$\frac{d\theta}{dz} = 2k_u\eta - \frac{k_1}{2}(p^2 + k_\beta^2 x^2),$$

$$\frac{d\eta}{dz} = \chi_1 \int d\nu \; e^{i\nu\theta} E_\nu(x;z) + c.c.,$$

$$\frac{dx}{dz} = p, \qquad\qquad \frac{dp}{dz} = -k_\beta^2 x,$$



- In the transverse plane, the electrons perform betatron oscillations, which can be described in the context of the smooth approximation.

- In the longitudinal dimension, one obtains the 3D generalization of the 1D pendulum equations.

Vlasov-Maxwell formalism

- The interaction between the electron beam and the FEL radiation can be described in a self-consistent fashion in the framework of the Vlasov-Maxwell equations.

- The e-beam is described in terms of a distribution function $F = F(\theta, \eta, \boldsymbol{x}, \boldsymbol{p}; z)$ in 6D-phase space. In view of the importance of stochastic effects such as shot noise, we use the Klimontovich distribution:

$$F(\theta, \eta, \boldsymbol{x}, \boldsymbol{p}; z) = \frac{k_1}{n_e} \sum_{j=1}^{N_e} \delta[\theta - \theta_j(z)] \delta[\eta - \eta_j(z)] \times \delta[\boldsymbol{x} - \boldsymbol{x}_j(z)] \delta[\boldsymbol{p} - \boldsymbol{p}_j(z)],$$

n_e: on-axis electron number density

- The evolution of the distribution is governed by the continuity equation

$$\frac{\partial F}{\partial z} + \frac{d\theta}{dz}\frac{\partial F}{\partial \theta} + \frac{d\eta}{dz}\frac{\partial F}{\partial \eta} + \frac{d\boldsymbol{x}}{dz} \cdot \frac{\partial F}{\partial \boldsymbol{x}} + \frac{d\boldsymbol{p}}{dz} \cdot \frac{\partial F}{\partial \boldsymbol{p}} = 0,$$

- On the other hand, the coherent radiation field generated by the microbunching satisfies a *driven wave equation*

$$\nabla^2 E_r - \frac{1}{c^2}\frac{\partial^2 E_r}{\partial t^2} = \frac{1}{\varepsilon_0}\left(\frac{\partial \rho_e}{\partial x} + \frac{1}{c^2}\frac{\partial j_x}{\partial t}\right) \qquad E_r \to E_x$$

  - ✓ The charge/current densities can be expressed in terms of the distribution function *F*. This leads to closed set of self-consistent, nonlinear equations.

- Up to the linear, exponential-gain regime, a **perturbation approach** is applicable. As in the 1D case, this process involves:

  - ✓ Decomposing the distribution function into a background distribution function $\bar{F}$ and a small perturbation δF i.e. $F = \bar{F} + \delta F$. We then introduce the Fourier amplitude F_ν through $F_\nu = (1/2\pi)\int d\theta(\delta F)e^{-i\nu\theta}$ and $\delta F = \int d\nu F_\nu e^{i\nu\theta}$.

  - ✓ Treating F_ν and E_ν as first order (small) quantities.

- After some manipulation (which involves using the equations of motion), we obtain a linearized Vlasov equation:

$$\left\{\frac{\partial}{\partial z} + p\cdot\frac{\partial}{\partial x} - k_\beta^2 x\cdot\frac{\partial}{\partial p} + i\nu\left[2\eta k_u - \frac{k_1}{2}(p^2 + k_\beta^2 x^2)\right]\right\} F_\nu = -\chi_1 E_\nu \frac{\partial}{\partial \eta}\bar{F}$$

- As expected, frequencies are not coupled in the linear regime. This greatly simplifies the analysis as it allows us to concentrate on a single frequency ν (which we do in what follows).

- On the other hand, the background-or unperturbed-distribution evolves according to the zeroth-order Vlasov equation

$$\left\{\frac{\partial}{\partial z} + p\cdot\frac{\partial}{\partial x} - k_\beta^2 x\cdot\frac{\partial}{\partial p}\right\}\bar{F} = 0$$

- To close the loop, we obtain a driven paraxial wave equation for the radiation field:

$$\left(\frac{\partial}{\partial z} + i\Delta\nu k_u + \underbrace{\frac{\nabla_\perp^2}{2ik_1}}_{\text{extra 3D term due to radiation diffraction}}\right) E_\nu(x;z) = -\chi_2 \frac{k_1}{2\pi}\sum_{j=1}^{N_e} e^{-i\nu\theta_j(z)}\delta[x - x_j(z)]$$

$$\chi_2 \equiv eK[\mathrm{JJ}]/2\varepsilon_0\gamma_r$$

- In terms of the distribution function amplitude, the driven paraxial becomes

$$\left(\frac{\partial}{\partial z} + i\Delta\nu k_u + \frac{\nabla_\perp^2}{2ik_1}\right) E_\nu = -\chi_2 n_e \underbrace{\int dp d\eta\, F_\nu}_{\text{current term now includes momentum integration}}$$

- These linearized Vlasov-Maxwell equations accurately describe the FEL operation up to the onset of nonlinear, saturation effects.

