X-Ray Free Electron Laser Theory

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Lecture outline

- **1D theory (~3 hrs)**
- **3D theory (~2 hrs)**
- **Seeding (~1 hr)**
- **Tutorials (demonstrate problems/simulations)**

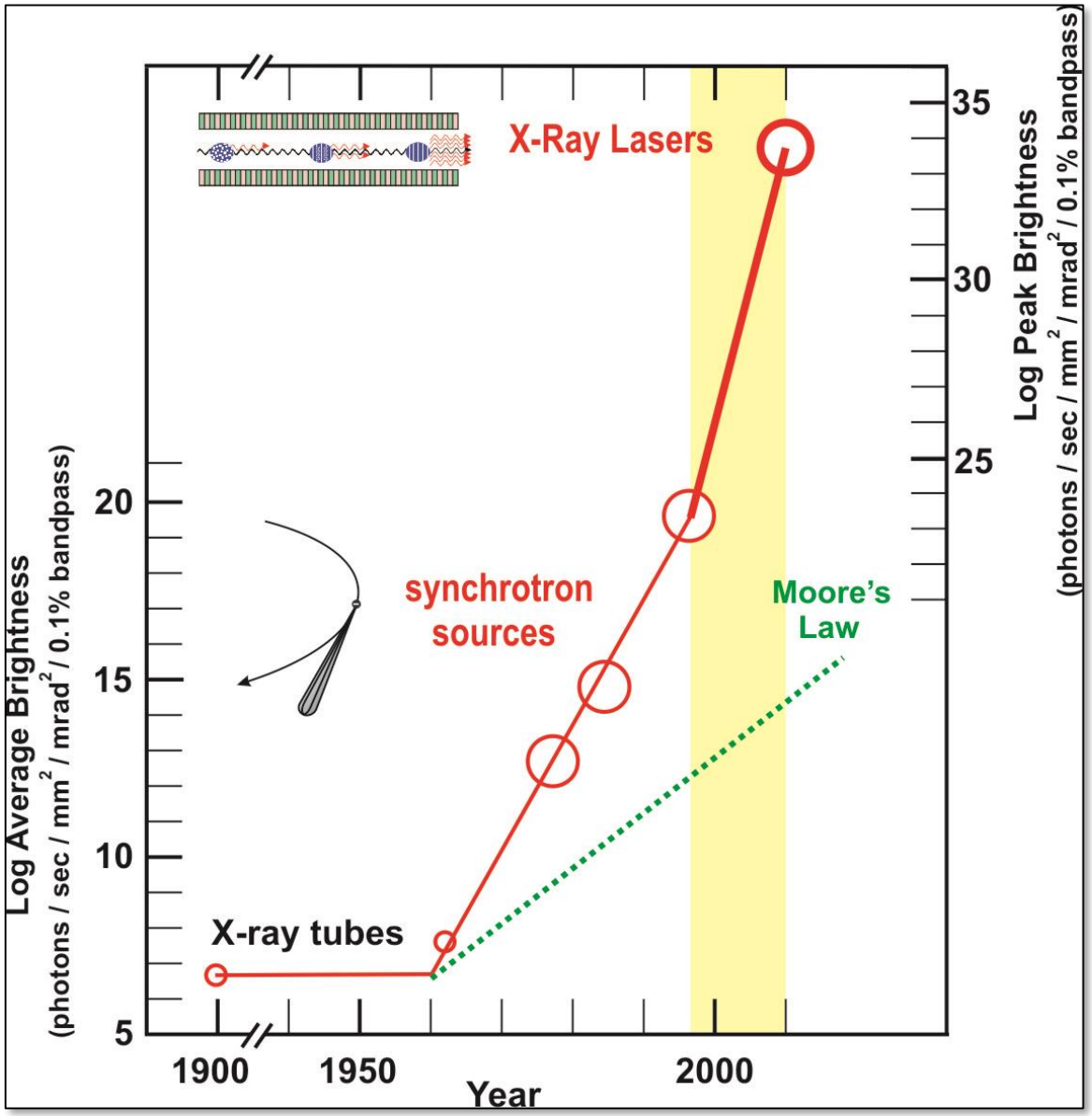
Selected references

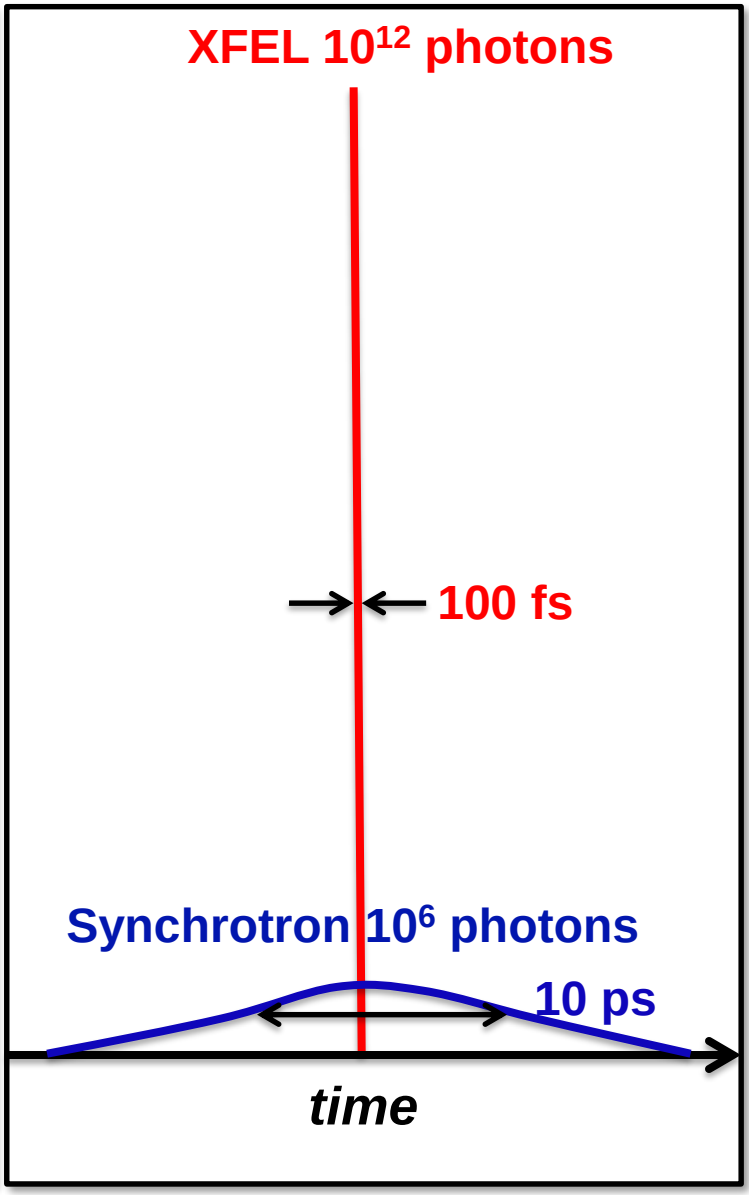
- J. Murphy and C. Pellegrini, Introduction to the Physics of Free Electron Laser, Laser handbook 6 (North-Holland, 1990)
- E.L. Saldin, E.A. Schneidmiller and M.V. Yurkov, Physics of Free Electron Lasers (Springer, 2000).
- Z. Huang, K.-J. Kim, Phys. Rev. ST Accel. Beams **10**, 034801, 2007.
- P. Schmser, M. Dohlus, and J. Rossbach, Free Electron Lasers in the UV and X-Ray Regime (Springer, 2014).
- K.-J. Kim, Z. Huang, R. Lindberg, Synchrotron Radiation and Free Electron Lasers for Bright, USPAS2013 lecture note, to be published by Cambridge Press, 2016.

1D FEL Theory

- Introduction
- Electron equation of motion
- High-gain theory
- Self Amplified Spontaneous Emission
- Nonlinear harmonic generation

XFELs are Extremely Bright and Ultrafast

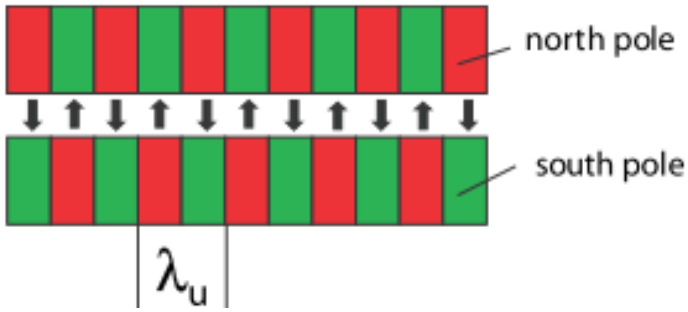




Note: synchrotron sources are much higher rep. rate than XFELs

Electron trajectory in a planar undulator

Undulator magnetic field near axis



$$B_y = -B_0 \sin(k_u z) \;\text{ for }\; 0 \le z \le N_u \lambda_u$$

$$\frac{d}{dt}\gamma mc\boldsymbol{\beta} = -e\boldsymbol{E} - e\boldsymbol{v}\times\boldsymbol{B} = -mck_u K \frac{dz}{dt}\sin(k_u z)\hat{x}$$

$$= mcK\frac{d}{dt}\cos(k_u z)\hat{x}.$$

$$\boxed{K = \frac{eB_0}{mck_u} = 0.934\,\lambda_u[\mathrm{cm}]B_0[\mathrm{T}].}$$

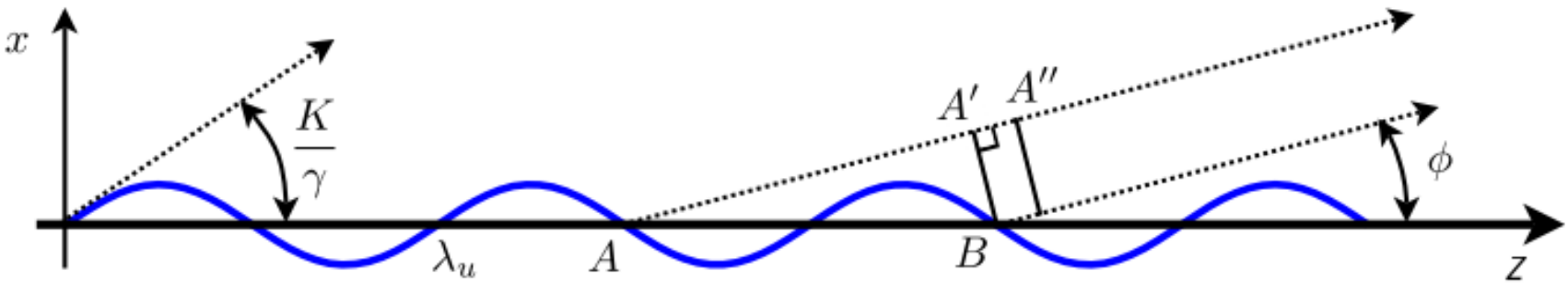
$$\beta_x = \frac{K}{\gamma}\cos(k_u z) \qquad x = \frac{K}{\gamma k_u}\sin(k_u z)$$

$$\beta_z = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2} \approx 1 - \frac{1}{2\gamma^2} - \frac{K^2}{2\gamma^2}\cos^2(k_u z)$$

$$= 1 - \frac{1 + K^2/2}{2\gamma^2} - \frac{K^2}{4\gamma^2}\cos(2k_u z)$$

In beam frame 8

Undulator radiation resonant wavelength



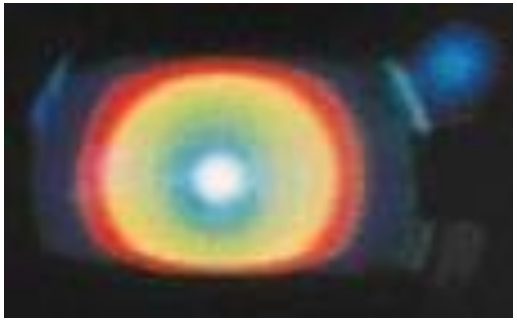
$$\frac{\lambda_1(\phi)}{c} = \frac{\widetilde{AB}}{v} - \frac{\overline{AA'}}{c}$$

$$\widetilde{AB} = \int_0^{\lambda_u} dz\, \sqrt{1+(x')^2} \approx \int_0^{\lambda_u} dz\, \left(1+\frac{1}{2}x'^2\right) = \lambda_u\left(1+\frac{K^2}{4\gamma^2}\right)$$

$$\overline{AA'} = \lambda_u \cos\phi \approx \lambda_u\left(1-\frac{\phi^2}{2}\right)$$

$$\frac{\lambda_1(\phi)}{c} = \frac{\lambda_u}{c}\left[\frac{1+K^2/(4\gamma^2)}{\beta} - \left(1-\frac{\phi^2}{2}\right)\right]$$
$$\approx \frac{\lambda_u}{c}\frac{1+K^2/2+\gamma^2\phi^2}{2\gamma^2},$$

UVSOR, Okazaki, Japan

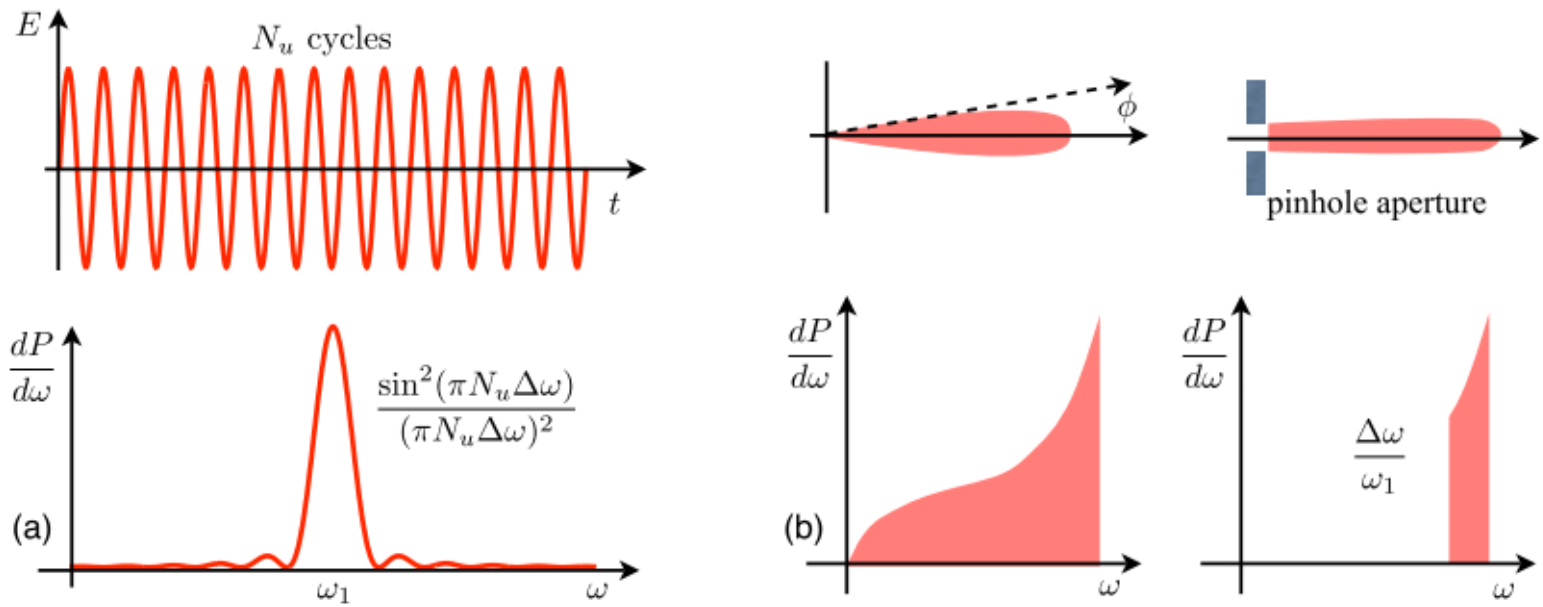


Works for harmonics
$\lambda_h = \lambda_1/h$

Radiation spectrum

Since any electron makes N_u oscillations in an undulator composed of N_u periods, the resulting wave train has N_u cycles. Thus, the spectrum of the undulator radiation at observation angle ϕ is peaked around $\omega_1(\phi)$ with intrinsic bandwidth

$$\frac{\Delta\omega}{\omega_1} = \frac{\Delta\lambda}{\lambda_1} \sim \frac{1}{N_u}.$$

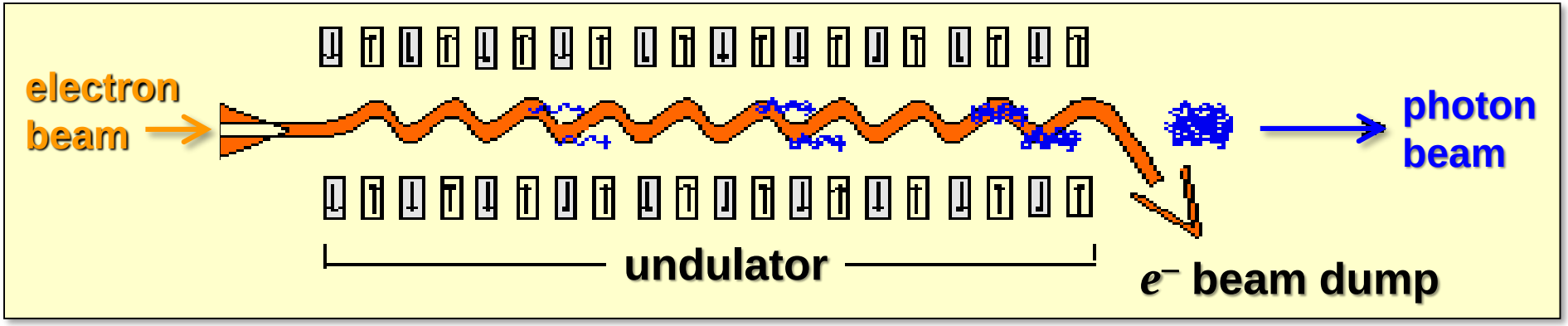


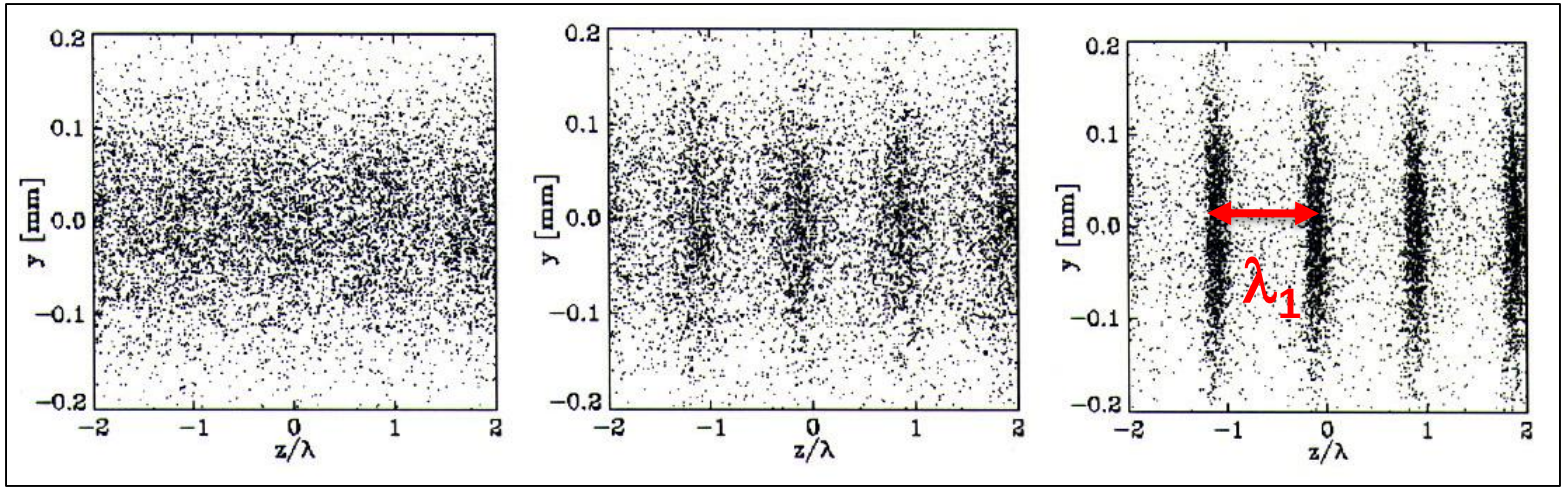
This is illustrated in Fig. 1.5(a). As the observation angle ϕ increases from zero, the wavelength is "red" shifted because

$$\frac{\lambda_1(\phi) - \lambda_1(0)}{\lambda_1(0)} = \frac{\gamma^2\phi^2}{1 + K^2/2} = \frac{\lambda_u}{2\lambda_1(0)}\phi^2 > 0.$$

