Homework Set 2 solutions (10/31/15), C1 class (X-Ray Free Electron Laser Theory), 9th ILC school, 2015

1 Parabolic model

For a trial solution of the form

$$A(\hat{\mathbf{x}}) = \exp(-a\hat{r}^2)$$

we have

$$\begin{aligned}
&\frac{1}{2\hat{r}}\frac{\partial}{\partial \hat{r}}(\hat{r}\frac{\partial A}{\partial \hat{r}}) = \frac{1}{2\hat{r}}\frac{\partial}{\partial \hat{r}}[-2a\hat{r}^2 \exp(-a\hat{r}^2)] \\
&= -\frac{a}{\hat{r}}\frac{\partial}{\partial \hat{r}}[\hat{r}^2 \exp(-a\hat{r}^2)] = -\frac{a}{\hat{r}}[2\hat{r} - 2a\hat{r}^3]\exp(-a\hat{r}^2) \\
&= -\frac{a}{\hat{r}}[2\hat{r} - 2a\hat{r}^3]\exp(-a\hat{r}^2) = -2a[1 - a\hat{r}^2]\exp(-a\hat{r}^2)
\end{aligned}$$

so the mode equation becomes

$$\begin{aligned}
&\frac{1}{2\hat{r}}\frac{\partial}{\partial \hat{r}}(\hat{r}\frac{\partial}{\partial \hat{r}})A(\hat{\mathbf{x}}) + \left[\mu_l - \frac{\Delta\nu}{2\rho} - \frac{U(\hat{\mathbf{x}})}{\mu_l^2}\right]A(\hat{\mathbf{x}}) \\
&= -2a[1 - a\hat{r}^2]\exp(-a\hat{r}^2) - \frac{1}{\mu_l^2}(1 - \frac{\hat{r}^2}{2\hat{\sigma}_x^2})\exp(-a\hat{r}^2) \\
&+ \left(\mu_l - \frac{\Delta\nu}{2\rho}\right)\exp(-a\hat{r}^2) = 0
\end{aligned}$$

Collecting the powers of $\hat{r}$ on the LHS and equating to zero yields two equations:

$$\mu_l - \frac{\Delta\nu}{2\rho} - \frac{1}{\mu_l^2} - 2a = 0$$

and

$$2a^2 + \frac{1}{2\hat{\sigma}_x^2\mu_l^2} = 0 \rightarrow a^2 = -\frac{1}{4\hat{\sigma}_x^2\mu_l^2}$$

The latter relation yields

$$a = \pm \frac{i}{2\hat{\sigma}_x \mu_l} = \pm \frac{i\mu_l^*}{2\hat{\sigma}_x |\mu_l|^2}$$

Since

$$\mathrm{Re}[a] = \pm \frac{\mathrm{Im}[\mu_l]}{2\hat{\sigma}_x |\mu_l|^2}$$

we choose the plus sign since we need the real part of a to be positive (recall that the imaginary part of μ_l is positive for a growing mode). Thus, we have

$$a = +\frac{i}{2\hat{\sigma}_x \mu_l}$$

and the dispersion relation becomes

$$\mu_l - \frac{\Delta\nu}{2\rho} - \frac{1}{\mu_l^2} = 2a = \frac{i}{\hat{\sigma}_x \mu_l}$$

or

$$\left(\mu_l - \frac{\Delta\nu}{2\rho}\right) \mu_l^2 - 1 = \frac{i\mu_l}{\hat{\sigma}_x}$$

2 LCLS 3D optimization

The results for the optimization curves are shown below:

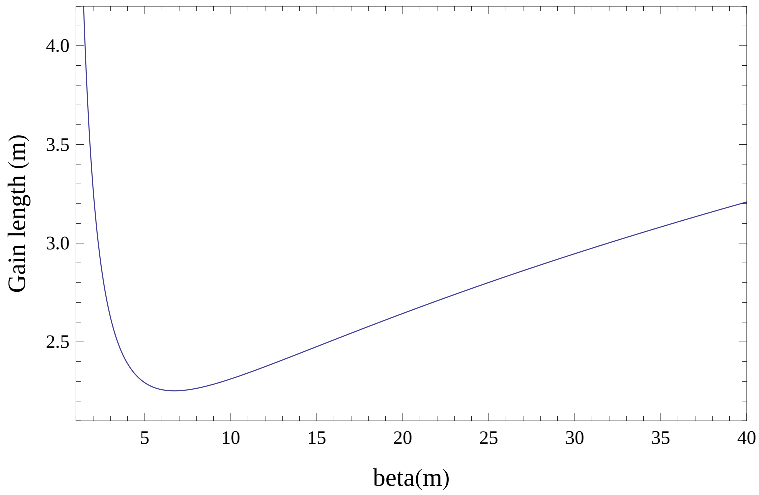

Figure 1: Gain length.

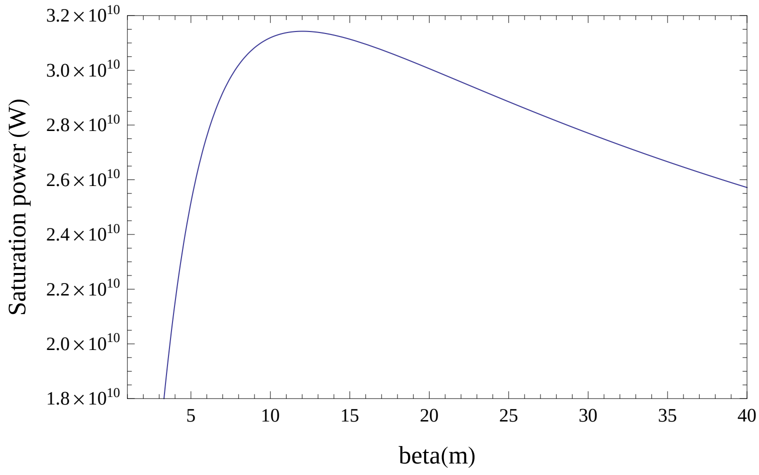

Figure 2: Saturation power.

The optimum beta is about 10 m, resulting in a minimum gain length of 2.3 m and a saturation power of about 30 GW.