Eigenmode equation

- We introduce a set of convenient scaled quantities

$$\hat{z} = 2\rho k_u z \qquad \hat{\eta} = \frac{\eta}{\rho}, \qquad a_\nu = \frac{\chi_1}{2k_u\rho^2} E_\nu = \frac{eK\,[\mathrm{JJ}]}{4\gamma_r^2 mc^2 k_u \rho^2} E_\nu,$$

$$\hat{x} = x\sqrt{2k_1 k_u \rho} \qquad \hat{p} = p\sqrt{\frac{k_1}{2k_u\rho}}. \qquad f_\nu = \frac{2k_u\rho^2}{k_1} F_\nu. \qquad \hat{k}_\beta = k_\beta/(2k_u\rho)$$

- The linearized FEL equations become

$$\left(\frac{\partial}{\partial \hat{z}} + i\frac{\Delta\nu}{2\rho} + \frac{\hat{\nabla}_\perp^2}{2i}\right) a_\nu(\hat{x};\hat{z}) = -\int d\hat{\eta}d\hat{p}\; f_\nu(\hat{\eta},\hat{x},\hat{p};\hat{z})$$

$$\left(\frac{\partial}{\partial \hat{z}} + i\dot{\theta} + \hat{p}\cdot\frac{\partial}{\partial \hat{x}} - \hat{k}_\beta^2 \hat{x}\frac{\partial}{\partial \hat{p}}\right) f_\nu = -a_\nu \frac{\partial \bar{f}_0}{\partial \hat{\eta}},$$

phase derivative

$$\dot{\theta} = \frac{d\theta}{d\hat{z}} = \hat{\eta} - \frac{\hat{p}^2 + \hat{k}_\beta^2 \hat{x}^2}{2}$$

- We have again introduced the **Pierce-or FEL-parameter**

$$\rho = \left[\frac{n_e \chi_1 \chi_2}{(2k_u)^2}\right]^{1/3} = \left(\frac{e^2 K^2 [\mathrm{JJ}]^2 n_e}{32\epsilon_0 \gamma_r^3 m c^2 k_u^2}\right)^{1/3}$$
$$= \left[\frac{1}{8\pi}\frac{I}{I_A}\left(\frac{K[\mathrm{JJ}]}{1+K^2/2}\right)^2 \frac{\gamma\lambda_1^2}{2\pi\sigma_x^2}\right]^{1/3}$$

I : e-beam peak current
$I_A \approx 17\ kA$ (Alfven current)

- As far as the background distribution is concerned, we assume no z-dependence for $\bar{f}_0$. Specifically, we select a Gaussian transverse and energy profile and a uniform current profile.

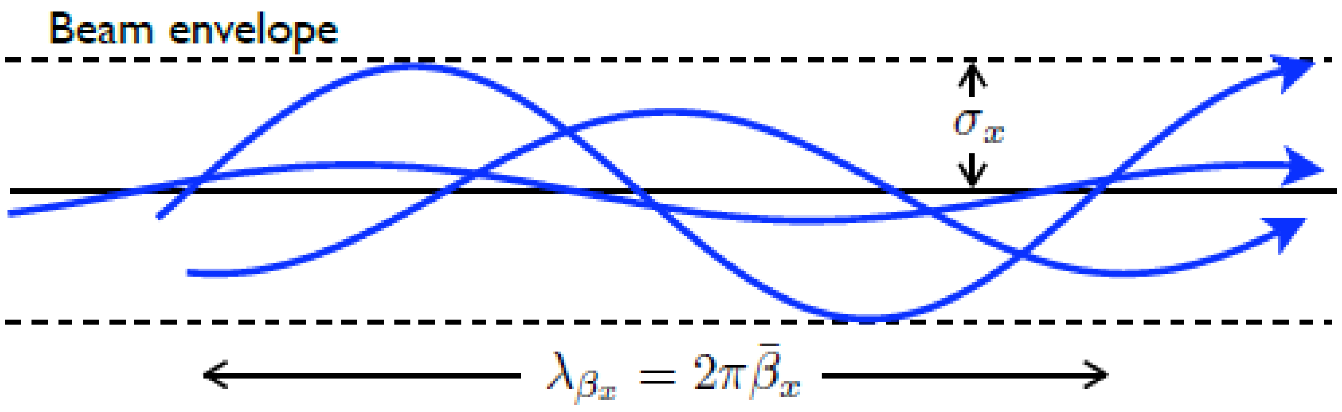
$$\bar{f}_0(\hat{p}^2 + \hat{k}_\beta^2 \hat{x}^2) = \frac{1}{2\pi \hat{k}_\beta^2 \hat{\sigma}_x^2} \exp\left(-\frac{\hat{p}^2 + \hat{k}_\beta^2 \hat{x}^2}{2\hat{k}_\beta^2 \hat{\sigma}_x^2}\right) \frac{1}{\sqrt{2\pi}\hat{\sigma}_\eta} \exp\left(-\frac{\hat{\eta}^2}{2\hat{\sigma}_\eta^2}\right)$$