Free Electron Lasers

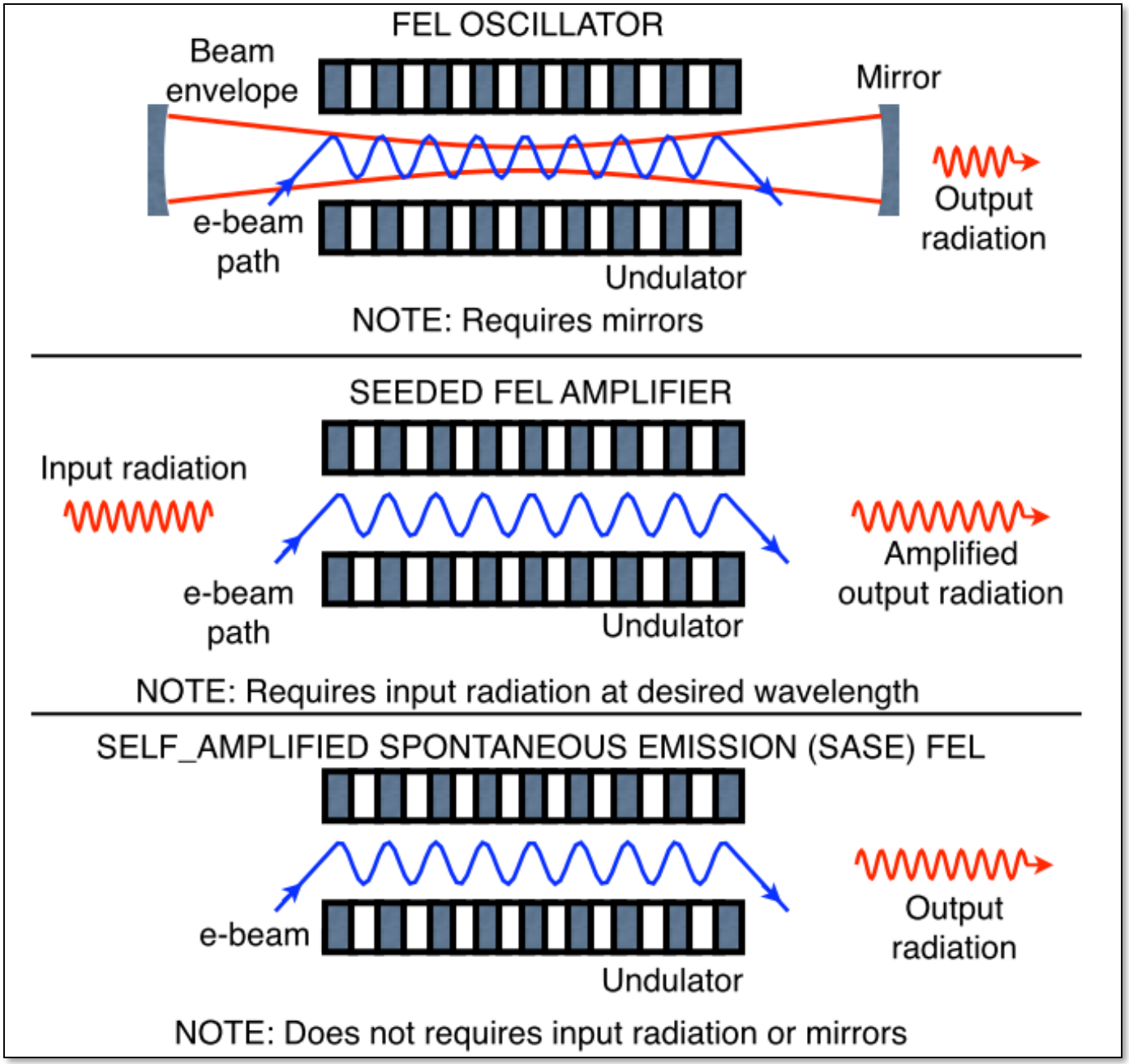
- Produced by **resonant interaction** of a relativistic electron beam with EM radiation in an undulator (J. Madey, 1971)





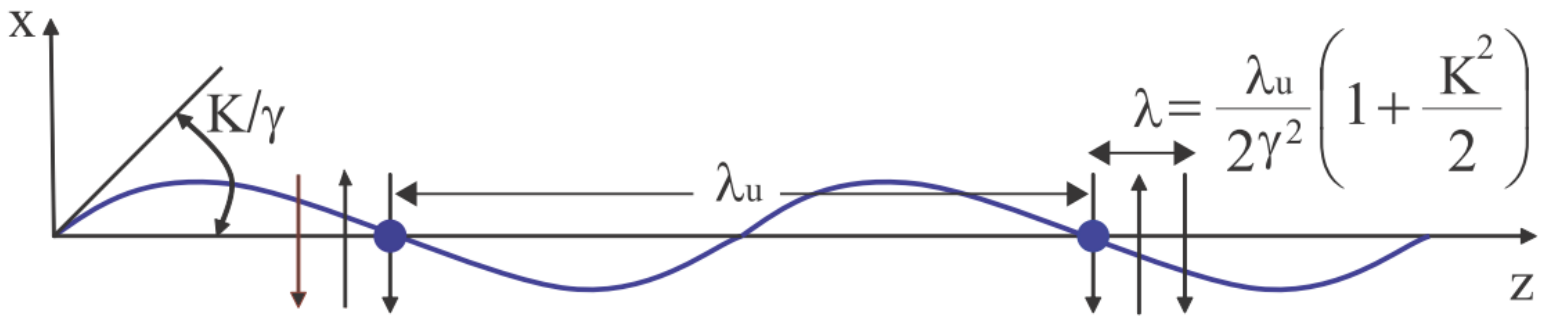
- **Radiation intensity ∝ N^2**
- **Tunable, Powerful, Coherent radiation sources**

Three FEL modes



FEL longitudinal dynamics

- Resonant interaction



- Energy exchange in the undulator

$$v_x = \frac{Kc}{\gamma}\cos(k_u z)$$

$$\boldsymbol{E}(z,t) = \hat{x}E_0\cos(kz - \omega t + \phi), \qquad \omega = ck = \frac{2\pi c}{\lambda}$$

$$\frac{d\gamma}{dt} = -\frac{eE_0Kc}{mc^2\gamma}\cos(k_u z)\cos(kz - \omega t + \phi)$$

$$= -\frac{eE_0Kc}{2mc^2\gamma}\left\{\cos\underbrace{[(k+k_u)z - \omega t + \phi]}_{\equiv\theta-\pi/2} + \underbrace{\cos[(k-k_u)z - \omega t + \phi]}_{\text{neglect, will justify later}}\right\},$$

where we have introduced the particle phase $\theta \equiv (k+k_u)z - \omega t + \phi - \pi/2$.

- Time rate of change in the phase is

$$\frac{d\theta}{dt} = (k + k_u)v_z - ck. \qquad v_z \to \bar{v}_z = c\left(1 - \frac{1 + K^2/2}{2\gamma^2}\right)$$

$$\frac{d\theta}{dt} = ck\left(\frac{k_u}{k} - \frac{1 + K^2/2}{2\gamma^2}\right)$$

In order to have a stationary phase ($d\theta/dt = 0$) and significant energy exchange, we need

$$\frac{k_u}{k} = \frac{\lambda}{\lambda_u} = \frac{1 + K^2/2}{2\gamma^2},$$

- This is resonant condition if $\lambda=\lambda_1$, and $\gamma=\gamma_r$

$$\frac{\lambda_1}{\lambda_u} \equiv \frac{1 + K^2/2}{2\gamma_r^2}.$$

- Since γ a function of time, we introduce a normalized energy variable

$$\eta \equiv \frac{\gamma - \gamma_r}{\gamma_r} \ll 1$$

- The phase equation becomes $\frac{d\theta_j}{dt} = 2k_u c\eta_j$ (j for each individual electron)

- The energy equation becomes

$$\frac{d\eta_j}{dt} = \frac{1}{\gamma_r}\frac{d\gamma_j}{dt} = -\frac{eE_0Kc}{2\gamma_j\gamma_r mc^2}\sin\theta_j.$$

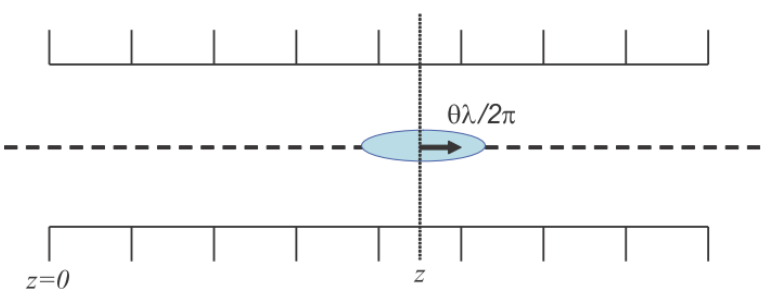
- It is convenient to use *z* as the independent variable *z* ~ *ct*

z: The independent variable giving the location inside the undulator.

$t(z)$: The time an electron arrives at z.

$\theta(z)$: The ponderomotive phase defined by

$$\theta = (k+k_u)z - \omega t + \text{const.} = \omega\left[\frac{k+k_u}{\omega}z - \bar{t}(z)\right] + \text{const.}$$
$$= \omega\left[z/\bar{v}_z - \bar{t}(z)\right] + \text{const.}$$



- If we keep track of the oscillatory part of $v_z = \bar{v}_z - \frac{cK^2}{4\gamma^2}\cos(2k_uz) \approx \bar{v}_z - \frac{ck_uK^2}{k_1(2+K^2)}\cos(2k_uz)$

$$K \longrightarrow K[\text{JJ}], \quad \text{with} \quad [\text{JJ}] \equiv J_0\left(\frac{K^2}{4+2K^2}\right) - J_1\left(\frac{K^2}{4+2K^2}\right)$$

see tutorial

FEL Pendulum equation

- Longitudinal electron motion in combined undulator and radiation fields described by pendulum equations

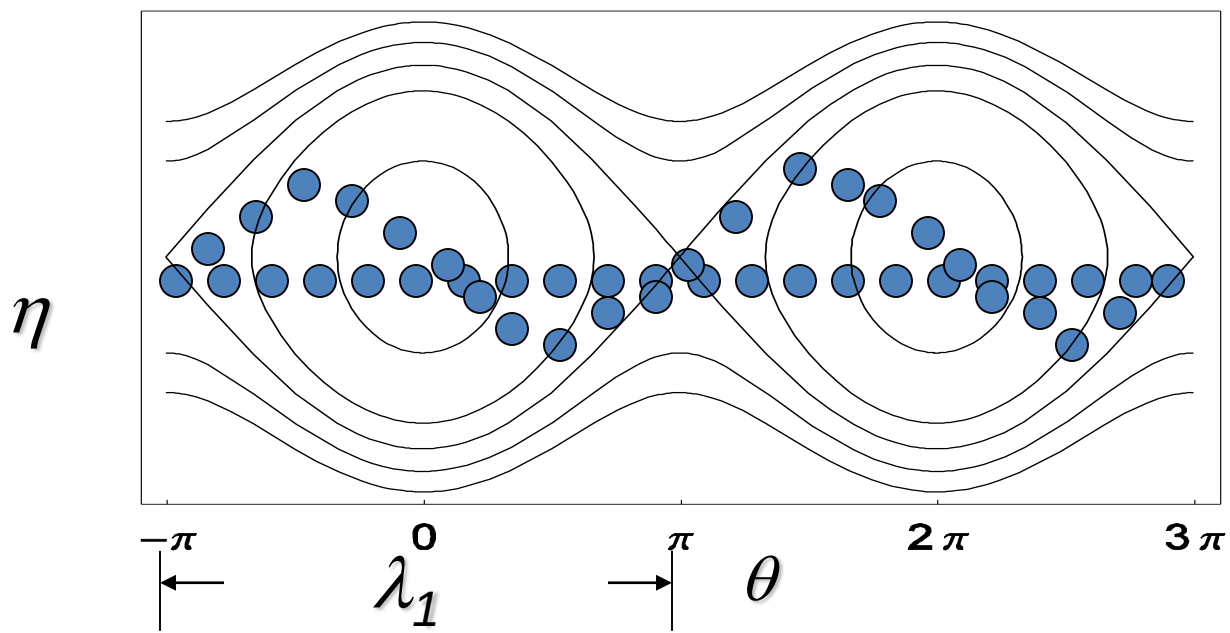
$$\frac{d\theta}{dz} = 2\eta k_u, \qquad \frac{d\eta}{dz} = \chi_1(\tilde{E}e^{i\theta} + \tilde{E}^* e^{-i\theta})$$

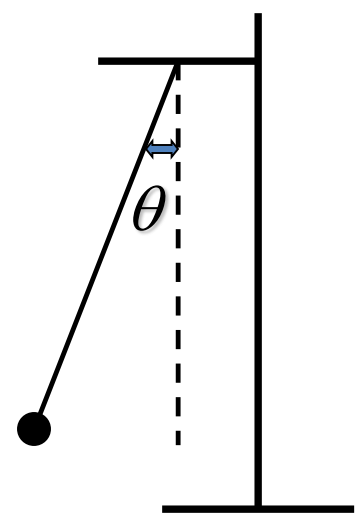
$$\chi_1 = \frac{eK[\mathrm{JJ}]}{(2\gamma_0^2 mc^2)}$$

$$[\mathrm{JJ}] = J_0\left(\frac{K^2}{4+2K^2}\right) - J_1\left(\frac{K^2}{4+2K^2}\right)$$

for planar undulator

=1 for helical undulator





Low-gain regime

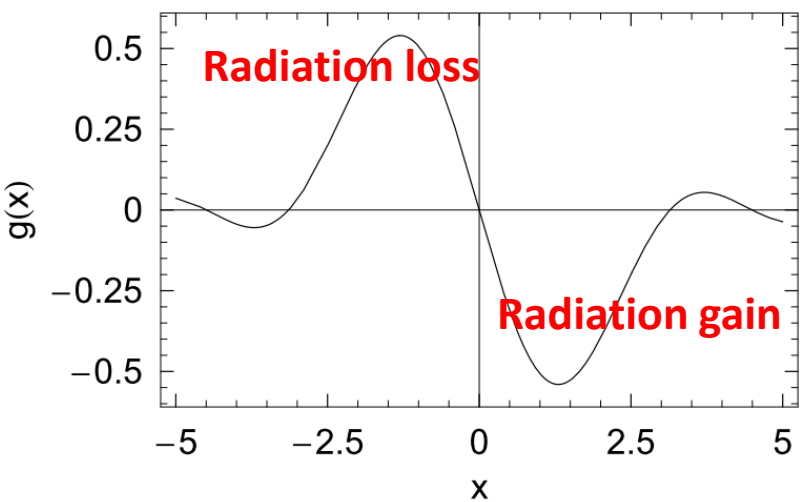
- Gain per pass is small, ignore Maxwell equation.
- Use only particle motion and energy conservation.

Therefore, the average net change in the electron beam energy at second order is given by

$$\langle \Delta\eta \rangle = \langle \epsilon^2 \eta_2(L_u) \rangle = \frac{e^2 E_0^2 K^2 [\mathrm{JJ}]^2}{4\gamma_r^4 (mc^2)^2} \frac{k_u L_u^3}{4} g(x),$$

where we have introduced the normalized gain function

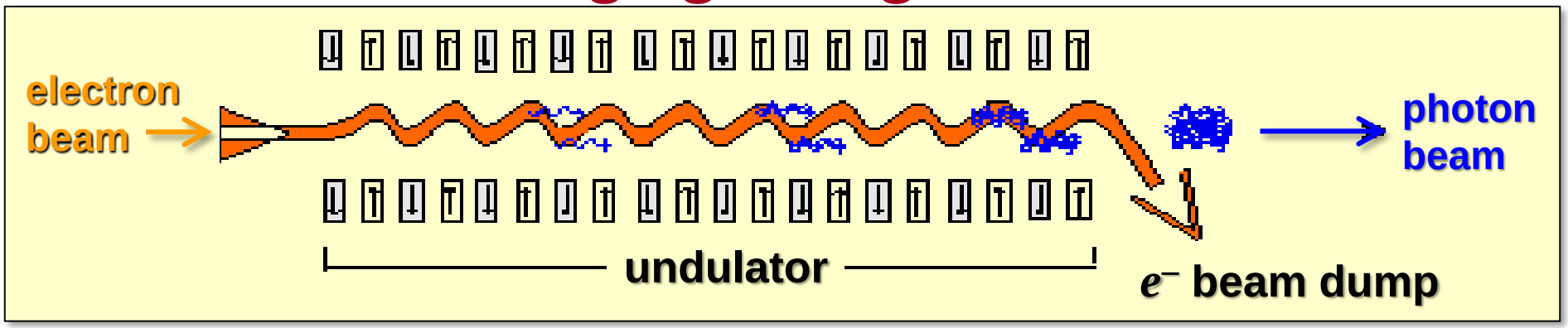
$$g(x) = \frac{d}{dx}\left(\frac{\sin x}{x}\right)^2$$

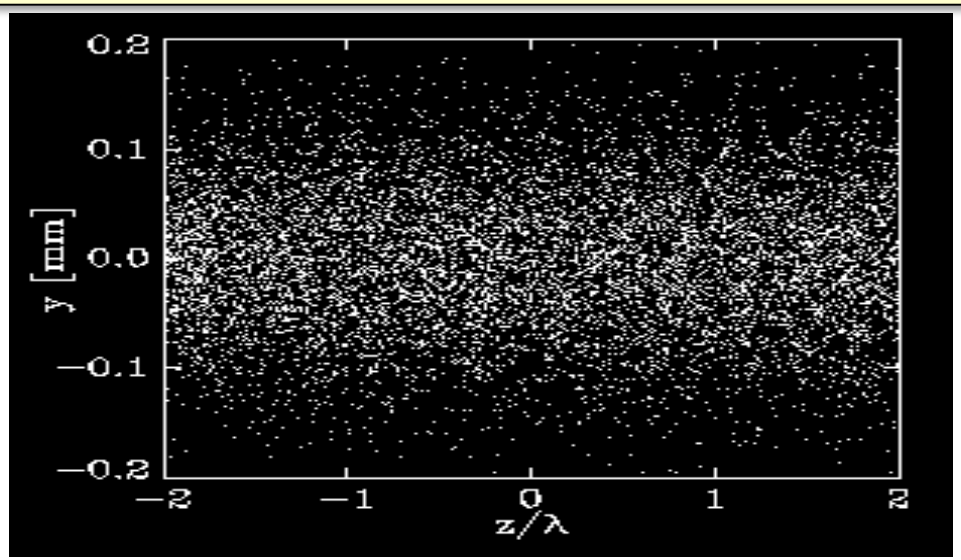


$$x = \pi N_u (\omega_0 - \omega_1)/\omega_1 = \pi N_u \Delta\nu.$$

Radiation frequency

High-gain regime





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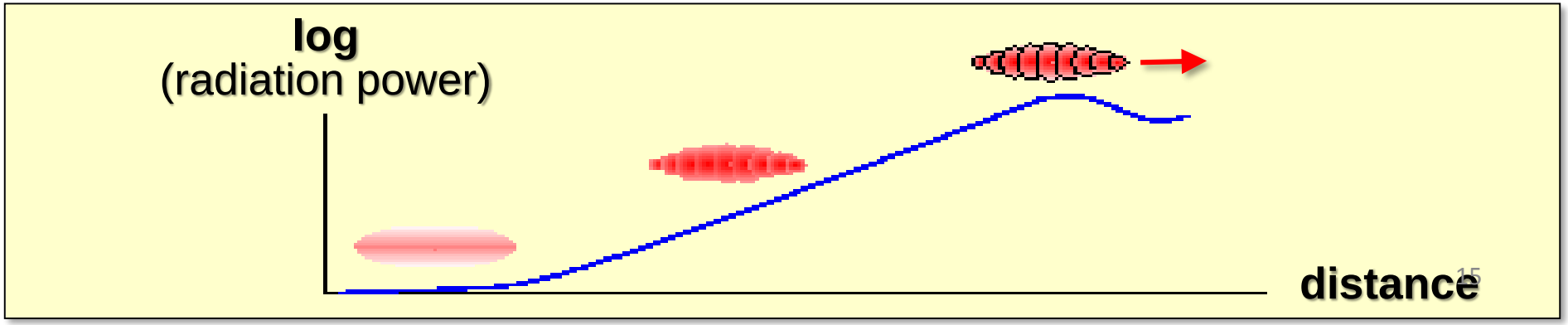
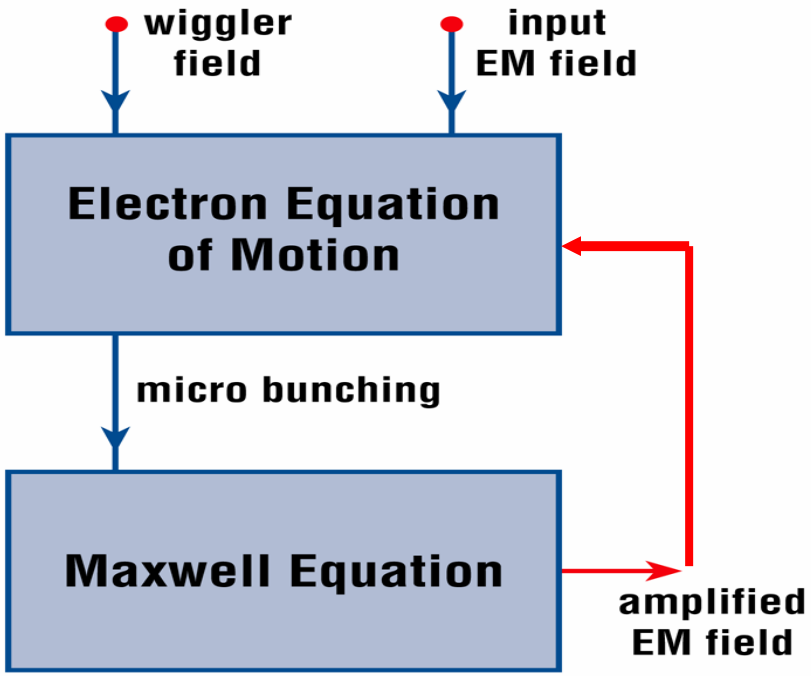


