Homework Set 1 solutions (10/29/15), C1 class (X-Ray Free Electron Laser Theory), 9th ILC school, 2015

1 Energy conservation in 1D FEL theory

In a slightly modified form, the energy conservation relation of slide 19 (including the correction for the typo) is

$$\int d\theta \frac{\partial}{\partial z}(2\varepsilon_0 E(\theta, z)E^*(\theta, z)) + \sum_j n_e\gamma_r mc^2\frac{d\eta_j}{dz} = 0$$

To begin with, we note that the integrand can be written as

$$2\varepsilon_0\left(\frac{\partial E}{\partial z}E^* + E\frac{\partial E^*}{\partial z}\right)$$

Using the Maxwell equation for the field, we get

$$\frac{\partial E}{\partial z} = -\chi_2 n_e\left\langle e^{-i\theta_j}\right\rangle_\Delta - k_u\frac{\partial E}{\partial\theta}$$

which yields

$$\begin{aligned}
&2\varepsilon_0\left(\frac{\partial E}{\partial z}E^* + E\frac{\partial E^*}{\partial z}\right)\\
&= -2\varepsilon_0\chi_2 n_e[E^*\left\langle e^{-i\theta_j}\right\rangle_\Delta + E\left\langle e^{+i\theta_j}\right\rangle_\Delta] - 2\varepsilon_0 k_u\frac{\partial(EE^*)}{\partial\theta}\\
&= -\gamma_r mc^2\chi_1 n_e[E^*\left\langle e^{-i\theta_j}\right\rangle_\Delta + E\left\langle e^{+i\theta_j}\right\rangle_\Delta] - 2\varepsilon_0 k_u\frac{\partial(EE^*)}{\partial\theta}
\end{aligned}$$

where we have used the fact that $2\varepsilon_0\chi_2 = eK[JJ]/2\gamma_r = \gamma_r mc^2\chi_1$. Using the above relation and the pendulum equation

$$\frac{d\eta_j}{dz} = \chi_1[E_j e^{i\theta_j} + {E_j}^* e^{-i\theta_j}]$$

we get

$$\begin{aligned}
&\int d\theta \frac{\partial}{\partial z}\left(2\varepsilon_0 E(\theta,z)E^*(\theta,z)\right) + \sum_j n_e\gamma_r mc^2 \frac{d\eta_j}{dz} \\
&= -2\varepsilon_0 k_u \int d\theta \frac{\partial (EE^*)}{\partial \theta} \\
&- \gamma_r mc^2 \chi_1 n_e \int d\theta [E^* \langle e^{-i\theta_j}\rangle_\Delta + E\langle e^{+i\theta_j}\rangle_\Delta] \\
&+ n_e\gamma_r mc^2\chi_1 \sum_j [E_j e^{i\theta_j} + E_j{}^* e^{-i\theta_j}] = -\,2\varepsilon_0 k_u \left.|E|^2\right|_{\theta=-\infty}^{\theta=+\infty} = 0
\end{aligned}$$

since the field vanishes at $\theta = \pm\infty$ (we have also used the relation $\int d\theta E\langle e^{i\theta_j}\rangle_\Delta = \sum_j E_j e^{i\theta_j}$, along with its conjugate).

2 1D SASE parameters

From $D(\mu) = 0$, the case of zero energy spread gives $\mu^3 = 1$ (by setting $V(\eta) = \delta(\eta)$ and $\Delta\nu = 0$ in the expression of slide 27, we have $D(\mu) = \mu - \mu^{-2}$). Again using $V(\eta) = \delta(\eta)$, we then find

$$g_A = \frac{1}{|D'(\mu)|^2} = \frac{1}{|1+2\mu^{-3}|^2} = \frac{1}{9} \tag{1}$$

and similarly

$$g_S = \int \frac{d\eta V(\eta)}{|\eta/\rho - \mu|^2} = \frac{1}{|\mu|^2} = 1\,. \tag{2}$$

From the cubic equation, $\mu^3 = 1$, we find a positive imaginary part $\mu_I = \sqrt{3}/2$.

The gain length is $L_G = (4\mu_I\rho k_u)^{-1} = \lambda_u/4\pi\sqrt{3}\rho$, as we found before.

If we write $\mu = \mu_0 + a\Delta\nu + b\Delta\nu^2 + ...$, we can use $\mu_0^3 = 1$ to find

$$\begin{aligned}
&\mu^3 - \frac{\Delta\nu\mu^2}{2\rho} = 1 \\
\Rightarrow &\mu_0^3 + 3\mu_0^2 a\Delta\nu + 3\mu_0 a^2\Delta\nu^2 + 3\mu_0^2 b\Delta\nu^2 - \frac{\Delta\nu\mu_0^2 + 2\mu_0 a\Delta\nu^2}{2\rho} = 1 \\
\Rightarrow &\mu_0^3 + \Delta\nu(3\mu_0^2 a - \mu_0^2/2\rho) + \Delta\nu^2(3\mu_0 a^2 + 3\mu_0^2 b - 2\mu_0 a/2\rho) = 1
\end{aligned} \tag{3}$$

where we've kept terms only to $\Delta\nu^2$. Collecting terms in powers of $\Delta\nu$ (which must all vanish), we find $a = 1/6\rho$ and $3\mu_0 b = 2a/2\rho - 3a^2$ which gives

$$
\begin{aligned}
b = \frac{1}{36\rho^2\mu_0} &= \frac{1}{36\rho^2(1/2+\sqrt{3}i/2)} \\
&= -\frac{1}{2}\frac{1}{36\rho^2} - \frac{\sqrt{3}}{2}\frac{1}{36\rho^2}i
\end{aligned} \tag{4}
$$

where we've taken the solution for μ_0 with exponential growth. Plugging back in to μ we find

$$
\mu = -\frac{1}{2}\left(1 - \frac{1}{3\rho}\Delta\nu + \frac{1}{36\rho^2}\Delta\nu^2 + ...\right) + \frac{\sqrt{3}}{2}i\left(1 - \frac{1}{36\rho^2}\Delta\nu^2 + ...\right). \tag{5}
$$

Then taking only the imaginary part, μ_I, of μ, we write the growth term as $e^{4\mu_I\rho k_u z} = e^{\tau}S(\omega-\omega_m)$, with $\tau = 2\sqrt{3}\rho k_u z$. That leaves

$$
\begin{aligned}
S(\omega-\omega_m) &= e^{-4\rho k_u z\frac{\sqrt{3}}{2}\frac{\Delta\nu^2}{36\rho^2}} \\
&= e^{-\frac{1}{2}\frac{k_u z\sqrt{3}}{9\rho}\left(\frac{\omega-\omega_m}{\omega_m}\right)^2}
\end{aligned} \tag{6}
$$

So we find

$$
\begin{aligned}
\sigma_\omega &= \sqrt{\frac{9\rho}{k_u z\sqrt{3}}} \\
&= \sqrt{\frac{9\rho}{2\pi\sqrt{3}N_u}}
\end{aligned} \tag{7}
$$

where we used $z = N_u\lambda_u$ in the final step.