$n_e = I/(2\pi\sigma_x^2 ec)$

$$\hat{\sigma}_x = \sigma_x \sqrt{2k_1 k_u \rho}$$
$$\hat{\sigma}_\eta = \sigma_\eta / \rho$$

σ_x : rms beam size in x and y (round beam)
σ_η : rms relative energy spread

- This distribution corresponds to a matched beam with a constant beam size.



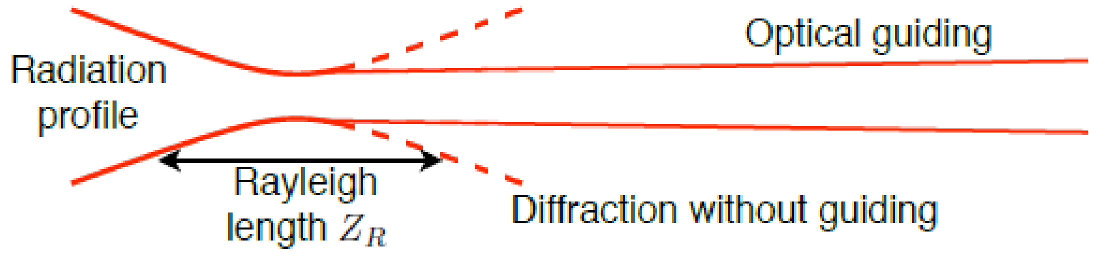
$\sigma_x' = \sigma_x k_\beta$: rms angular divergence

$\varepsilon_x = \sigma_x \sigma_x'$: transverse emittance

- For such a z-independent case, we seek the self-similar, guided eigenmodes of the FEL. These are solutions of the form:

$$\Psi = \begin{bmatrix} a_\nu(\hat{x};\hat{z}) \\ f_\nu(\hat{\eta},\hat{x},\hat{p},\hat{z}) \end{bmatrix} = e^{-i\mu_\ell \hat{z}} \begin{bmatrix} \mathcal{A}_\ell(\hat{x}) \\ \mathcal{F}_\ell(\hat{x},\hat{p},\hat{\eta}) \end{bmatrix}$$

- They are characterized by a constant growth rate μ_l and a z-independent radiation/density mode profile A_l/F_l.



- Substituting into the Vlasov-Maxwell (FEL) equations, we obtain two coupled relations for the growth rate and the mode amplitudes:

$$\begin{bmatrix} \mu_\ell \mathcal{A}_\ell + \left(-\frac{\Delta\nu}{2\rho} + \frac{1}{2}\hat{\boldsymbol{\nabla}}_\perp^2\right)\mathcal{A}_\ell + i\int d\hat{p}d\hat{\eta}\,\mathcal{F}_\ell \\ \mu_\ell \mathcal{F}_\ell + i\mathcal{A}_\ell \frac{\partial \bar{f}_0}{\partial\hat{\eta}} + \left\{-\nu\dot{\theta} + i\left(\hat{p}\cdot\frac{\partial}{\partial\hat{x}} - \hat{k}_\beta^2\hat{x}\cdot\frac{\partial}{\partial\hat{p}}\right)\right\}\mathcal{F}_\ell \end{bmatrix} = 0.$$