High Gain FEL – Exponential Growth



1D Theory: exponential growth, proposes SASE

Self Amplified Spontaneous Emission

Saldin et al. (1980)
Bonifacio, Pellegrini et al. (1984)

Maxwell equation

- Transverse electric field:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right] E_x = -\frac{1}{\epsilon_0 c^2}\frac{\partial J_x}{\partial t}.$$

- Transverse current

$$J_x = -\frac{ecK}{2\pi\sigma_x^2}\cos(k_u z)\sum_{j=1}^{N_e}\frac{1}{\gamma_j}\delta[z - z_j(t)],$$

Beam cross section area

- We apply slowly-varying phase and amplitude (SVPA) approximation

$$E_x = E(z,t)e^{i(k_1 z - \omega_1 t)} + E^*(z,t)e^{-i(k_1 z - \omega_1 t)}.$$

slowly varying fast oscillatory

- We then decompose the wave operator as

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} = D_+ D_- \quad \text{with} \quad D_\pm = \frac{1}{c}\frac{\partial}{\partial t} \pm \frac{\partial}{\partial z},$$

- Note that $D_-\left[Ee^{i(k_1 z-\omega_1 t)}\right] = -2ikEe^{i(k_1 z-\omega_1 t)} + \cancel{e^{i(k_1 z-\omega_1 t)}D_-E}.$

neglect because $|D_-E| \ll k_1|E|$

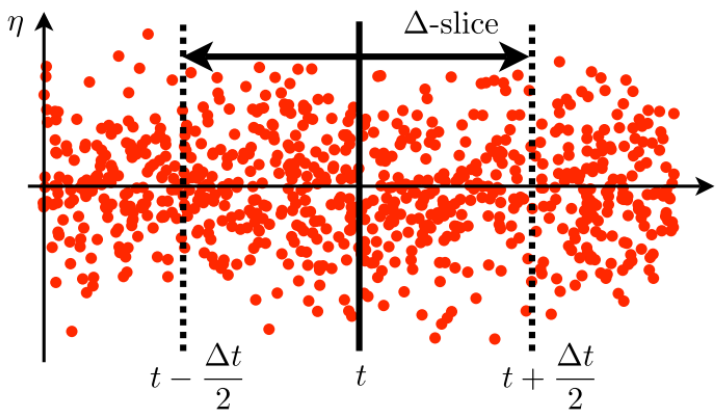
- In addition, we have

$$D_+\left[Ee^{i(k_1 z-\omega_1 t)}\right] = e^{i(k_1 z-\omega_1 t)}D_+E$$

- Maxwell equation becomes to

$$D_+E - e^{-2i(k_1 z-\omega_1 t)}D_+E^* = -\frac{i}{2\epsilon_0 k_1 c^2}e^{-ik_1(z-\omega_1 t)}\frac{\partial J_x}{\partial t}.$$

- Average both sides by $\frac{1}{\Delta t}\int_{t-\Delta t/2}^{t+\Delta t/2} dt' \Big|_{\text{at fixed } z}$



$$D_+E = -\frac{1}{2\epsilon_0 c}\frac{1}{\Delta t}\frac{ecK}{2\pi\sigma_x^2}\sum_{j\in[z,\Delta t]}\frac{1}{\gamma_j}e^{-i[k_1 z-\omega_1 t_j(z)]}\cos(k_u z),$$

$$\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]E = -\chi_2 n_e\langle e^{-i\theta_j}\rangle_\Delta \qquad \chi_2 = \frac{eK[\mathrm{JJ}]}{4\epsilon_0\gamma_r}$$

FEL equations and energy conservation

- Write the pendulum equations in terms of the slowly varying E:

$$\frac{d\theta_j}{dz} = 2k_u\eta,$$
$$\frac{d\eta_j}{dz} = \chi_1\left(Ee^{i\theta_j} + E^* e^{-i\theta_j}\right)$$

$$\chi_1 = \frac{eK[\mathrm{JJ}]}{2\gamma_r^2 mc^2}$$

- Change the field equation variables from (t , z) to (θ , z)

Using $\theta = (k_1 + k_u)z - ck_1 t$, we have

$$\left.\frac{\partial}{\partial z}\right|_t + \frac{1}{c}\left.\frac{\partial}{\partial t}\right|_z = \left.\frac{\partial}{\partial z}\right|_\theta + k_u\left.\frac{\partial}{\partial \theta}\right|_z,$$

- We obtain

$$\left[\frac{\partial}{\partial z} + k_u\frac{\partial}{\partial \theta}\right]E(\theta; z) = -\chi_2 n_e\langle e^{-i\theta_j}\rangle_\Delta \qquad \chi_2 = \frac{eK[\mathrm{JJ}]}{4\epsilon_0\gamma_r}$$

- 1D FEL equations conserves total energy (particle+field)

$$\frac{d}{dz}\left[\sum_j n_e(1+\eta_j)\gamma_r mc^2 + \int d\theta\,\frac{\epsilon_0}{2}\left|E(\theta; z)\right|^2\right] = 0$$

Particle energy density

Field energy density

FEL scaling parameter ρ

We introduce the as yet unspecified parameter ρ by defining the scaled longitudinal coordinate $\hat{z} \equiv 2k_u \rho z$ that leads to the phase equation

$$\frac{d\theta_j}{d\hat{z}} = \hat{\eta}_j \quad \text{for} \quad \hat{\eta}_j \equiv \frac{\eta_j}{\rho} \quad \text{(the new "momentum" variable).}$$

To simplify the energy equation for $\hat{\eta}_j$, we define the dimensionless complex field amplitude

$$a = \frac{\chi_1}{2k_u \rho^2} E,$$

in terms of which the energy equation reduces to

$$\frac{d\hat{\eta}_j}{d\hat{z}} = a(\theta_j, \hat{z}) e^{i\theta_j} + a(\theta_j, \hat{z})^* e^{-i\theta_j}.$$

Writing the field equation (4.72) in terms of $\hat{z}$ and a, we have

set to 1

$$\left[\frac{\partial}{\partial \hat{z}} + \frac{1}{2\rho}\frac{\partial}{\partial \theta}\right] a(\theta, \hat{z}) = -\frac{\chi_1}{2k_u \rho^2}\frac{n_e \chi_2}{2k_u \rho} \langle e^{-i\theta_j} \rangle_\Delta.$$

$$\rho = \left[\frac{n_e \chi_1 \chi_2}{(2k_u)^2}\right]^{1/3} = \left(\frac{e^2 K^2 [\mathrm{JJ}]^2 n_e}{32\epsilon_0 \gamma_r^3 m c^2 k_u^2}\right)^{1/3}$$

$$= \left[\frac{1}{8\pi}\frac{I}{I_A}\left(\frac{K[\mathrm{JJ}]}{1+K^2/2}\right)^2 \frac{\gamma \lambda_1^2}{2\pi\sigma_x^2}\right]^{1/3},$$

I_A=17 kA is Alfven current

FEL efficiency

- At saturation electrons are fully bunched so that $|\langle e^{-i\theta_j}\rangle_\Delta| \rightarrow 1$, and radiation amplitude $|a| \rightarrow 1$ at saturation to $|E| \rightarrow 2k_u\rho^2/\chi_1$, so that the maximum field energy density

$$2\epsilon_0 |E|^2 \sim 2\epsilon_0\rho\frac{4k_u^2\rho^3}{\chi_1^2} = 2\epsilon_0\rho\frac{\chi_2}{\chi_1} = \rho n_e\gamma_r mc^2.$$

Because $n_e mc^2\gamma_r$ is the electron energy density, we see that ρ also gives the FEL efficiency at saturation:

$$\boxed{\rho = \frac{\text{field energy generated}}{\text{e-beam kinetic energy}}.}$$

Therefore, the FEL (or Pierce) parameter ρ determines the main characteristics of high-gain FEL systems, including

$$\text{Saturation power} \sim \rho\times(\text{e-beam power}),$$

saturation efficiency ($\rho \sim 10^{-3}$ for short-wavelength FELs)

1D solution

- Illustrate FEL gain by neglecting θ dependence of E field (slippage)

$$\frac{d\theta_j}{d\hat{z}} = \hat{\eta}_j \qquad \boxed{\hat{z} = 2\rho k_u z}$$

$$\frac{d\hat{\eta}_j}{d\hat{z}} = a e^{i\theta_j} + a^* e^{-i\theta_j} \qquad \boxed{\hat{\eta} = \frac{\eta}{\rho}}$$

$$\frac{da}{d\hat{z}} = -\langle e^{-i\theta_j} \rangle_\Delta \,. \qquad a = \frac{\chi_1}{2k_u\rho^2} E$$

These are $2N_\Delta + 2$ coupled first order ordinary differential equations, $2N_\Delta$ for the particles, and 2 equations for the complex amplitude a. In general, these can only be solved via computer simulation. However, the system can be linearized in terms of three collective variables

$$a \qquad \text{(field amplitude)}$$

$$b = \langle e^{-i\theta_j} \rangle_\Delta \qquad \text{(bunching parameter)}$$

$$P = \langle \hat{\eta}_j e^{-i\theta_j} \rangle_\Delta \qquad \text{(collective momentum).}$$

Cubic equation

$$\frac{da}{d\hat{z}} = -b$$

Bunching produces coherent radiation.

$$\frac{db}{d\hat{z}} = -iP$$

Energy modulation becomes density bunching.

$$\frac{dP}{d\hat{z}} = a$$

Coherent radiation drives energy modulation.

These are three coupled first order equations, which can be reduced to a single third-order equation for a as

$$\frac{d^3a}{d\hat{z}^3} = ia.$$

We solve the linear equation by assuming that the field dependence is $\sim e^{-i\mu\hat{z}}$, which results in the following dispersion relation for μ:

$$\mu^3 = 1.$$

$$\mu_1 = 1, \qquad \mu_2 = \frac{-1-\sqrt{3}i}{2}, \qquad \mu_3 = \frac{-1+\sqrt{3}i}{2}$$

Exponential solution

- General solution

$$a(\hat{z}) = \sum_{\ell=1}^{3} C_\ell e^{-i\mu_\ell \hat{z}}$$

- Solving for the initial conditions

$$a(\hat{z}) = \frac{1}{3}\sum_{\ell=1}^{3}\left[a(0) - i\frac{b(0)}{\mu_\ell} - i\mu_\ell P(0)\right] e^{-i\mu_\ell \hat{z}}.$$

- At large undulator distance, the exponential solution (μ_3) dominates

$$a(\hat{z}) \approx \frac{1}{3}\left[a(0) - i\frac{b(0)}{\mu_3} - i\mu_3 P(0)\right] e^{-i\mu_3 \hat{z}}.$$

seeded SASE or pre-bunched

- Consider the case of SASE (no seed and no energy modulation)

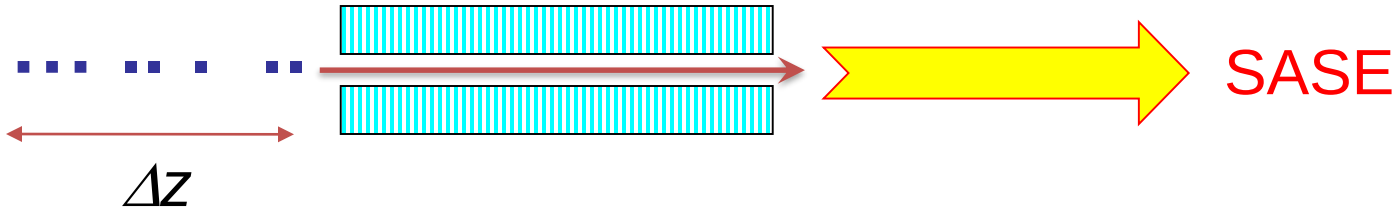
$$\langle |a(\hat{z})|^2 \rangle \approx \frac{1}{9}\langle |b(0)|^2 \rangle e^{\sqrt{3}\hat{z}}.$$

Here, the scaled propagation distance $\sqrt{3}\hat{z} = \sqrt{3}(2k_u z\rho) = z/L_{G0}$, and the ideal 1D power gain length is

$$L_{G0} \equiv \frac{\lambda_u}{4\pi\sqrt{3}\rho}.$$

What is SASE?

- Shot noise originates from discrete nature of electrons



electron arrival time t is random ➔ spontaneous emission
➔ amplified by FEL interaction
➔ quasi-coherent x-rays

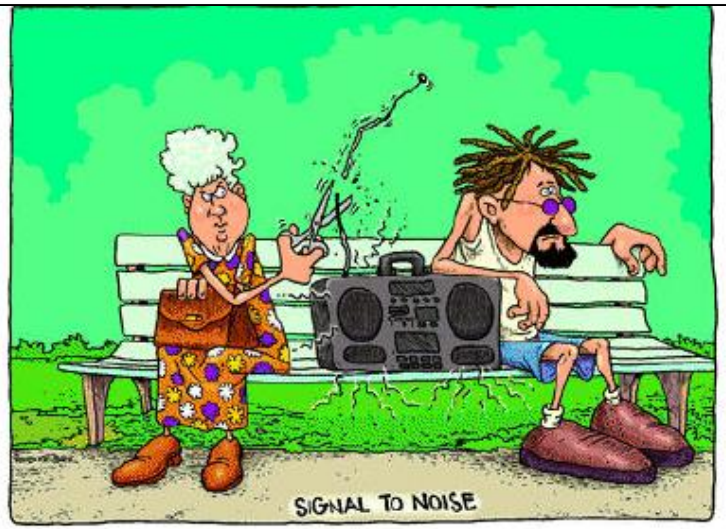


FIG. 1: Whether noise is a nuisance or a signal may depend on whom you ask. (Cartoon by Rand Kruback. Used with permission of Agilent Technologies.)

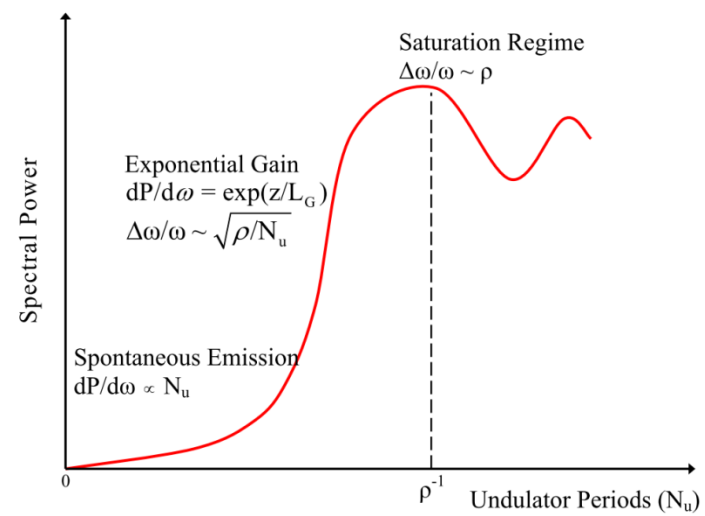
Qualitative Analysis

The bunching at the undulator entrance $\langle |b(0)|^2 \rangle$ derives from the initial shot noise of the beam, which is subsequently amplified by the FEL process. This input noise turns out to be approximately given by the spontaneous undulator radiation generated in the first gain length L_{G0} of the undulator.

$$\langle |b(0)|^2 \rangle = \left\langle \frac{1}{N_{l_{\rm coh}}^2} \left| \sum_{j \in l_{\rm coh}} e^{-i\theta_j} \right| \right\rangle \approx \frac{1}{N_{l_{\rm coh}}},$$

where $N_{l_{\rm coh}}$ is the number of electrons in a coherence length $l_{\rm coh}$. As we have mentioned, the normalized bandwidth of SASE is $\Delta\omega/\omega \sim \rho$, so that the coherence length $l_{\rm coh} \sim \lambda_1/\rho$; alternatively, one can recognize the coherence length as approximately given by the amount the radiation slips ahead of the electron beam in one gain length. Hence, the startup noise of a SASE FEL is characterized by

$$N_{l_{\rm coh}} \sim \frac{I}{ec} \frac{\lambda_1}{\rho}.$$



Quantitative Analysis

- Solving Maxwell-Vlasov equation in the linear regime before saturation. The Vlasov equation describes the evolution of particle longitudinal distribution under the influence of the electric field the particles generate.

- The initial particle energy distribution is described by $V(\eta)$ with $\int d\eta\, V(\eta) = 1$.

$$\frac{dP}{d\omega} = e^{4\mu_I \rho k_u z} g_A \left(\frac{dP}{d\omega}\bigg|_0 + g_S \frac{\rho\, \gamma_r m c^2}{2\pi} \right)$$

seeded SASE

$$\frac{dP}{d\omega}\bigg|_0 \equiv \text{input power spectrum},$$

$$g_A \equiv \frac{1}{|D'(\mu)|^2},$$

$$g_S \equiv \int d\eta\, \frac{V(\eta)}{|\eta/\rho - \mu|^2}.$$

$$D(\mu) \equiv \mu - \frac{\Delta\nu}{2\rho} - \int d\eta\, \frac{V(\eta)}{(\eta/\rho - \mu)^2}$$

frequency detune $\Delta\nu = (\omega - \omega_1)/\omega_1$

- Solutions to $D(\mu) = 0$ for which μ has a positive imaginary part (μ_I) give rise to exponential growing radiation power.

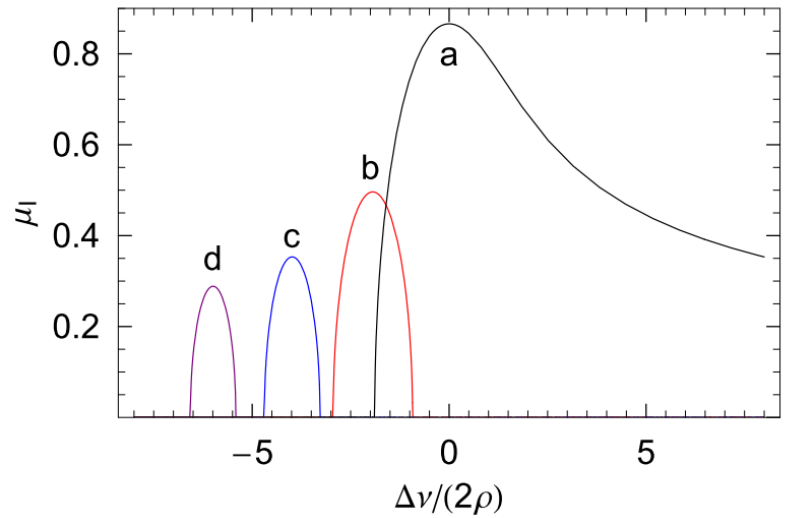
Dispersion relation

- Cubic equation is now generalized to a dispersion relation (μ as a function of $\Delta\nu$) for an arbitrary energy distribution.

$$D(\mu) = \mu - \frac{\Delta\nu}{2\rho} - \int d\eta \frac{V(\eta)}{(\eta/\rho - \mu)^2} = 0$$

- For a flattop energy distribution

$$\left(\mu - \frac{\Delta\nu}{2\rho}\right)\left(\mu^2 - \frac{\zeta^2}{4}\right) = 1$$

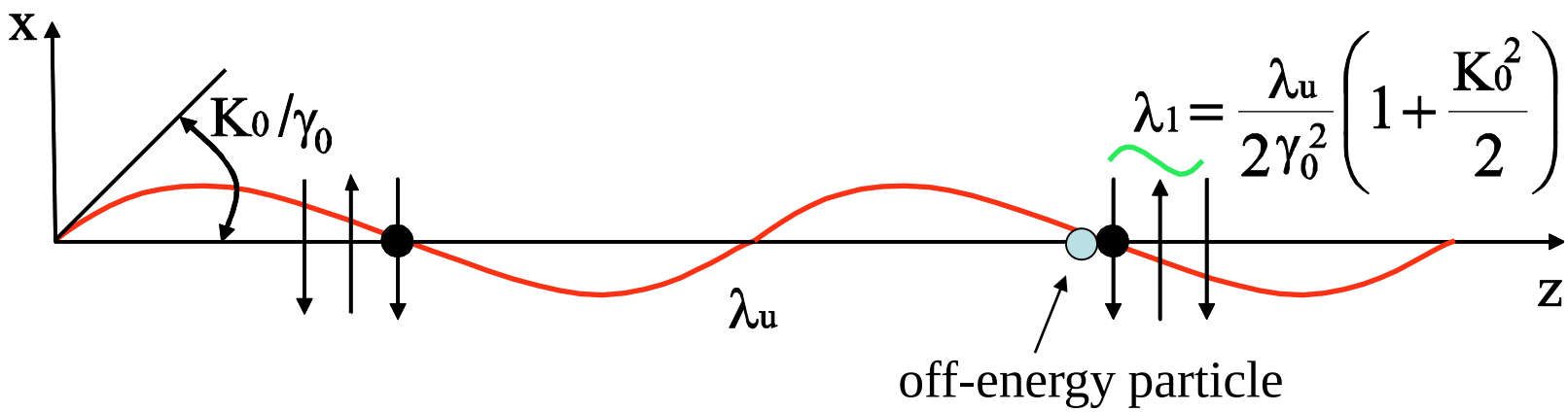


for a flattop energy distribution with the full width $\Delta\eta = \zeta\rho$, where (a) $\zeta = 0$ (black), (b) $\zeta = 2$ (red), (c) $\zeta = 4$ (blue), and (d) $\zeta = 6$ (purple).

- For a cold beam with vanishing energy spread

$$D(\mu) = \mu - \frac{\Delta\nu}{2\rho} - \frac{1}{\mu^2} = 0,$$

Effects of energy spread



➢ For efficient FEL interaction, the resonant wavelength spread caused by the energy spread over a gain length << 1

$$\frac{\Delta\gamma}{\gamma_0} = \frac{2\Delta\lambda}{\lambda_1} \ll \frac{\lambda_u}{L_G} \sim 4\pi\rho$$

$\sigma_\eta \ll \rho \sim 10^{-3}$ for short-wavelength FELs

This effect for a Gaussian energy spread can be approximated as affecting the gain length via

$$L_G(\sigma_\eta) \approx L_{G0}\left[1 + (\sigma_\eta/\rho)^2\right],$$

➢ This is a local energy spread requirement not a global one

➢ Concepts of "slice" energy spread (later)

SASE Bandwidth

- For the case of vanishing energy spread (cold beam)

$$D(\mu) = \mu - \frac{\Delta\nu}{2\rho} - \frac{1}{\mu^2} = 0,$$

$$g_A = \frac{1}{9}, \qquad g_S = 1,$$

$$L_G = L_{G0} = \frac{\lambda_u}{4\pi\sqrt{3}\rho}, \qquad \mu_{Im} = \frac{\sqrt{3}}{2}.$$

- Assuming $\Delta\nu/\rho \ll 1$, we can expand the solution in Taylor series

$$\mu \approx -\frac{1}{2}\left[1 - \frac{\Delta\nu}{3\rho} + \frac{(\Delta\nu)^2}{36\rho^2}\right] + i\frac{\sqrt{3}}{2}\left[1 - \frac{(\Delta\nu)^2}{36\rho^2}\right]$$

$$e^{4\mu_I \rho k_u z} = e^{z/L_G} \exp\left[-\frac{1}{2}\left(\frac{\omega - \omega_m}{\omega_m \sigma_\nu}\right)^2\right]$$

SASE bandwidth

$$\boxed{\sigma_\nu = \sigma_{\Delta\omega/\omega} = \sqrt{\frac{3\sqrt{3}\rho}{k_u z}} = \rho\sqrt{\frac{18}{N_G}} \approx \sqrt{\frac{0.83\rho}{z/\lambda_u}} \sim \rho}$$

SASE shot noise power

- Integrating over the SASE term over the frequency, obtain power

$$\frac{dP}{d\omega} = e^{4\mu_I \rho k_u z} g_A \left(\cancel{\left.\frac{dP}{d\omega}\right|_0} + g_S \frac{\rho\, \gamma_r mc^2}{2\pi} \right)$$

$$P = \int d\omega\, \frac{dP}{d\omega} = g_S g_A \frac{\rho\, \gamma_r mc^2}{2\pi} \sqrt{2\pi}\omega_1 \sigma_\nu e^{z/L_G}$$

$$= g_S g_A \rho P_{\text{beam}} \frac{e^{z/L_G}}{\sqrt{2} N_{l_{\text{coh}}}}.$$

Here $P_{\text{beam}} = (I/e)\gamma_r mc^2$ is the e-beam power and $N_{l_{\text{coh}}} = (I/ec)l_{\text{coh}}$ is the number of electrons in one coherence length $l_{\text{coh}} \equiv ct_{\text{coh}} = \lambda_1/(2\sqrt{\pi}\sigma_\nu)$. Since we expect the saturation power to be about ρP_{beam}, the total amplification factor will be about $N_{l_{\text{coh}}}$, which is a large number whose typical magnitude is 10^5 to 10^7.

- Example: LCLS shot noise power

  $\rho = 5\text{x}10^{-4}$, P_{beam} = 14 GV x 3 kA = 40 TW

  $P_{\text{sat}} \sim \rho\, P_{\text{beam}}$ = 20 GW

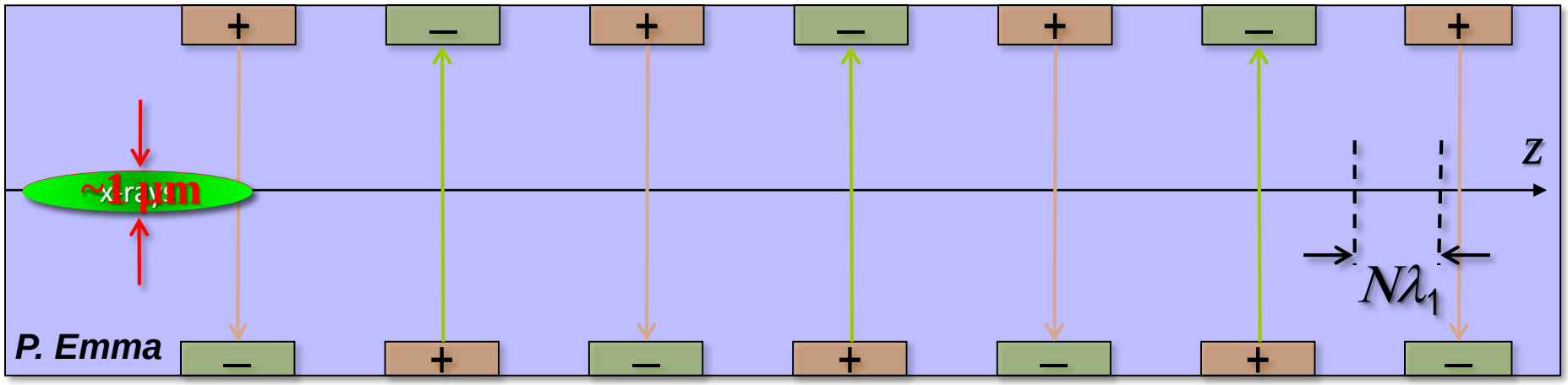
  l_c = 1.5e-10/(2.5x5x10^{-4}) = 0.12 μ m = 400 attosecond

  $N_{lc} \sim 10^7$,

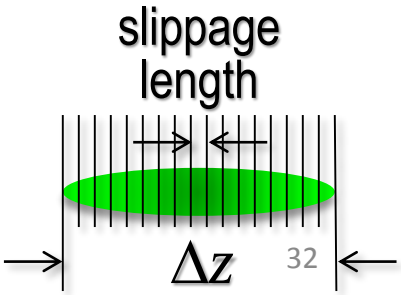
  $P_{\text{noise}} \sim \rho P_{\text{beam}}/N_{lc}$ = 2 kW

Slippage leads to coherence length and spiky structure

Due to resonant condition, light overtakes e^- beam by one radiation wavelength λ_1 per undulator period (interaction length = undulator length)



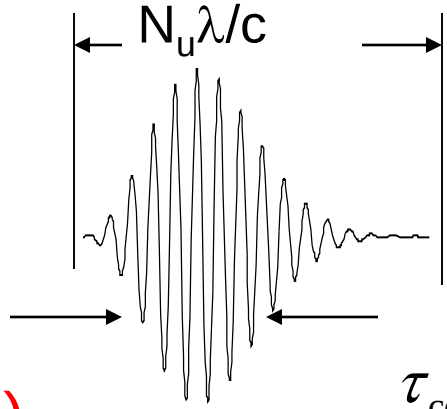
- Slippage length = $\lambda_1 \times N$ undulator periods:
  (at 1.5 Å, *LCLS* slippage length is: $l_s \approx 1.5$ fs << 100-fs pulse length)
- Each part of optical pulse is amplified by those electrons within a slippage length (an FEL slice)
- Coherence length is slippage over ~$2L_G$ ($l_c \approx l_s/10$)
- $M_L \approx \Delta z/l_c$ independent radiation sources (modes)



Temporal coherence

- $E(t)=\sum_j E_0(t-t_j)$, t_j is the random arrival time of jth e^-

E_0: wave packet of a single e^-

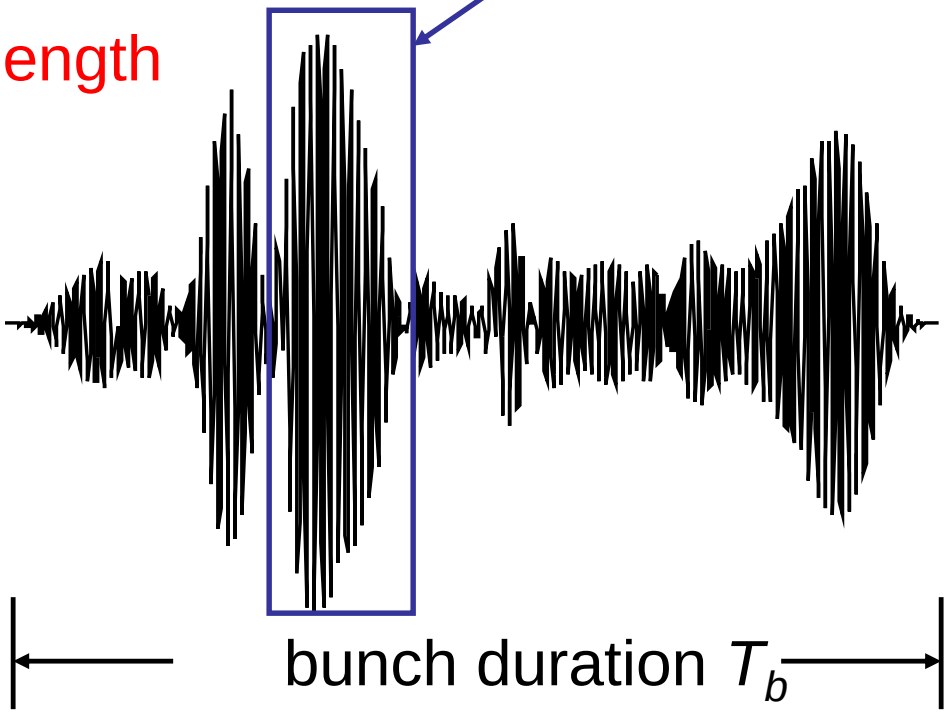


$$\tau_{coh} \approx \frac{2}{\sigma_\omega} \approx \frac{\lambda}{\pi c}\sqrt{\frac{N_u}{\rho}}$$

- coherence time (one FEL slice)

$l_c = c\,\tau_{coh} \sim 1000\ \lambda \ll$ bunch length

- Sum of all packets ➔ E(t)



SASE Statistics
(Saldin, Schneidmiller, Yurkov)

$$I(t) = |E(t)|^2 \qquad \tilde{I}(\omega) = |E(\omega)|^2$$

Probability distribution describing both the spectral intensity $\tilde{I}(\omega)$ and the time-domain intensity $I(t)$ is the exponential distribution

$$p(I) = \frac{1}{\langle I \rangle} e^{-I/\langle I \rangle}$$

SASE Pulse Energy

$$W(z) \propto \int_0^{T_b} |E(t,z)|^2 \, dt$$

Energy Fluctuation σ_W

Number of Modes M

$$\frac{\sigma_W^2}{W^2} = \frac{(W - \langle W \rangle)^2}{\langle W \rangle^2} \equiv \frac{1}{M} \equiv \frac{\tau_{coh}}{T_b}$$

Energy per pulse is described by the Gamma Distribution

$$\boxed{p_W(W) = \frac{M^M}{\Gamma(M)} \left(\frac{W}{\langle W \rangle} \right)^{M-1} \frac{1}{\langle W \rangle} \exp\left(-M \frac{W}{\langle W \rangle} \right)}$$

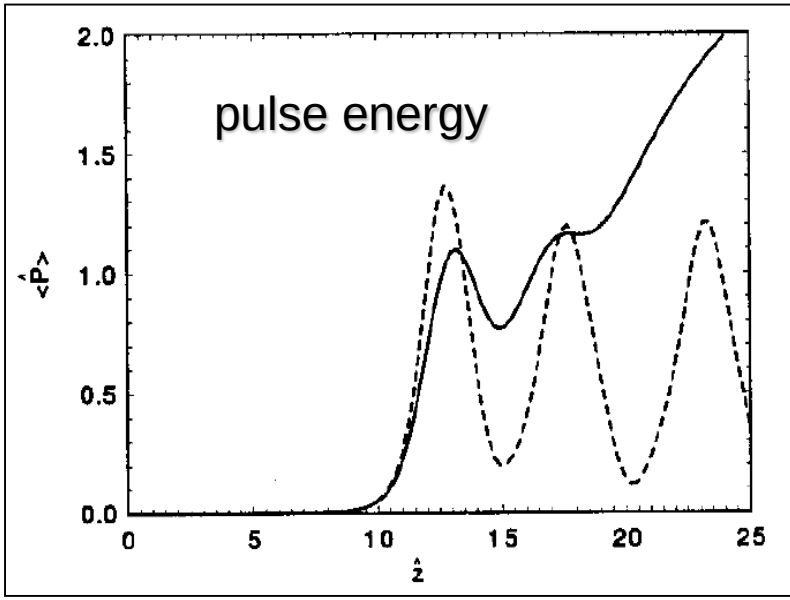
Field Correlation

$$g_1(t_1 - t_2) = \frac{\langle E(t_1) E^*(t_2) \rangle}{\sqrt{\langle |E(t_1)|^2 |E(t_2)|^2 \rangle}}$$

Intensity Correlation

$$g_2(t_1 - t_2) = \frac{\left\langle |E(t_1)|^2 |E(t_2)|^2 \right\rangle}{\left\langle |E(t_1)|^2 |E(t_2)|^2 \right\rangle}$$

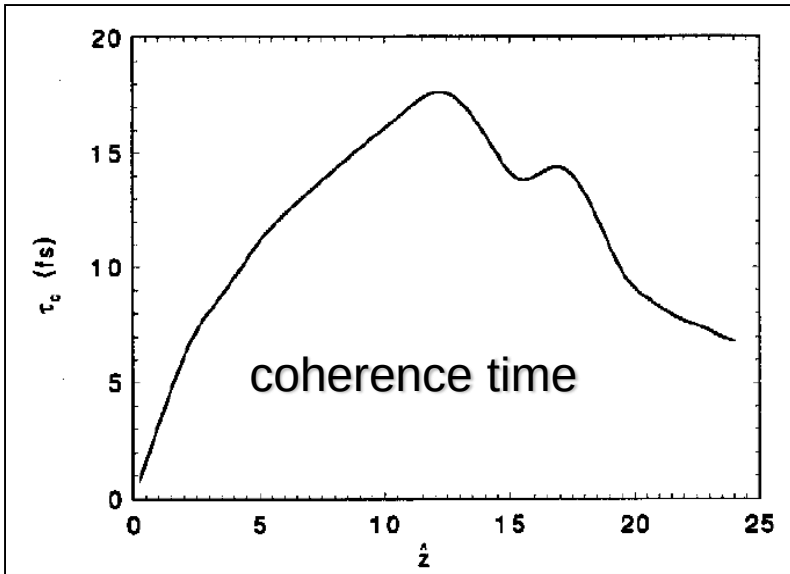
$$g_2(t) = 1 + |g_1(t)|^2$$



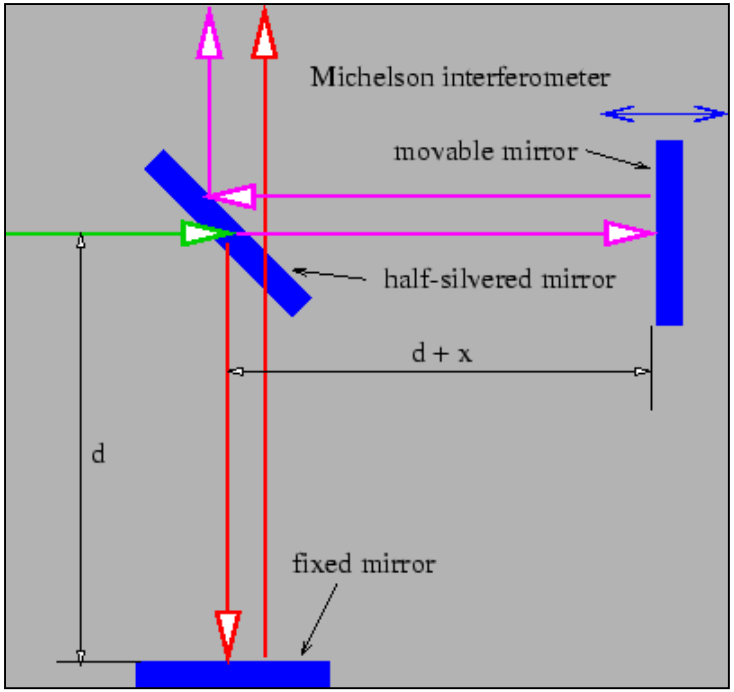
$$\frac{\sigma_W^2}{W^2} = \frac{1}{T_b^2} \int_0^{T_b} dt_1 \int_0^{T_b} dt_2 \left[g_2(t_1 - t_2) - 1 \right] = \frac{1}{T_b} \int_{-T_b}^{T_b} dt \, |g_1(t)|^2$$

$$\boxed{\frac{\sigma_W^2}{W^2} = \frac{(W - \langle W \rangle)^2}{\langle W \rangle^2} \equiv \frac{1}{M} \equiv \frac{\tau_{coh}}{T_b}}$$

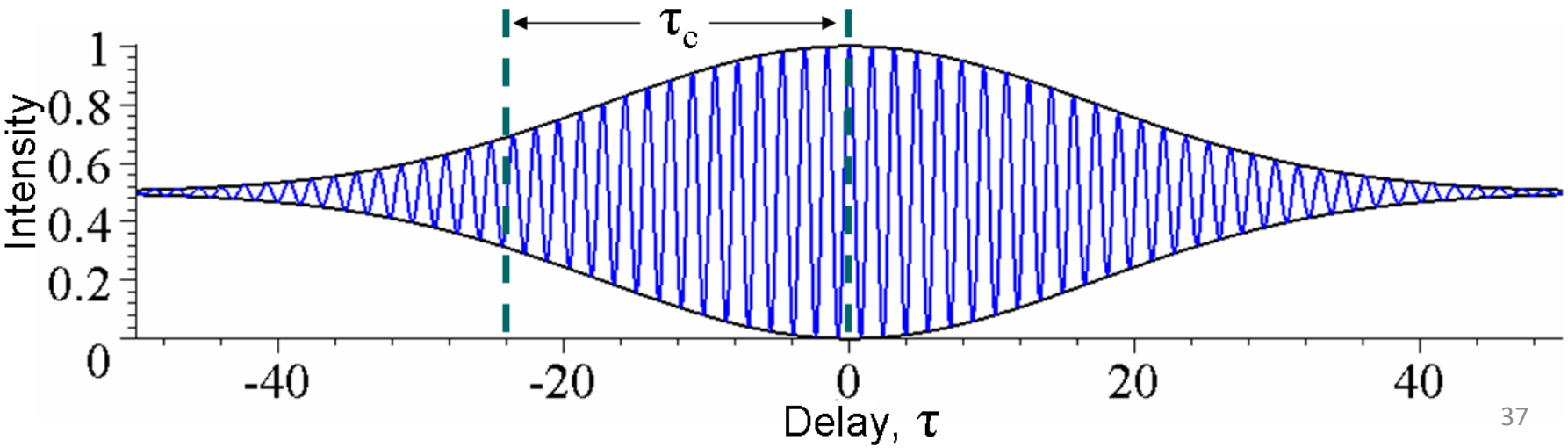
$$\boxed{\tau_{coh} = \int dt \, |g_1(t)|^2}$$