- The second equation can be solved analytically in terms of F_l:

$$\mathcal{F}_\ell = -\frac{\partial \bar{f}_0}{\partial\hat{\eta}}\int_{-\infty}^{0} d\tau\ \mathcal{A}_\ell(\hat{x}_+)e^{i(\nu\dot{\theta}-\mu_\ell)\tau}. \qquad \hat{x}_+(\tau) \equiv \hat{x}\cos(\hat{k}_\beta\tau) + (\hat{p}/\hat{k}_\beta)\sin(\hat{k}_\beta\tau)$$

- Inserting this into the first equation yields a single relation for the mode growth rate and profile:

$$\left(\mu_\ell - \frac{\Delta\nu}{2\rho} + \frac{1}{2}\hat{\boldsymbol{\nabla}}_\perp^2\right)\mathcal{A}_\ell(\hat{x}) - i\int d\hat{p}d\hat{\eta}\int_{-\infty}^{0} d\tau\ e^{i(\nu\dot{\theta}-\mu_\ell)\tau}\frac{d\bar{f}_0}{d\hat{\eta}}\,\mathcal{A}_\ell(\hat{x}_+) = 0.$$

- Using the specific form of $\overline{f}_0$, we obtain a more explicit relation:

$$\left(\mu - \frac{\Delta\nu}{2\rho} + \frac{1}{2}\hat{\nabla}_\perp^2\right)\mathcal{A}(\hat{x}) - \frac{1}{2\pi\hat{k}_\beta^2\hat{\sigma}_x^2}\int_{-\infty}^{0} d\tau\,\tau e^{-\hat{\sigma}_\eta^2\tau^2/2 - i\mu\tau}$$
$$\times \int d\hat{p}\,\mathcal{A}[\hat{x}_+(\hat{x},\hat{p},\tau)]\exp\left[-\frac{1+i\tau\hat{k}_\beta^2\hat{\sigma}_x^2}{2\hat{k}_\beta^2\hat{\sigma}_x^2}\left(\hat{p}^2+\hat{k}_\beta^2\hat{x}^2\right)\right] = 0.$$

- From the above equation, it follows that there are four basic dimensionless parameters that affect the growth rate:

  - $\hat{\sigma}_x$ is a quantitative measure of the diffraction effect

$$\hat{\sigma}_x^2 = \sigma_x^2 2k_1k_u\rho = \frac{2\pi\sigma_x^2}{\lambda_1}\frac{4\pi\rho}{\lambda_u} = \frac{2}{\sqrt{3}}\frac{Z_R}{L_{G0}}, \qquad Z_R = \pi\sigma_x^2/\lambda_1$$

($L_{G0} = \frac{\lambda_u}{4\pi\sqrt{3}\rho}$ is the 1D gain length)

- $\hat{\sigma}_x \hat{k}_\beta$ is a measure of the emittance effect

$$(\hat{\sigma}_x \hat{k}_\beta)^2 \approx \frac{4\pi L_G}{\lambda_1 \bar{\beta}} \varepsilon_x < 1.$$

($\bar{\beta} = 1/k_\beta$ is the average beta function)

- $\hat{\sigma}_\eta$ represents the energy spread effect and gives the ratio of the energy spread-induced wavelength spread versus the bandwidth of the FEL effect given by ρ

$$\hat{\sigma}_\eta = \frac{\Delta\gamma}{\gamma\rho} \rightarrow \frac{\Delta\lambda_1}{2\lambda_1} \frac{1}{\rho} = 2\pi\sqrt{3} \frac{\Delta\lambda_1}{\lambda_1} \frac{L_{G0}}{\lambda_u} \leq 1.$$

- The scaled frequency detuning parameter is $\sigma_\nu = \Delta\nu/(2\rho)$

$$\sigma_\nu \sim \frac{\Delta\lambda_1}{\lambda_1} \sim \rho.$$

- Given the FEL eigenmodes, the general solution of the initial value problem can be constructed as their superposition, for instance

$$a_\nu(\boldsymbol{x}, z) = \sum_l c_l A_l(\boldsymbol{x})\, e^{-i\mu_l z}$$

- The constants c_l can be calculated through overlap integrals involving the initial field and density modulation.

- However, it needs to be emphasized that the FEL eigenmodes are (in general) not power-orthogonal.