Measurement of temporal coherence



Michelson interferometer



Statistical fluctuation

- Due to noise start-up, SASE is a chaotic light temporally with M_L coherent modes (M_L spikes in intensity profile)

$$M_L \approx \frac{\text{bunch length}}{\text{coherence length}} = \frac{T_b}{\tau_{\text{coh}}}$$

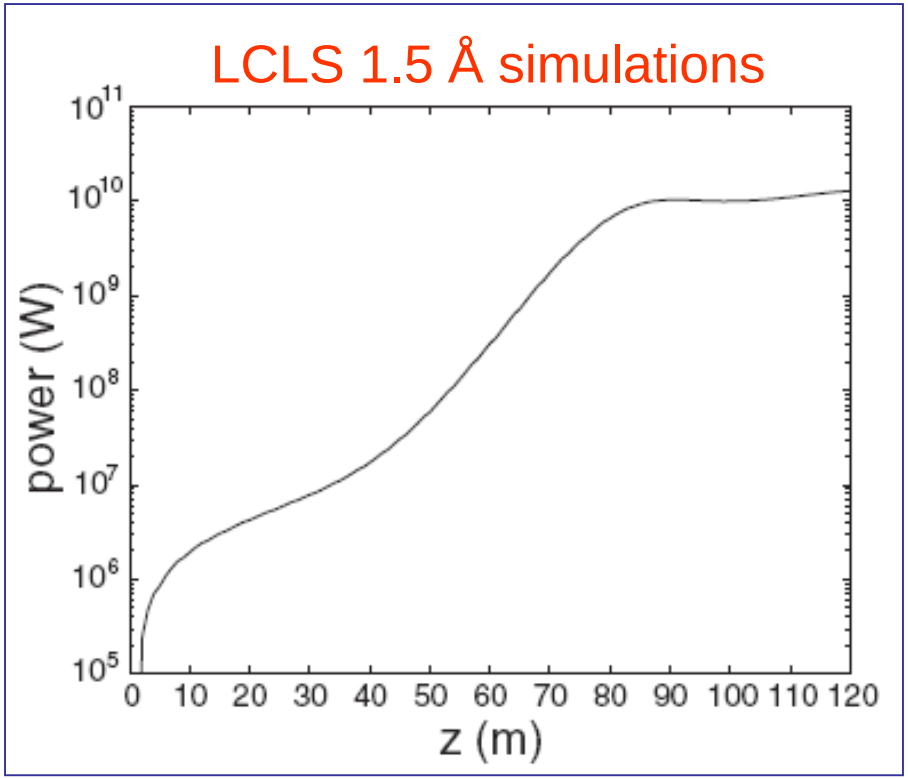
- Its longitudinal phase space is M_L larger than FT limit (rooms for improve!)

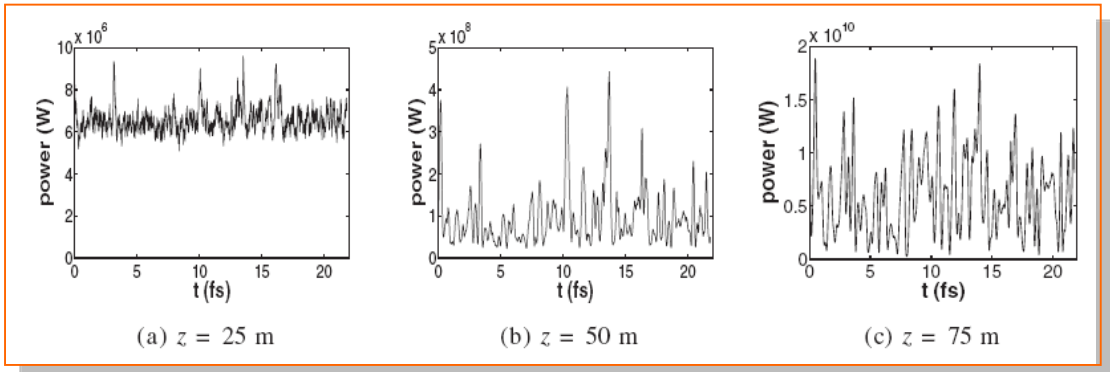
- Integrated energy fluctuation $\frac{\Delta W}{W} = \frac{1}{\sqrt{M_L}}$

- Singe spike intensity fluctuates 100%

- M_L is NOT a constant, decreases due to increasing coherence in the exponential growth, increases due to decreasing coherence after saturation)

Statistical fluctuation is large for long-wavelength exps since M_L is only a few, but much smaller for X-ray FELs





LCLS near saturation

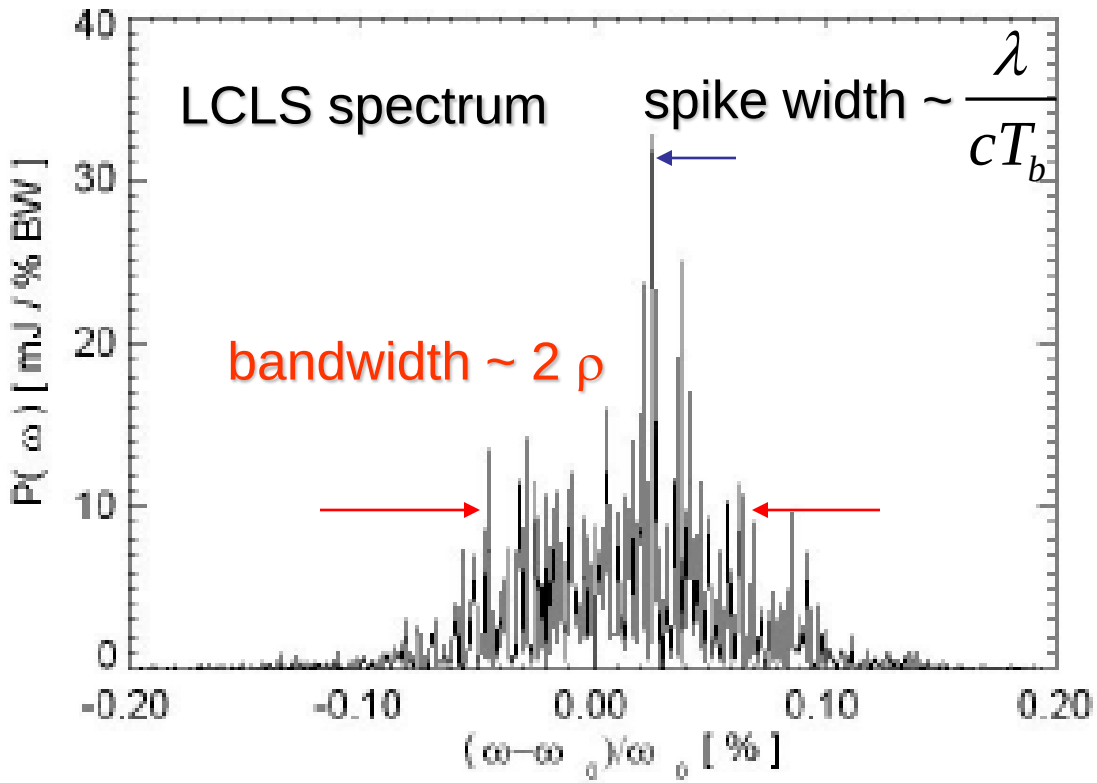
$M_L \sim 200$ ➔

$\Delta W/W \sim 7\ \%$

30 % of X-Ray Pulse Length

SASE spectrum

- Spectral properties are similar to temporal domain, except that everything is inverted



- Example, LCLS relative spectral spike width
  for 100 fs bunch length is 5×10^{-6},
  for 10 fs bunch length is 5×10^{-5},
  for 1 fs is 5×10^{-4}

SASE 1D Summary

- Power gain length $L_G = \dfrac{\lambda_u}{4\pi\sqrt{3}\rho}$

- Exponential growth $P = P_n \exp(z/L_G)$

- Startup noise power $P_n \sim \rho^2 \gamma mc^3/\lambda_r$

~ spontaneous radiation in two power gain length

- Saturation power ~ ρ x e-beam power

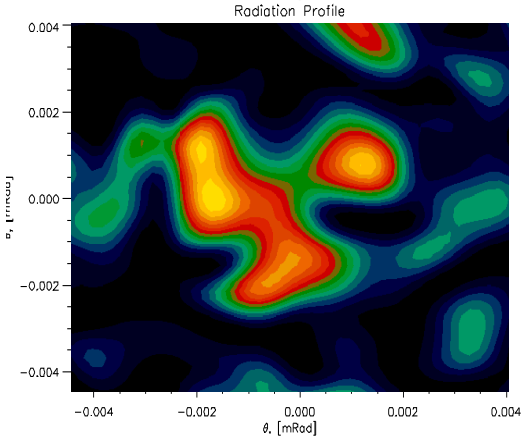
- Saturation length ~ λ_u/ρ ~ 20 L_G

- RMS bandwidth at saturation ~ ρ

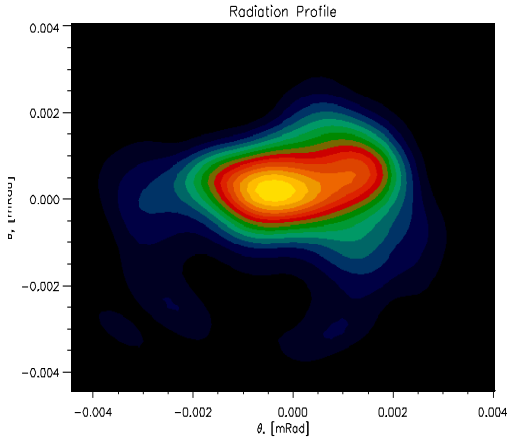
- Temporal coherence length at saturation ~ λ_r/ρ

Transverse coherence

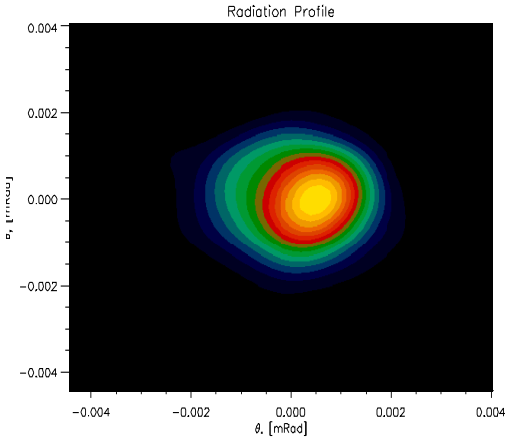
Z=25 m



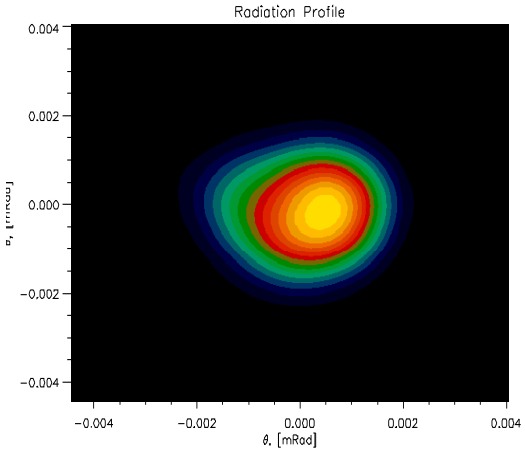
Z=37.5 m



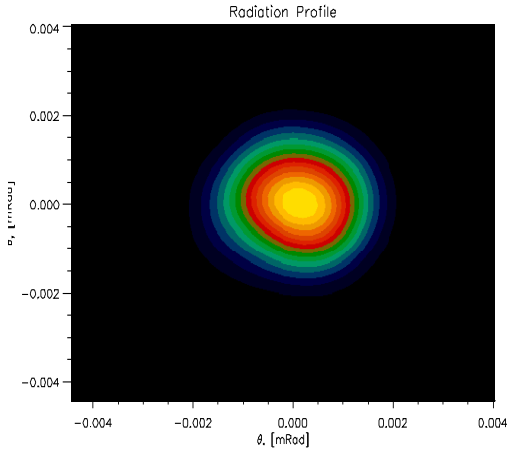
Z=50 m



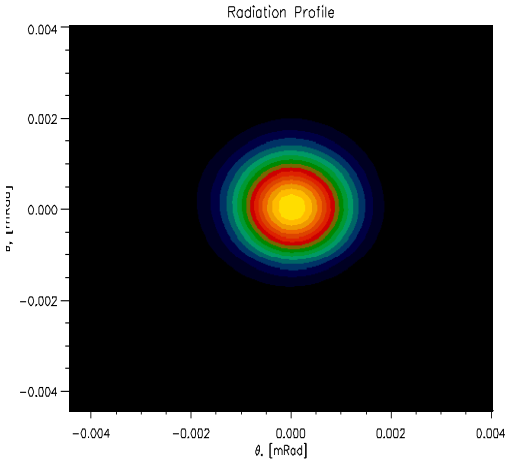
Z=62.5 m



Z=75 m



Z=87.5



Single mode dominates ➔ close to 100% transverse coherence

Peak Brightness Enhancement From Storage Ring Light Sources To SASE

$$B = \frac{\#\text{of photons}}{\Omega_x\,\Omega_y\,\Omega_z} \qquad (\Omega_i\text{- phase space area})$$

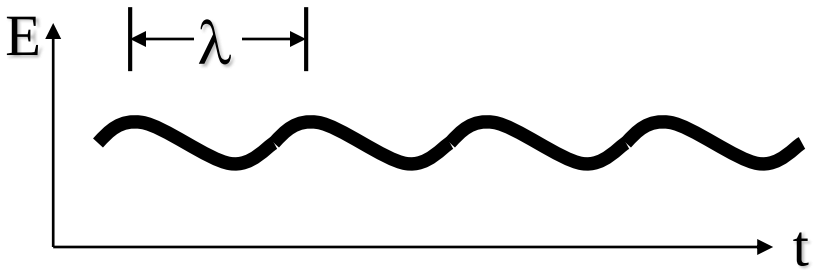
	Undulator in SR	SASE	Enhancement Factor
# of photons	αN_e	$\alpha N_e N_{l_c}$	$N_{l_c} \sim 10^6$ to 10^7
$\Omega_x \Omega_y$	$(2\pi\varepsilon_x)\,(2\pi\varepsilon_y)$	$(\lambda/2)^2$	10^2
Ω_Z	$\frac{\Delta\omega}{\omega}\cdot\left(\frac{\sigma_Z}{c}\right) = 10^{-3}\times 10\,ps$	$\frac{\Delta\omega}{\omega}\cdot\left(\frac{\sigma_Z}{c}\right)_{\text{compressed}} = 10^{-3}\times 100\,fs$	10^2
B	10^{23}	10^{33}	**10^{10} to 10^{11}**

N_{lc}: number of electrons within a coherence length l_c

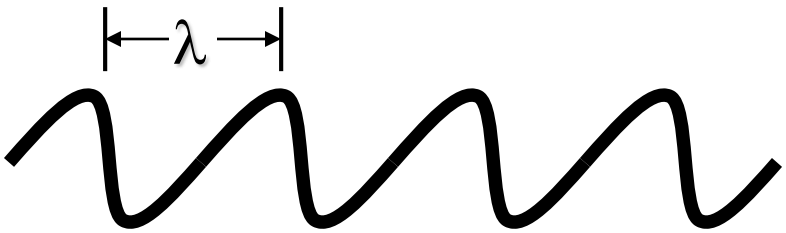
Nonlinear Harmonic Generation

- FEL instability creates energy and denstiy modulation at λ,
- Near saturation, strong bunching at fundamental produces rich harmonic components

small signal, linear regime



near saturation, nonlinear regime



- In a planar undulator, emissions of harmonics $\lambda_h=\lambda/h$ (h=1,2,3,…)

beam frame 8 ⟶ odd harmonics are usually favored

viewed off-axis ➔ broke the half-period symmetry ➔ even ones

Harmonic Interactions

- Harmonics are not independent: Maxwell-Vlasov ➔

$$\left(\frac{\partial}{\partial \bar{z}} + \frac{\bar{\nabla}^2}{2ih}\right) a_h = \underbrace{a_h \text{ term}}_{\text{linear bunching}} + \underbrace{\sum_{h_1+\ldots+h_m=h} a_{h_1} \times \ldots \times a_{h_m} \text{ term}}_{\text{nonliner harmonic interaction}}$$

- In general

$$\boxed{a_h = \underbrace{a_h^L}_{\text{Linear}} + \underbrace{a_h^{NL}}_{\text{Nonlinear Harmonic Generation}}}$$

- For fundamental, nonlinear term is weak until saturation
  For harmonics, nonlinear term dominates before saturation

- In general, $a_h^{NL} \sim a_1^h >> a_h^L$, with a nonlinear gain length L_g/h

Bonifacio, Salvo, Pierini, NIMA (1990)
Freund, Biedron, Milton, IEEE (2000)
Huang, Kim, PRE (2000)

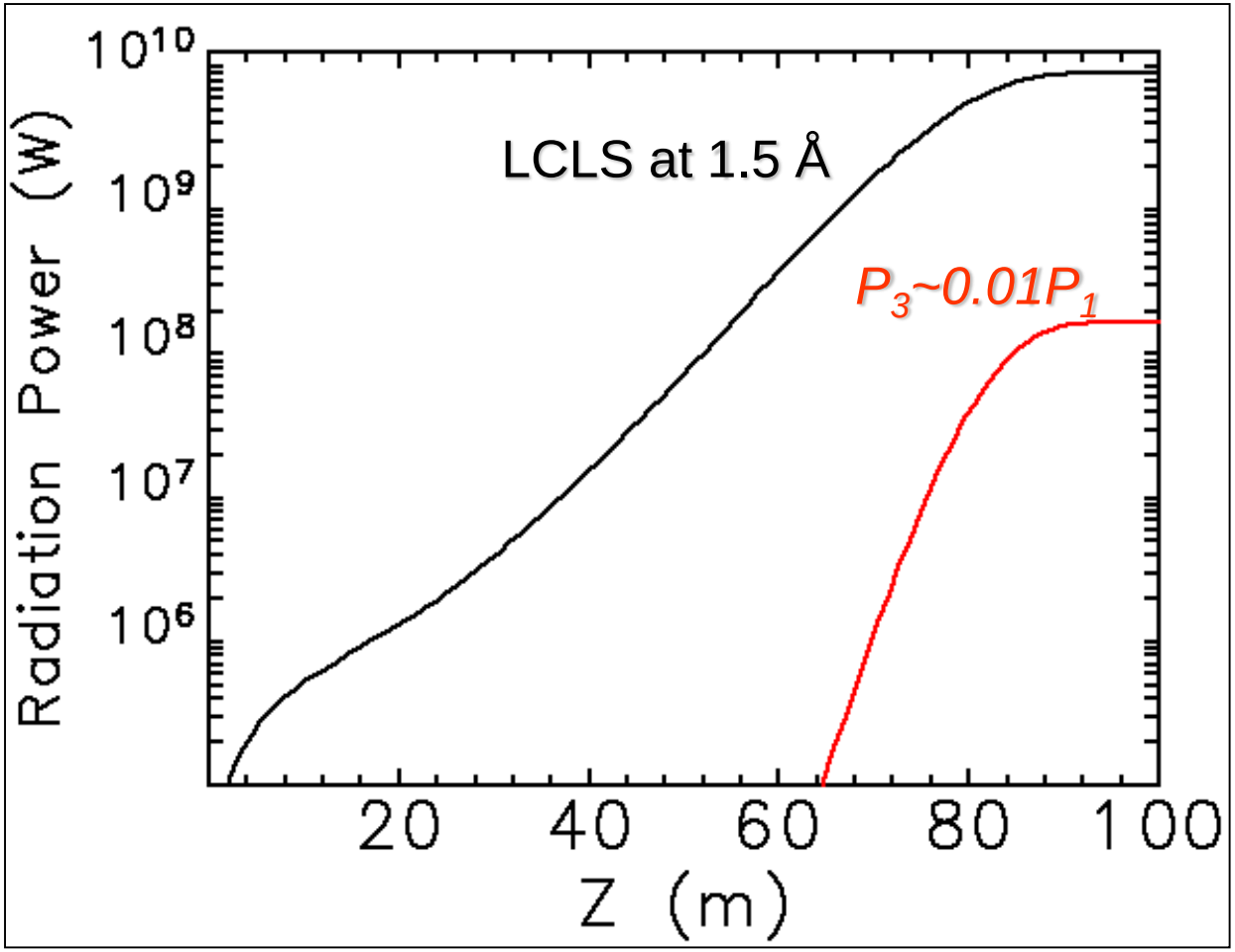
Nonlinear harmonic generation

- Coherent harmonics λ_r/h driven by fundamental λ_r

➔ gain length = L_G/h

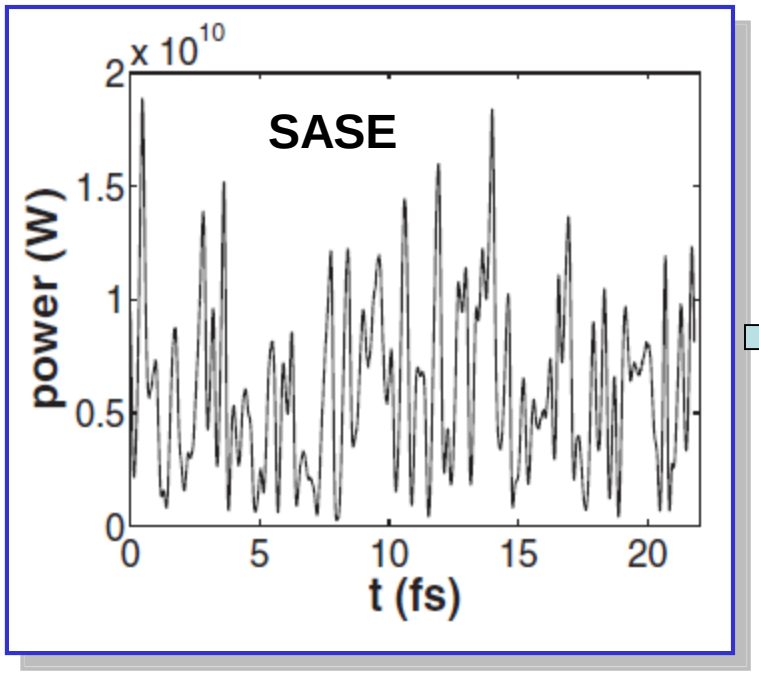
➔ Similar transverse coherence

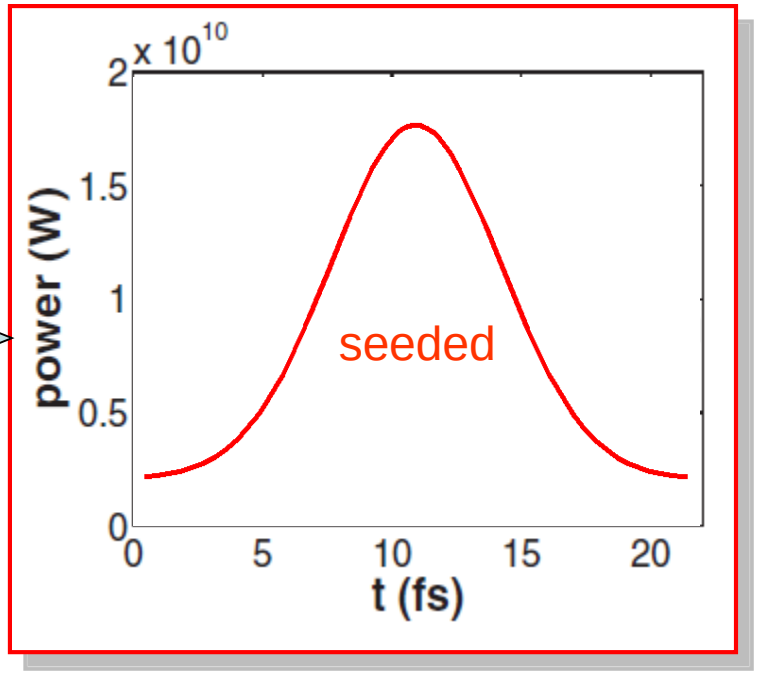
➔ Significant 3rd harmonic power



Seeding to improve temporal coherence

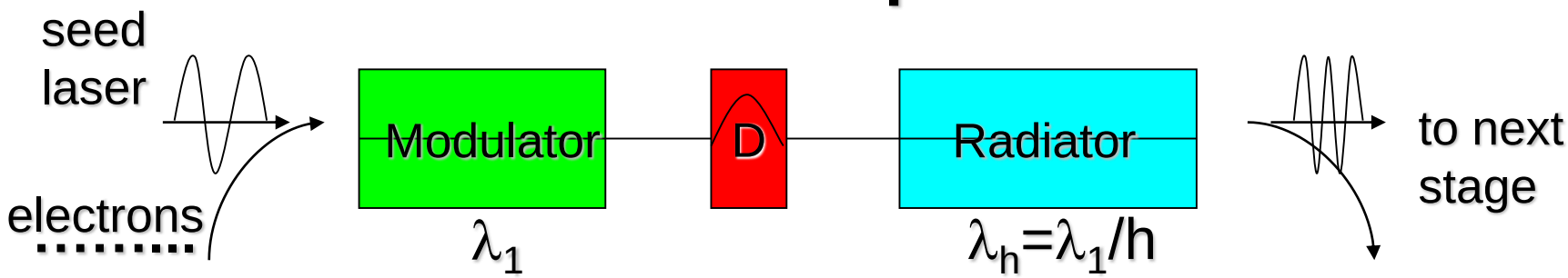
- High-gain harmonic generation (HGHG)
- Echo-enabled harmonic generation
- Self-seeding

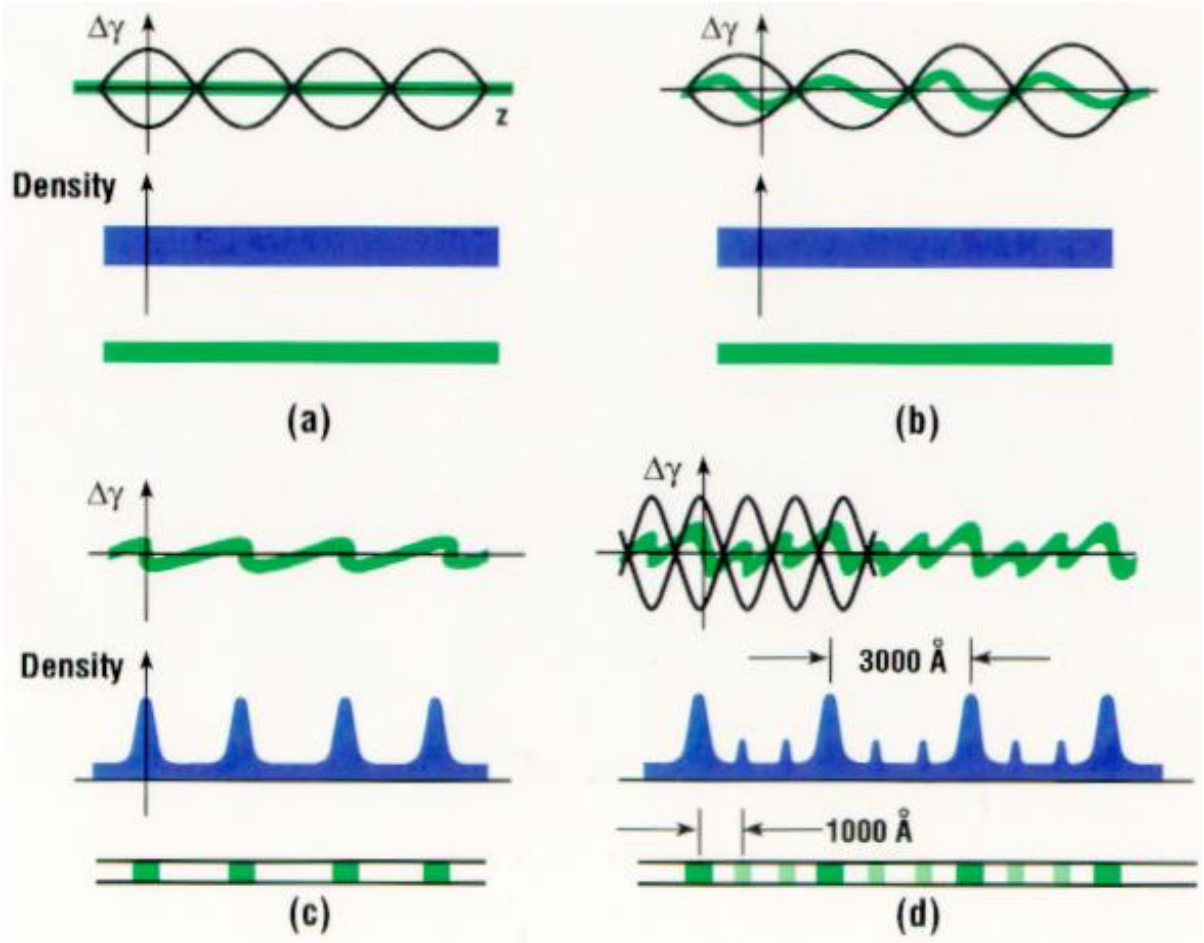




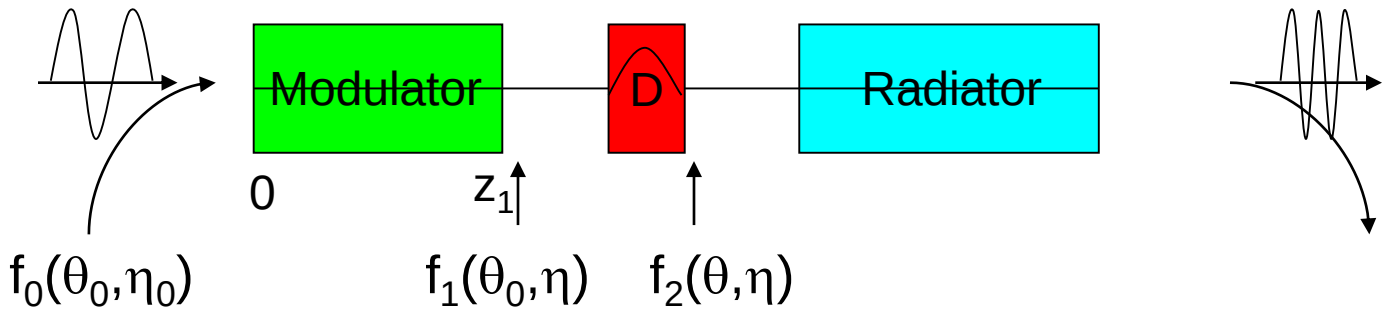
L.-H. Yu, PRA44, 5178 (1991)

HGHG Principle





High-gain harmonic generation



$\theta = ck_1 \Delta t$ is the long. position (phase) w.r.t. to the wave

$\eta = \Delta\gamma/\gamma$ is the relative energy variable

Energy modulation $\eta = \eta_0 + \Delta\eta \sin\theta_0$

Density modulation $\theta = \theta_0 + k_1 R_{56} \eta$

$$f_2(\theta,\eta) = f_1(\theta_0,\eta) = f_0(\theta_0,\eta_0)$$

Initial unmodulated distribution

$$
\begin{aligned}
b_h =& \langle e^{-ih\theta} \rangle = \int d\eta d\theta e^{-ih\theta} f(\eta, \theta) \\
=& \int d\eta d\theta_0 \exp\left[-ih(\theta_0 + k_1 R_{56}\eta)\right] f(\eta, \theta_0) \\
=& \int d\eta_0 d\theta_0 \exp\left[-ih(\theta_0 + k_1 R_{56}\eta_0 + k_1 R_{56}\Delta\eta \sin(\theta_0))\right] f_0(\eta_0) \\
=& \int d\eta_0 f_0(\eta_0) e^{-ihk_1 R_{56}\eta_0} \int d\theta_0 \exp\left[-ih(\theta_0 + k_1 R_{56}\Delta\eta \sin(\theta_0))\right] \\
=& J_h(hk_1 R_{56}\Delta\eta) \int d\eta_0 f_0(\eta_0) e^{-ihk_1 R_{56}\eta_0} .
\end{aligned}
$$

If we further assume that $f_0(\eta_0) \propto \exp[-\eta^2/(2\sigma_\eta^2)]$, then we obtain

$$
\boxed{b_h = J_h(hk_1 R_{56}\Delta\eta) \exp\left(-\frac{h^2 k_1^2 R_{56}^2 \sigma_\eta^2}{2}\right) .}
$$

and the current modulation amplitude is just $2|b_h|$. The Bessel function $J_h(x)$ maximizes when $x \approx 1.2h$, this yields the optimal chicane strength

$$
k_1 R_{56} \approx \frac{1.2}{\Delta\eta} .
$$

Then the exponential smearing factor becomes

$$
\boxed{\exp\left(-0.72 \frac{h^2 \sigma_\eta^2}{\Delta\eta^2}\right)}
$$

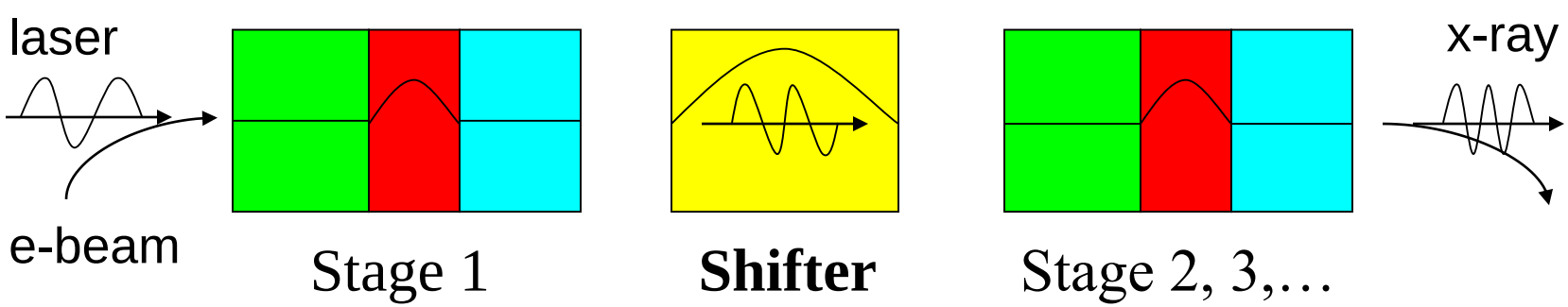
- In order to reach h^{th} harmonic

$$\Delta\eta \sim h\sigma_\eta \,.$$

- Energy modulation becomes effective energy spread in the radiator

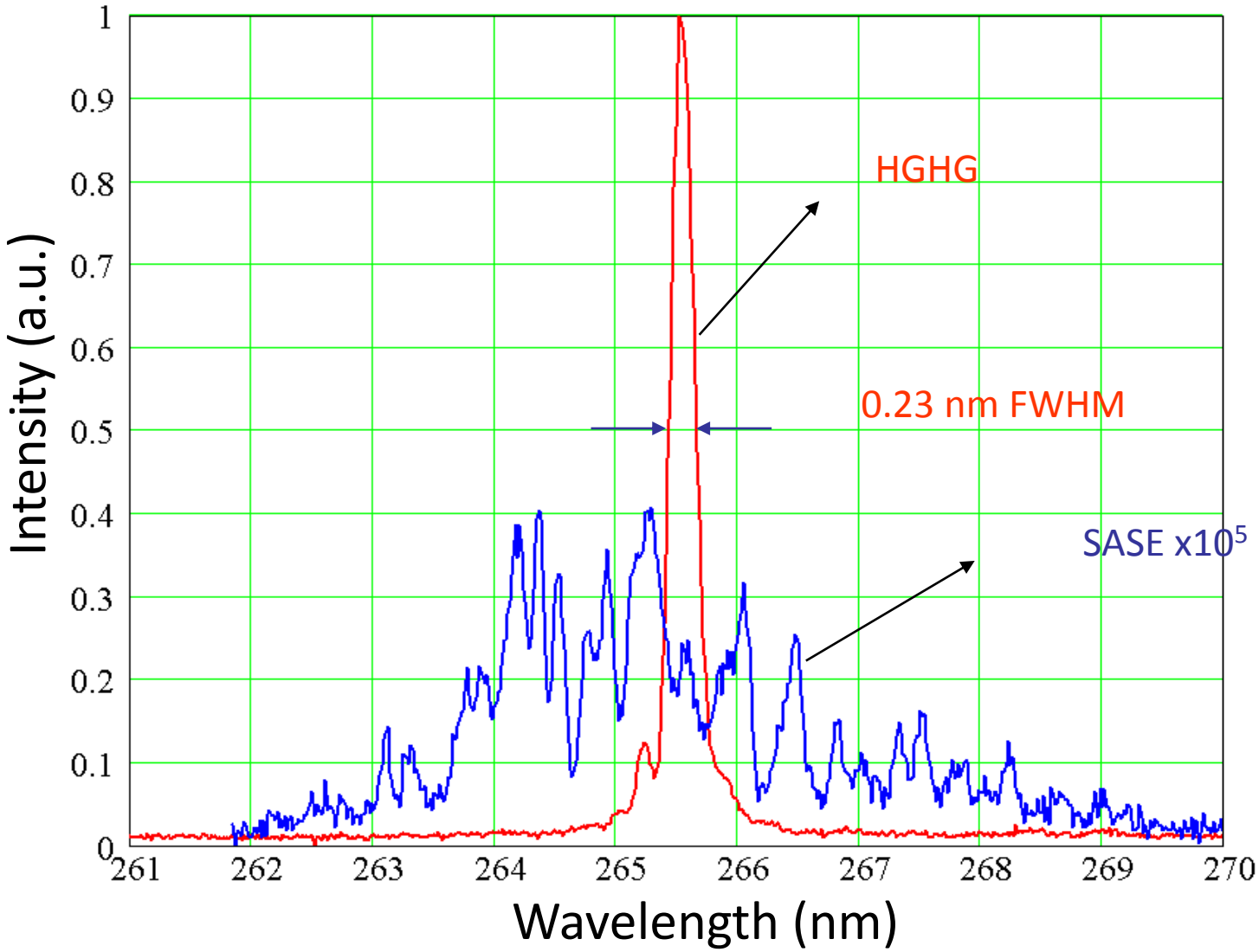
$$\sigma_{\eta 2} \approx \sqrt{\sigma_\eta^2 + \Delta\eta^2/2}\,,$$

➢ Need several stages to reach very short wavelengths
Each stage uses a fresh part of the beam



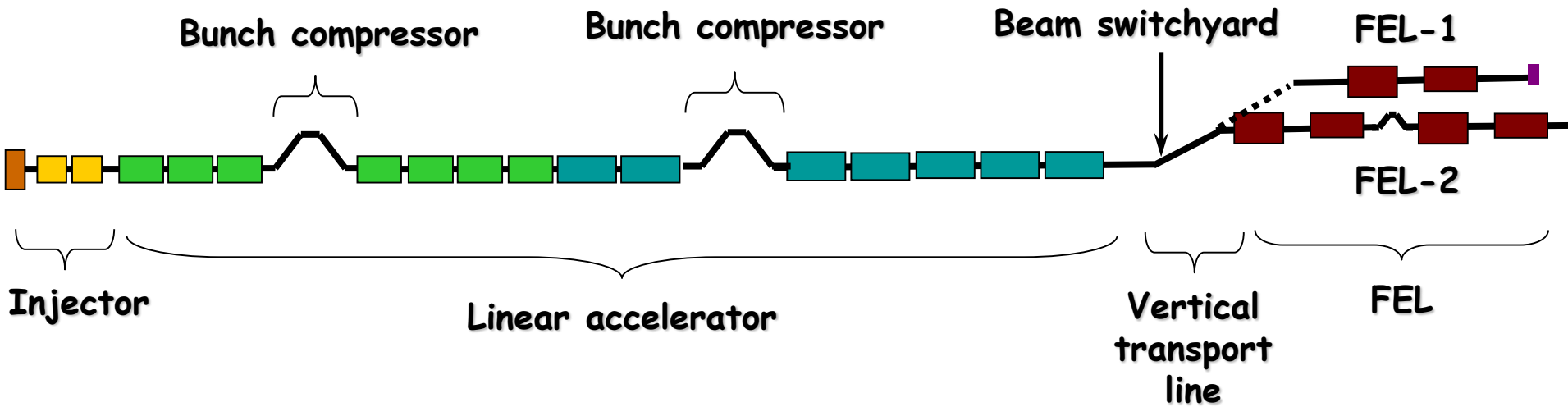
BNL SDL results

Spectrum of HGHG and SASE at 266 nm under the same electron beam condition



L.H. Yu, et al., *Phys. Rev. Lett*. **91**, 074801 (2003).