- The most important case is that of the high-gain regime, where a single mode (typically the fundamental or 00 mode) has the highest growth rate and dominates all the others [this happens when $z \gg L_G = L_{G0}\sqrt{3}/2Im(\mu_{00})$]

$$a_\nu(\boldsymbol{x}, z) \approx c_{00} A_{00}(\boldsymbol{x}) e^{-i\mu_{00} z}$$

Parabolic model

- We consider the simplified case of the parallel beam, where focusing and emittance effects are negligible ($k_\beta = 0$). Moreover, we take $\sigma_\eta = 0$:

$$\bar{f}_0(\hat{\eta}, \hat{x}, \hat{p}) = \delta(\hat{\eta})\delta(\hat{p})U(\hat{x}).$$

- The mode equation then becomes

$$\frac{1}{2}\left[\frac{1}{\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial}{\partial \hat{r}}\right) + \frac{1}{\hat{r}^2}\frac{\partial^2}{\partial \phi^2}\right]\mathcal{A}_\ell(\hat{x}) + \left[\mu_\ell - \frac{\Delta\nu}{2\rho} - \frac{U(\hat{x})}{\mu_\ell^2}\right]\mathcal{A}_\ell(\hat{x}) = 0.$$

- U is typically a Gaussian. In the limit of small diffraction ($\hat{\sigma}_x \gg 1$), the radiation size is smaller than the e-beam size. Then U can be approximated by a parabola:

$$U(\hat{x}) = 1 - \frac{\hat{r}^2}{2\hat{\sigma}_x^2} = 1 - \frac{|x|^2}{2\sigma_x^2}.$$

- The main advantage of this model is that it admits exact, analytical solutions:

$$\mathcal{A}_\ell(\hat{x}) = \mathcal{A}_{\ell,m}(\hat{r})e^{im\phi} \qquad \mathcal{A}_{\ell,m}(\hat{r}) = \left(\frac{i\hat{r}^2}{\mu_\ell\hat{\sigma}_x}\right)^{m/2} \mathrm{L}_\ell^m\left(\frac{i\hat{r}^2}{\mu_\ell\hat{\sigma}_x}\right)\exp\left(-\frac{i\hat{r}^2}{2\mu_\ell\hat{\sigma}_x}\right).$$

$$\mu_\ell^2\left(\mu_\ell - \frac{\Delta\nu}{2\rho}\right) - 1 = \underbrace{\frac{i\mu_\ell}{\hat{\sigma}_x}(2\ell+m+1)}_{\text{extra 3D term due to diffraction}}, \text{ for } \ell \geq 0,\ \ell \in \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$

- For $\hat{\sigma}_x \to \infty$, we recover the well-known 1D dispersion relation. For zero detuning $\Delta\nu = 0$, we also obtain the cubic relation $\mu_l^3 = 1$.

- For the fundamental mode (m=0,l=0), the radiation mode size is given by

$$\sigma_r^2 = \frac{|\mu_\ell|^2\sigma_x}{2\Im(\mu_\ell)\sqrt{2\rho k_1 k_u}}, \qquad \sigma_r/\sigma_x \propto \hat{\sigma}_x^{-1/2}$$

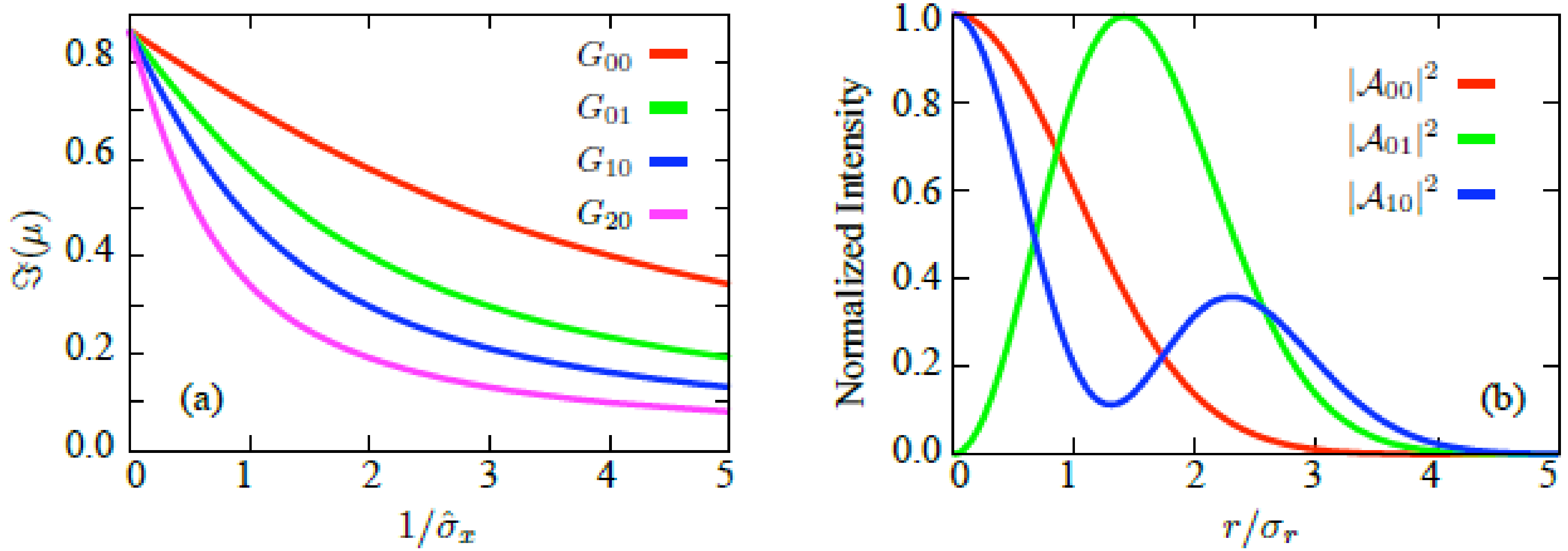


Figure 5.9: Transverse mode structure for the parabolic beam. (a) Growth rate as a function of the inverse of the scaled electron beam size, maximized with respect to $\Delta\nu$. All modes approach the 1D growth rate $\sqrt{3}/2 \approx 0.866$ as $\hat{\sigma}_x \to \infty$, and decrease as the beam size decreases. The gain of the lowest order Gaussian mode G_{00} decreases the slowest, so that at finite beam size it tends to dominate the other transverse modes. In panel (b) we plot the intensity profile of the three lowest order modes, each being normalized to its peak value with $\hat{\sigma}_x \gg 1$. The mode shapes look quite similar to those shown in Fig. 5.10, which were numerically computed for the LCLS.

Variational solution

- Changing the momentum variable from $\widehat{\boldsymbol{p}}$ to $\hat{\boldsymbol{x}}' = \hat{\boldsymbol{x}}\cos(\hat{k}_\beta\tau) + \frac{\hat{\boldsymbol{p}}}{\hat{k}_\beta}\sin(\hat{k}_\beta\tau)$, the general mode equation becomes

$$\left(\mu - \frac{\Delta\nu}{2\rho} + \frac{1}{2}\boldsymbol{\nabla}_\perp^2\right)A(\hat{\boldsymbol{x}}) - \int_{-\infty}^{0} d\tau\,\frac{\tau e^{-\hat{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{2\pi\hat{\sigma}_x^2\sin^2(\hat{k}_\beta\tau)}\int d\hat{\boldsymbol{x}}'\,A(\hat{\boldsymbol{x}}')$$
$$\times\exp\left[-\frac{1+i\hat{k}_\beta^2\hat{\sigma}_x^2\tau}{2\hat{\sigma}_x^2}\,\frac{\hat{\boldsymbol{x}}^2 + \hat{\boldsymbol{x}}'^2 - 2\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}}'\cos(\hat{k}_\beta\tau)}{\sin^2(\hat{k}_\beta\tau)}\right] = 0.$$

- The equation for azimuthal modes of the form $A(\hat{\boldsymbol{x}}) = A_m(R)e^{im\varphi}$ $\quad R = \frac{|\boldsymbol{x}|}{\sigma_x} = \frac{|\hat{\boldsymbol{x}}|}{\hat{\sigma}_x}$ is

$$\left\{\mu - \frac{\Delta\nu}{2\rho} + \frac{1}{2\hat{\sigma}_x^2}\left[\frac{1}{R}\frac{d}{dR}\left(R\frac{d}{dR}\right) - \frac{m^2}{R^2}\right]\right\}A_m(R)$$
$$= \int_0^\infty R'dR'\,G_m(R,R')A_m(R')$$

- The integral kernel G is given by

$$G_m(R,R') = \int_{-\infty}^{0} d\tau \, \frac{\tau}{\sin^2(\hat{k}_\beta \tau)} I_m \left[\frac{RR'(1 + i\hat{k}_\beta^2 \hat{\sigma}_x^2 \tau)\cos(\hat{k}_\beta \tau)}{\sin^2(\hat{k}_\beta \tau)} \right]$$
$$\times \exp\left[-\frac{\hat{\sigma}_x^2 \tau^2}{2} - i\mu\tau - \frac{(R^2 + R'^2)(1 + i\hat{k}_\beta^2 \hat{\sigma}_x^2 \tau)}{2\sin^2(\hat{k}_\beta \tau)} \right]$$

$$I_m(\xi) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{im\varphi + \xi\cos\varphi}$$

modified Bessel function

- An exact numerical solution of the above equation can be obtained through an integral transform technique, which eventually leads to a matrix equation.