Fermi FEL at Sincrotrone Trieste (Italy)



- **Spectral range covered by two undulator lines**

FEL 1: 100 – 20 nm (12–60eV) single stage

FEL 2: 20 – 4 nm (60–300eV) two stages

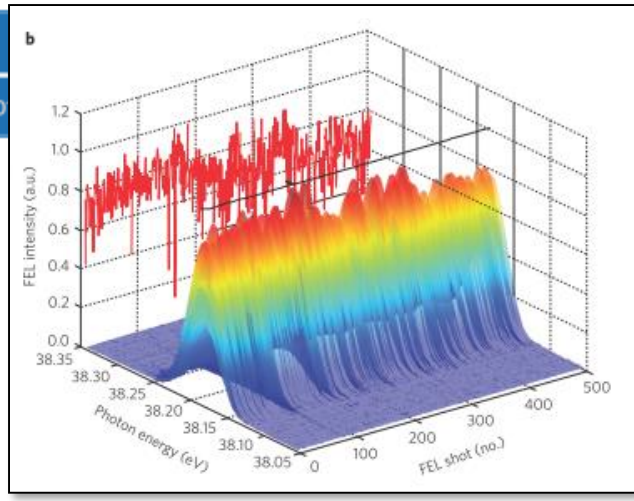
nature photonics

ARTICL

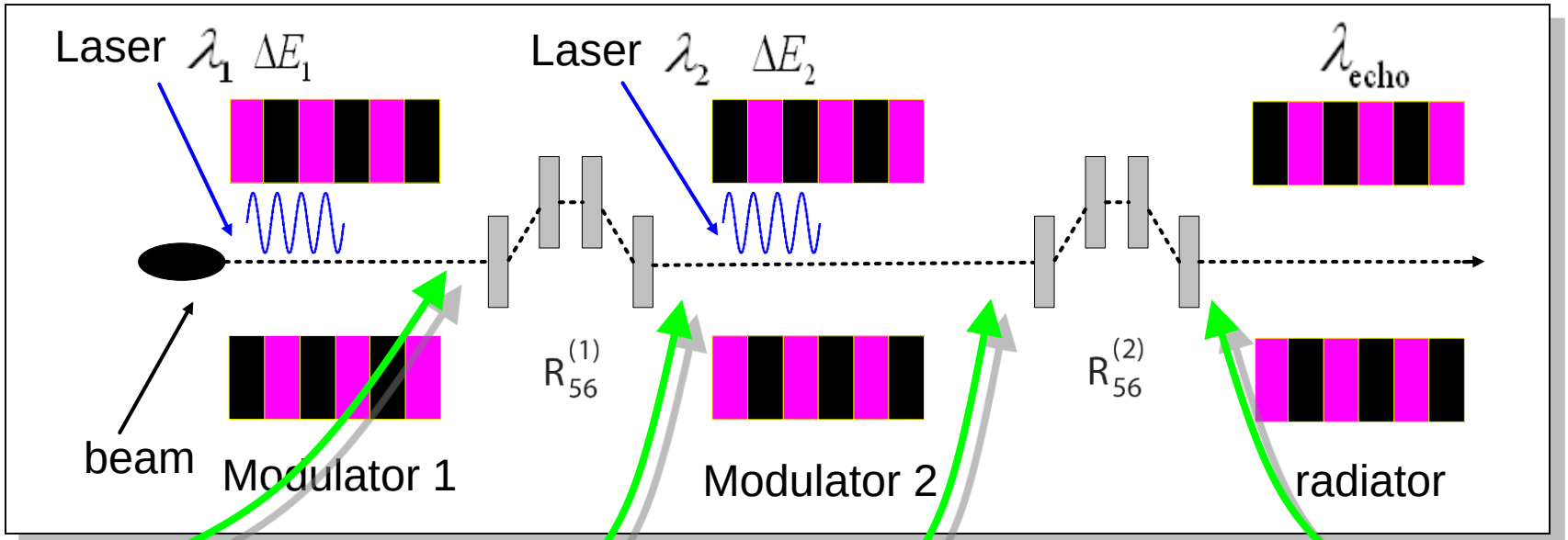
PUBLISHED ONLINE: 23 SEPTEMBER 2012 | DOI: 10.1038/NPHOTON.20

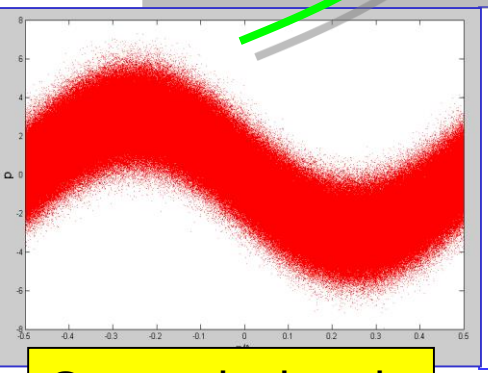
Highly coherent and stable pulses from the FERMI seeded free-electron laser in the extreme ultraviolet

E. Allaria *et al.**

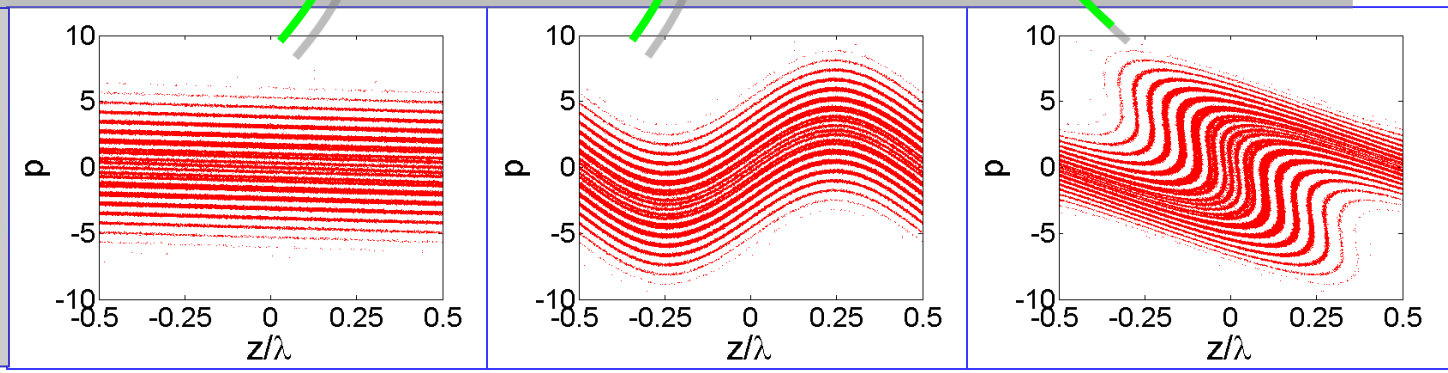


Echo-Enabled Harmonic Generation





One optical cycle



Separated energy bands → Separated current spikes

- Very high harmonic (~100) microbunching may be produced

G. Stupakov, PRL, 102, 074801 (2009)
D. Xiang, G. Stupakov, PRST-AB, 12, 030702 (2009)

Echo bunching theory

A general expression for the bunching factor b_k can be derived for arbitrary ω_1 and ω_2. Practically, the case $\omega_1 = \omega_2 = \omega$ can be realized with a single laser beam.

$$N(z)/N_0 = 1 + \sum_{k=1}^{\infty} 2b_k \cos(k\kappa_L z + \psi_k)\,.$$

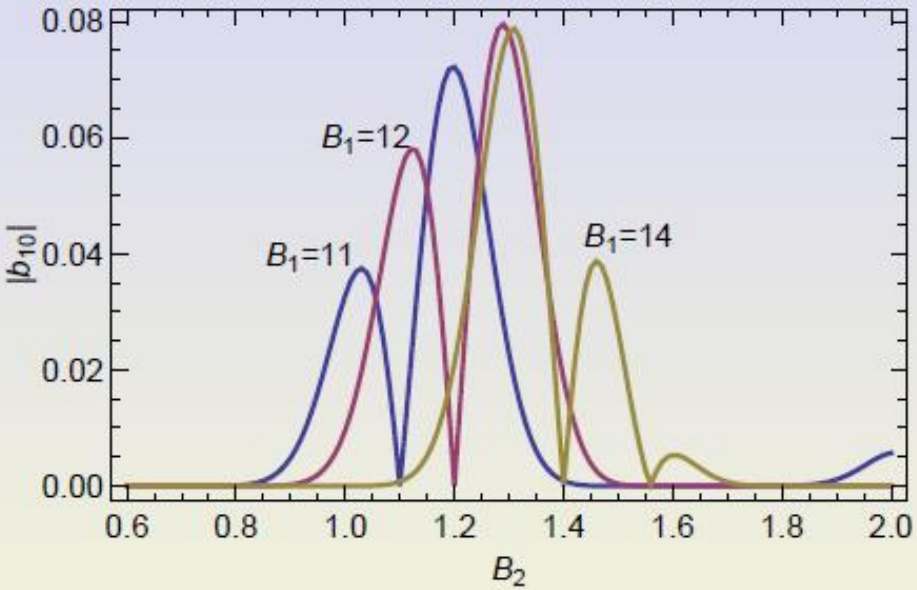
$$A = \Delta E_{mod}/\sigma_E \qquad B = R_{56}\kappa_L\sigma_E/E_0$$

$$b_k = \left| \sum_{m=-\infty}^{\infty} e^{im\phi} J_{-m-k}\left(A_1((m+k)B_1 + kB_2)\right) \times J_m(kA_2B_2)\, e^{-\frac{1}{2}((m+k)B_1+kB_2)^2} \right|$$

where ϕ is the phase between the laser beams 1 and 2. Remarkably, the maximized value of $|b_k|$ does not depend on ϕ!

An example, 10th harmonic

Assume $A_1, A_2 = 1$ and optimize the amplitude of the 10th harmonic.

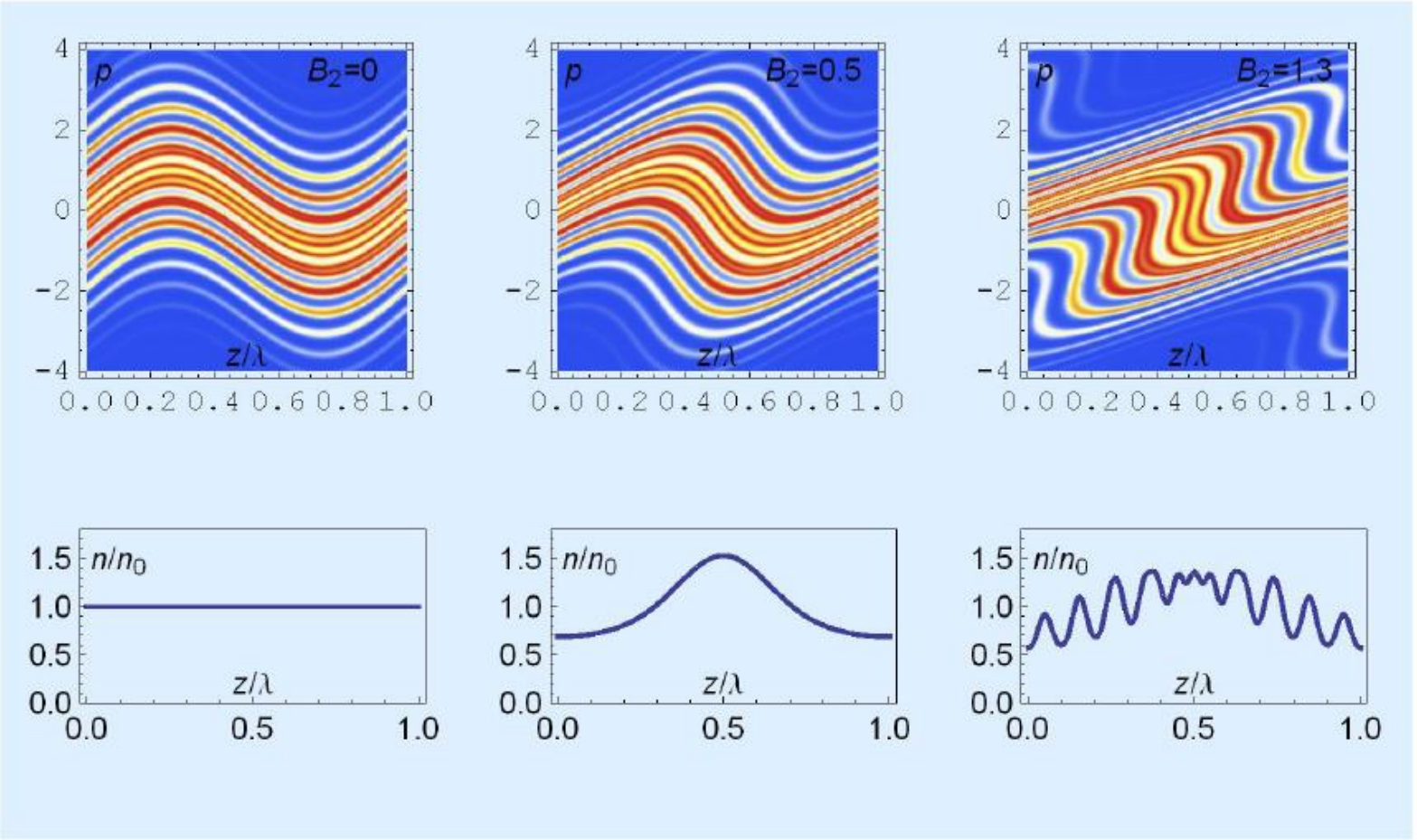


The bunching factor for $k = 10$ as a function of parameter B_2 for different values of B_1.

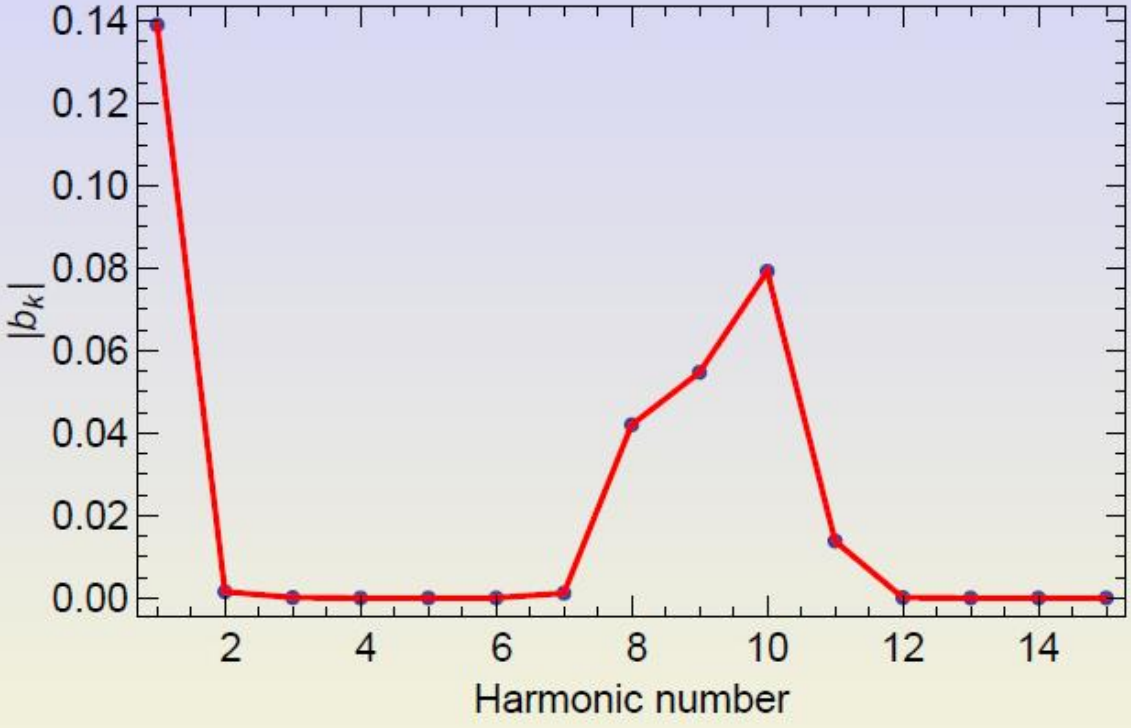
The maximal value of b_{10} is attained at $B_1 = 12.1$, $B_2 = 1.3$, and is equal to 0.08. RMS energy spread of the beam is increased by 40%.

Phase plots of echo induced modulation

Phase space after the second modulator for various dispersion strengths B_2 and $A_2 = 1$, $\omega_1 = \omega_2$.



Modulation amplitude versus harmonic number

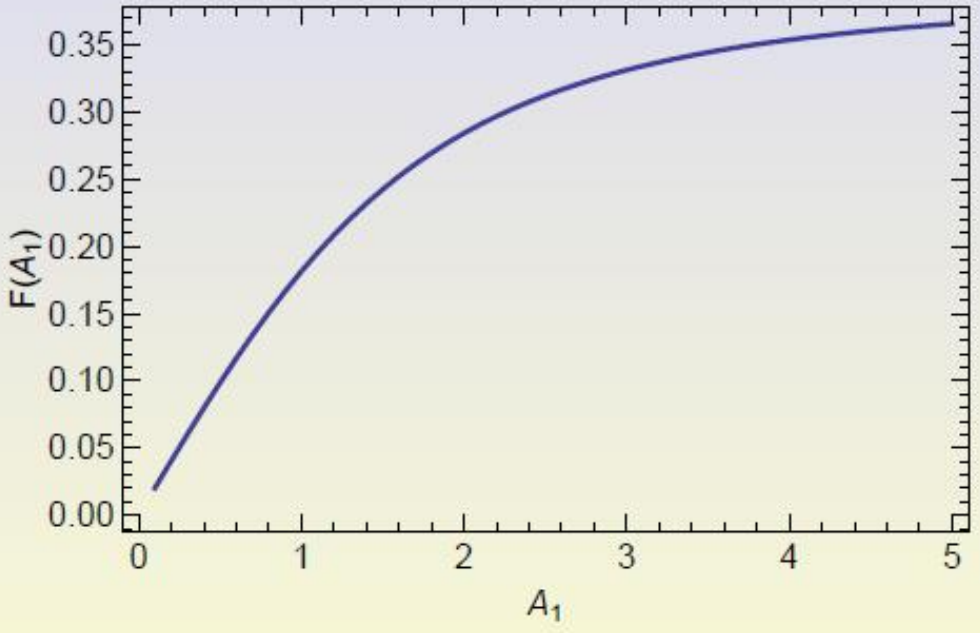


Echo excites a relatively narrow spectrum around the optimized harmonic.

Maximal echo modulation

What is the maximal echo modulation one can get with optimized amplitudes A_1, A_2 and dispersions?

$$|b_k| = \frac{F(A_1)}{k^{1/3}}$$



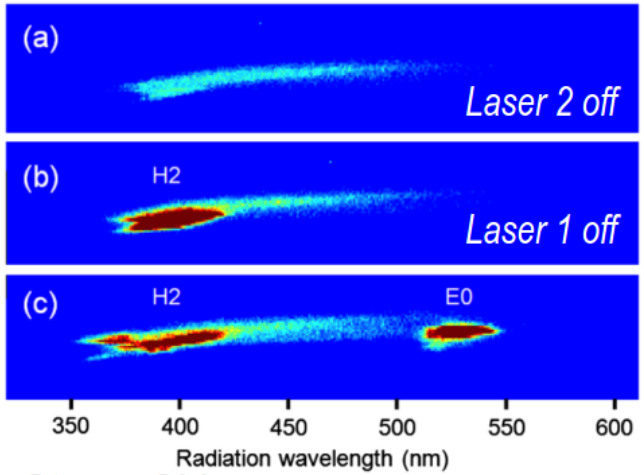
Remarkably, $|b_k|$ is a very slow function of k!