- A more flexible-and computationally faster-approximate solution can be derived through a variational method.

- We construct the variational functional

$$\mathcal{I}[\mathcal{A}] = \int_0^\infty R dR \left\{ \frac{1}{2\hat{\sigma}_x^2} \left[\frac{d\mathcal{A}_m(R)}{dR} \right]^2 - \left[\mu - \frac{\Delta\nu}{2\rho} - \frac{m^2}{R^2} \right] \mathcal{A}_m^2(R) \right\}$$
$$+ \int_0^\infty R dR \int_0^\infty R' dR' \, G(R, R') \mathcal{A}_m(R) \mathcal{A}_m(R') = 0$$

- Inserting a function A yields a complex number μ. If A is an actual mode profile, μ is a mode growth rate. Moreover, it can be shown that a first order variation from the mode profile yields only a second order variation from the growth rate.

- In view of the exact solution for the parabolic model, we choose a trial function of the form $A = \exp(-wr^2)$ for the fundamental mode. A similar process can be devised for the higher order modes.

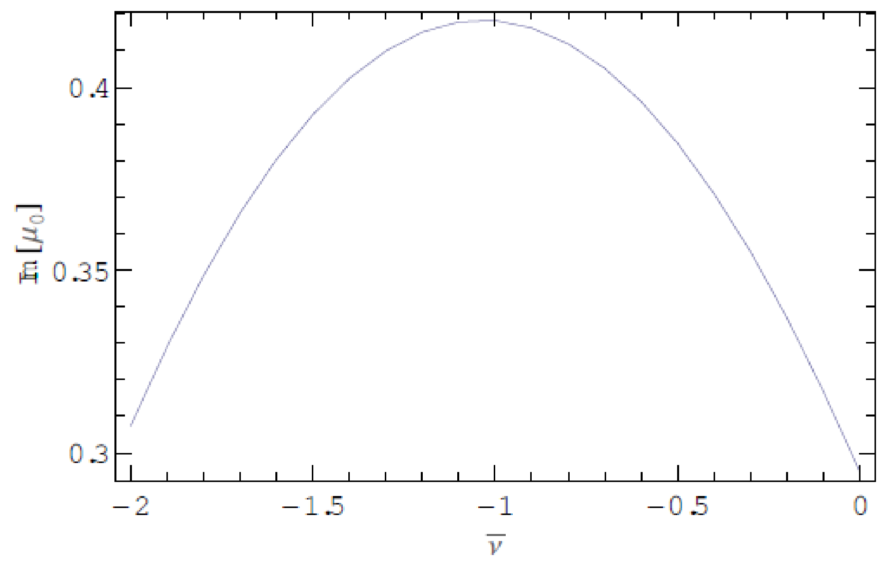
- This yields the relation

$$\left\{\frac{\mu_{00}-\Delta\nu/2\rho}{4w}-\frac{1}{4\hat{\sigma}_x^2}-\int_{-\infty}^{0}d\tau\,\frac{\tau e^{-\hat{\sigma}_\eta^2\tau^2/2-i\mu_{00}\tau}}{\left[(1+i\hat{k}_\beta^2\hat{\sigma}_x^2\tau)+2w\right]^2-4w^2\cos^2(\hat{k}_\beta\tau)}\right\}=0$$

- Using the stationary condition $\frac{\partial\mu_{00}}{\partial w}=0$, yields a second relation which completes the variational solution:

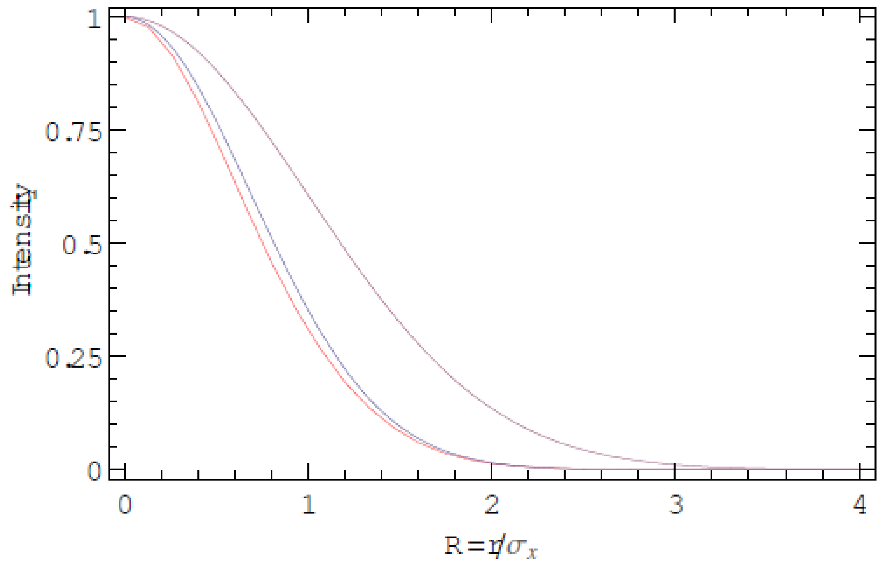
$$\frac{\mu_{00}-\Delta\nu/2\rho}{4w^2}=\int_{-\infty}^{0}d\tau\,\frac{\left[4(1+i\hat{k}_\beta^2\hat{\sigma}_x^2\tau)+8w\sin^2(\hat{k}_\beta\tau)\right]\tau e^{-\hat{\sigma}_\eta^2\tau^2/2-i\mu_{00}\tau}}{\left\{\left[(1+i\hat{k}_\beta^2\hat{\sigma}_x^2\tau)+2w\right]^2-4w^2\cos^2(\hat{k}_\beta\tau)\right\}^2}.$$

- Through this procedure, we obtain the growth rate μ and the mode parameter w as functions of the detuning $\Delta\nu$.



- LCLS fundamental mode growth rate versus the scaled detuning $\bar{\nu} = \Delta\nu/2\rho$

- optimum growth rate for negative detuning (wavelength longer than the resonant value)



LCLS fundamental mode intensity profile:
- from the exact solution (red)
- from the variational (blue)
- e-beam profile (purple)

Ming Xie's fitting formula

- Using data from the variational solution, a fitting formula can be found that relates the optimized power gain length L_G to the various scaled parameters of the FEL (Ming Xie, Nucl. Instr. A, 445, 59 (2000))

$$L_G = L_{G0}\frac{\sqrt{3}/2}{\Im(\mu_{00})} = L_{G0}(1+\Lambda) \qquad \Lambda = a_1\eta_d^{a_2} + a_3\eta_\varepsilon^{a_4} + a_5\eta_\gamma^{a_6} + a_7\eta_\varepsilon^{a_8}\eta_\gamma^{a_9}$$

$$+ a_{10}\eta_d^{a_{11}}\eta_\gamma^{a_{12}} + a_{13}\eta_d^{a_{14}}\eta_\varepsilon^{a_{15}} + a_{16}\eta_d^{a_{17}}\eta_\varepsilon^{a_{18}}\eta_\gamma^{a_{19}}$$

$$L_{G0} = \lambda_u/(4\sqrt{3}\pi\rho)$$

$$\eta_d = \frac{L_{G0}}{2k_1\sigma_x^2} = \frac{1}{2\sqrt{3}\hat{\sigma}_x^2} \qquad \text{diffraction parameter,}$$

$$\eta_\varepsilon = 2\frac{L_{G0}}{\bar{\beta}}k_1\varepsilon = \frac{2}{\sqrt{3}}\hat{k}_\beta^2\hat{\sigma}_x^2 \qquad \text{angular spread parameter,}$$

$$\eta_\gamma = 4\pi\frac{L_{G0}}{\lambda_u}\sigma_\eta = \frac{\hat{\sigma}_\eta}{\sqrt{3}} \qquad \text{energy spread parameter.}$$

$$a_1 = 0.45, \quad a_2 = 0.57, \quad a_3 = 0.55, \quad a_4 = 1.6, \quad a_5 = 3,$$
$$a_6 = 2, \quad a_7 = 0.35, \quad a_8 = 2.9, \quad a_9 = 2.4, \quad a_{10} = 51,$$
$$a_{11} = 0.95, \quad a_{12} = 3, \quad a_{13} = 5.4, \quad a_{14} = 0.7, \quad a_{15} = 1.9,$$
$$a_{16} = 1140, \quad a_{17} = 2.2 \quad a_{18} = 2.9, \quad a_{19} = 3.2.$$

- All the coefficients given above are positive. Thus, the fitting formula illustrates the increase of the gain length due to the various additional 3D effects.

- Another fitting formula exists for the saturation power:

$$P_{\text{sat}} \approx 1.6 \left(\frac{L_{G0}}{L_G}\right)^2 \rho P_{\text{beam}} = \frac{1.6}{(1+\Lambda)^2} \rho P_{\text{beam}}$$

$$P_{\text{beam}}[\text{TW}] = \gamma_r mc^2[\text{GeV}] I\,[\text{kA}]$$

ρ is roughly equal to the power transformer ratio of the FEL

Thank you for your attention!