EEHG demonstration experiments

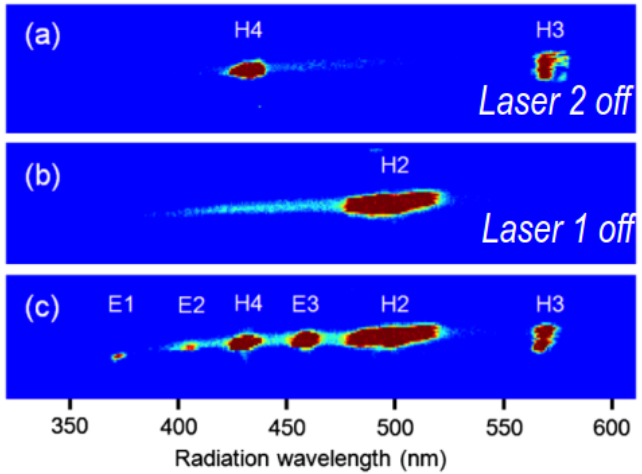
[D. Xiang et al PRL 105 (2010) 114801]

λ_1= 759 nm, λ_2 = 1590 nm

Little Chirp



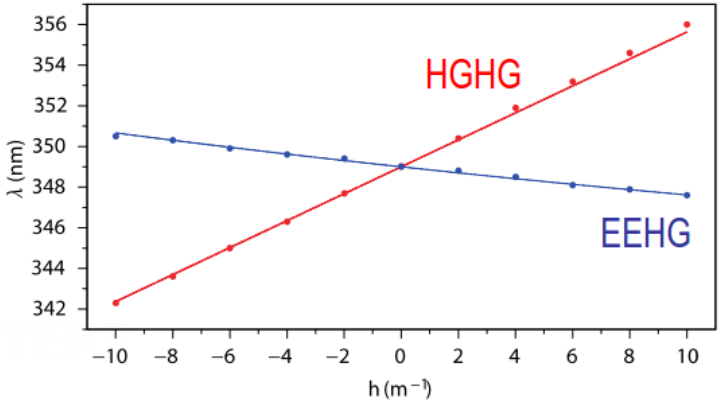
Strong Chirp

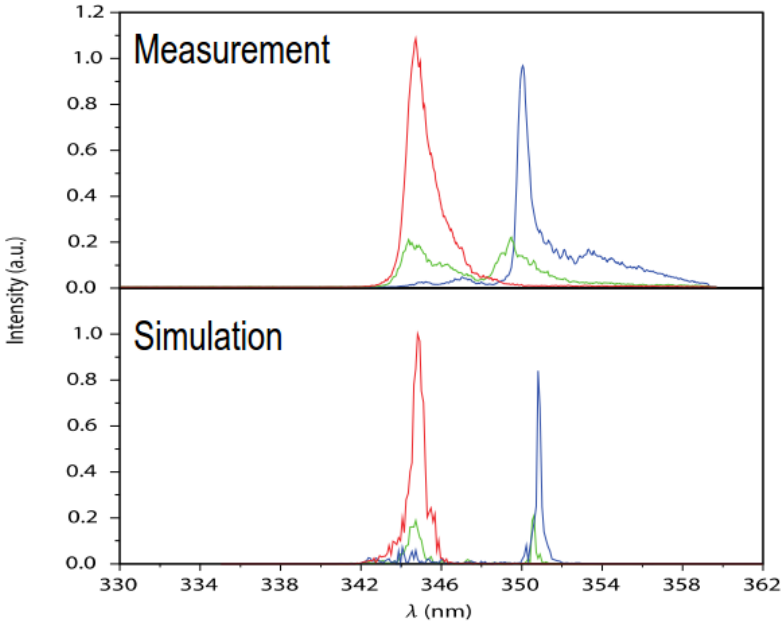


[Z.T. Zhao et al Nature Photonics 6 (2012) 360]

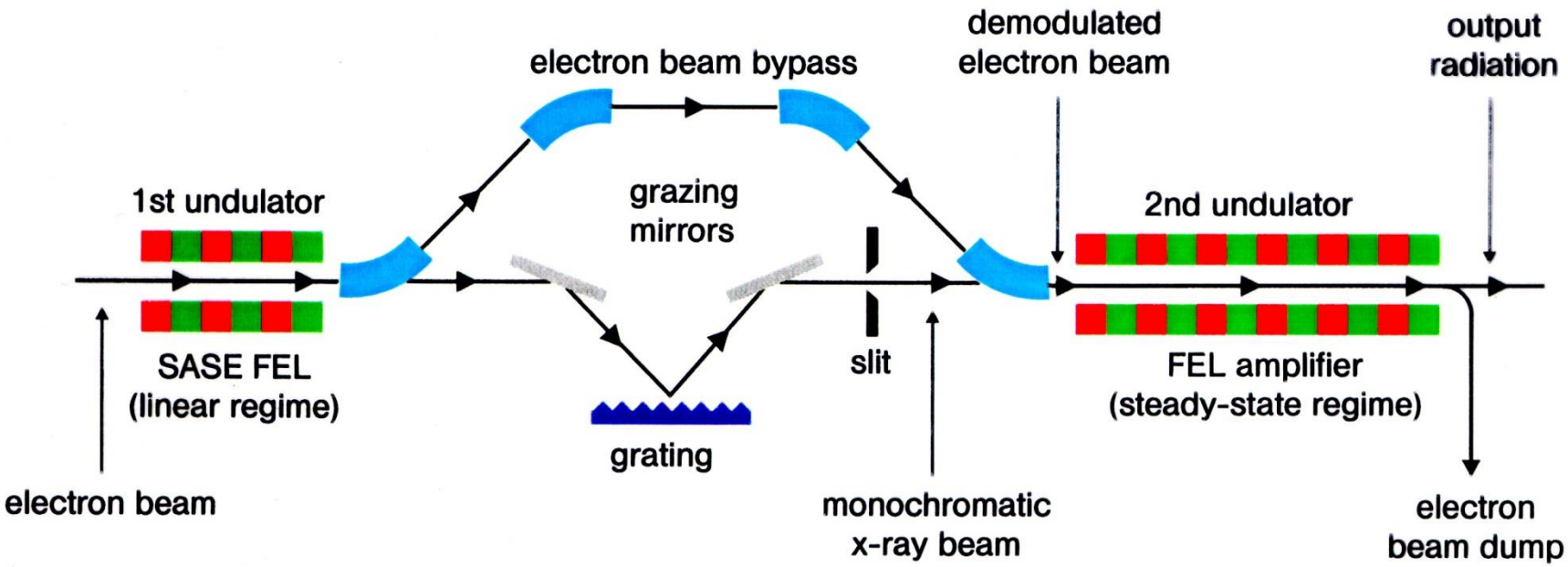
λ_1= 1047 nm, λ_2 = 1047 nm

Wavelength vs chirp





Two-stage self-seeding option*



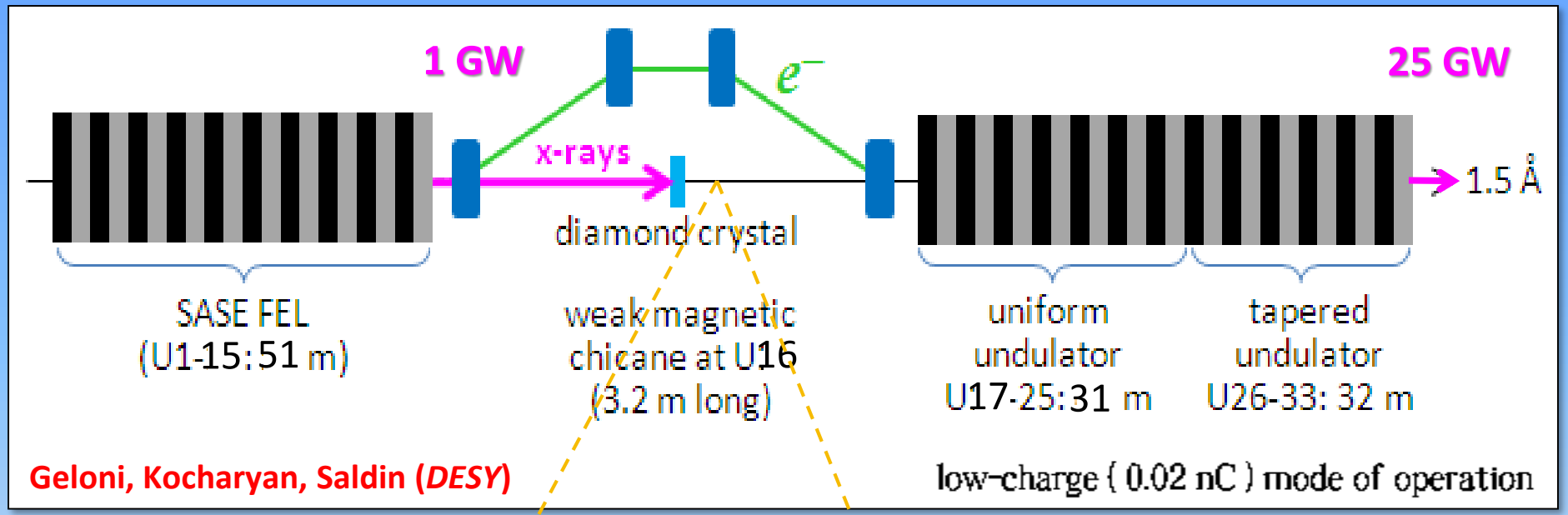
Schematic view of the seeding option for FLASH

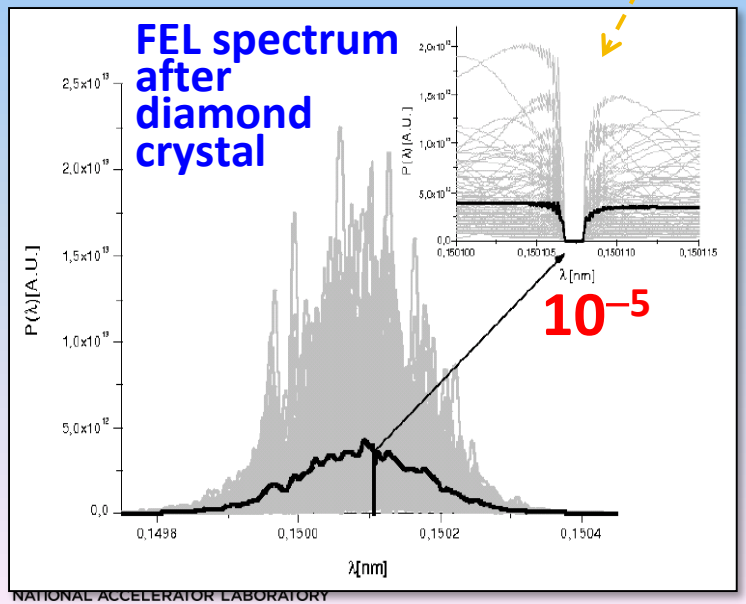
Basic requirements:

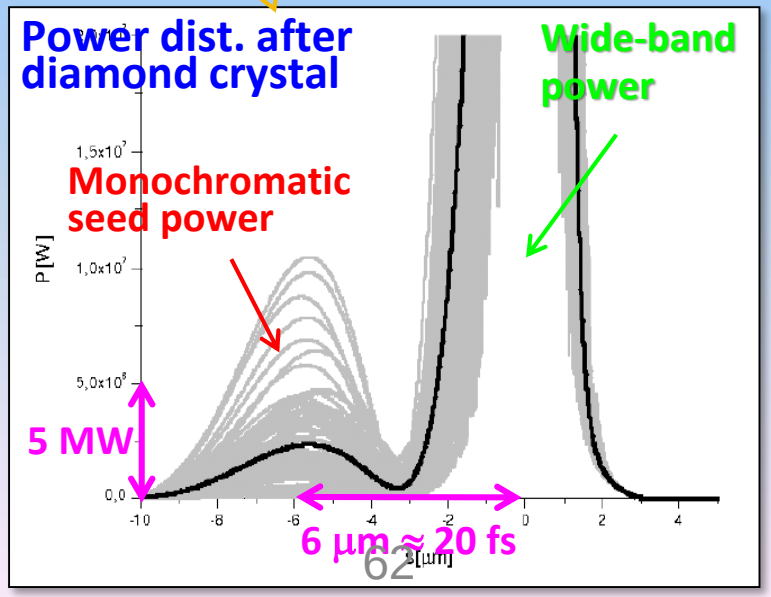
1) The 1st section operates in linear high-gain regime, $\langle P_{SASE} \rangle \sim 10$MW
2) The micro bunching is smeared out after the magnetic chicane
3) The monochromator resolution $\Delta\omega/\omega \approx 5 \cdot 10^{-5}$
4) The seeding power $P_{SEED} \sim 10$kW >> shot noise power $P_{SHOT} \sim 10$W
5) The seed pulse is amplified to saturation in the 2nd undulator section

* J. Feldhaus et al. / Optics Communications 140(1997) 341-352

Hard x-ray self-seeding @ *LCLS*

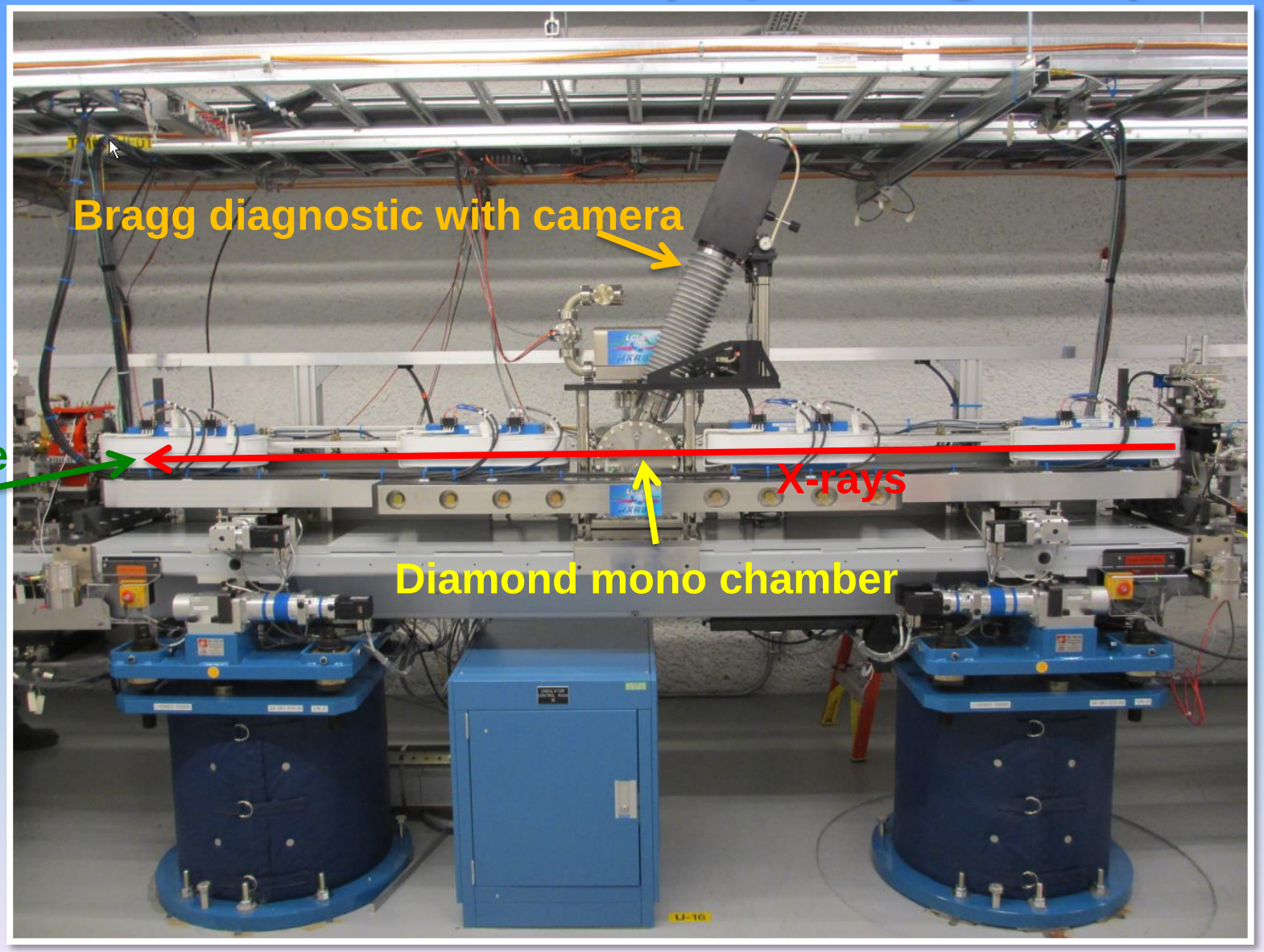






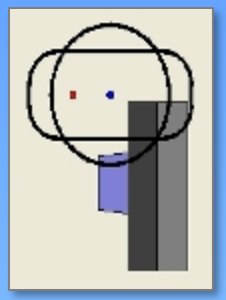
Self-seeding of 1-μm e^- pulse at 1.5 Å yields **10^{-4} BW** with low charge mode

HXRSS at LCLS (replacing U16)



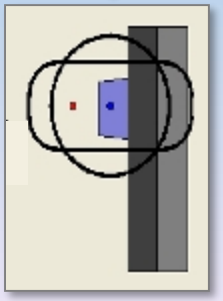
J. Amann et al., Nature Photonics 6, 693 (2012)

8.3 keV

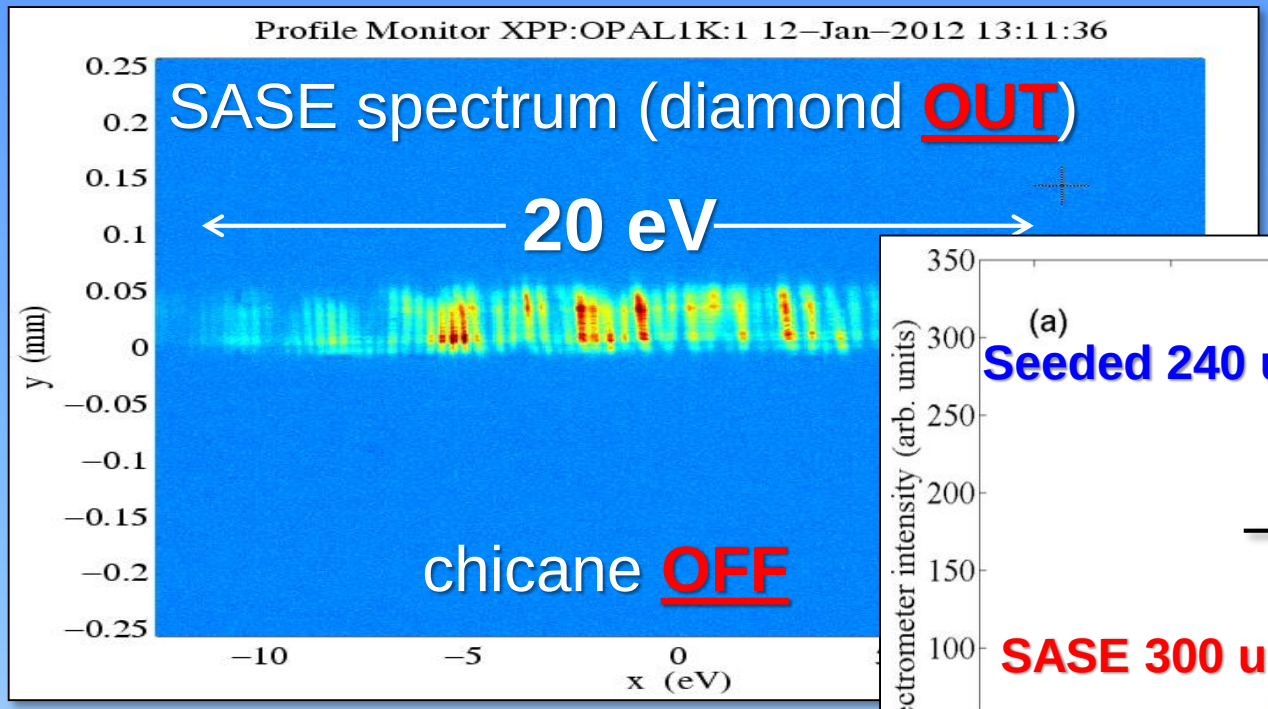


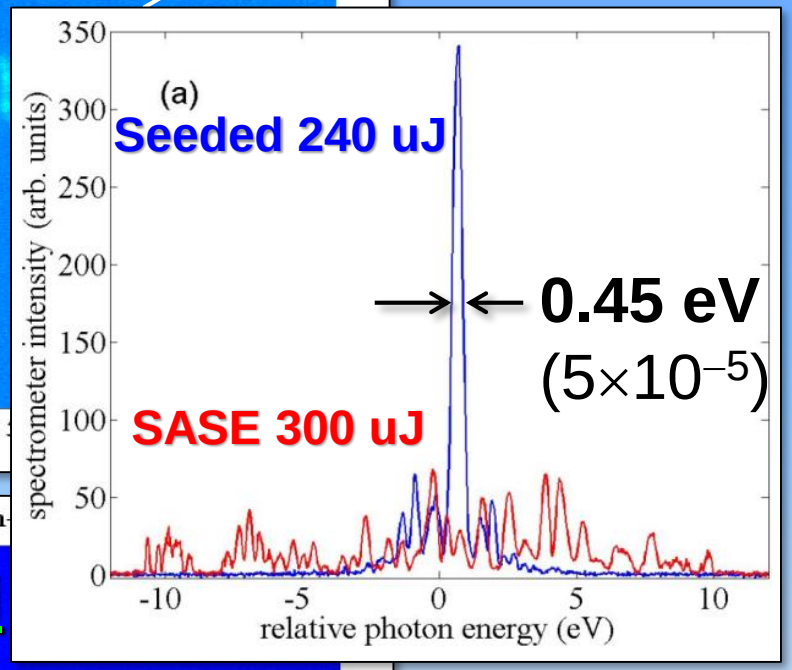
SASE

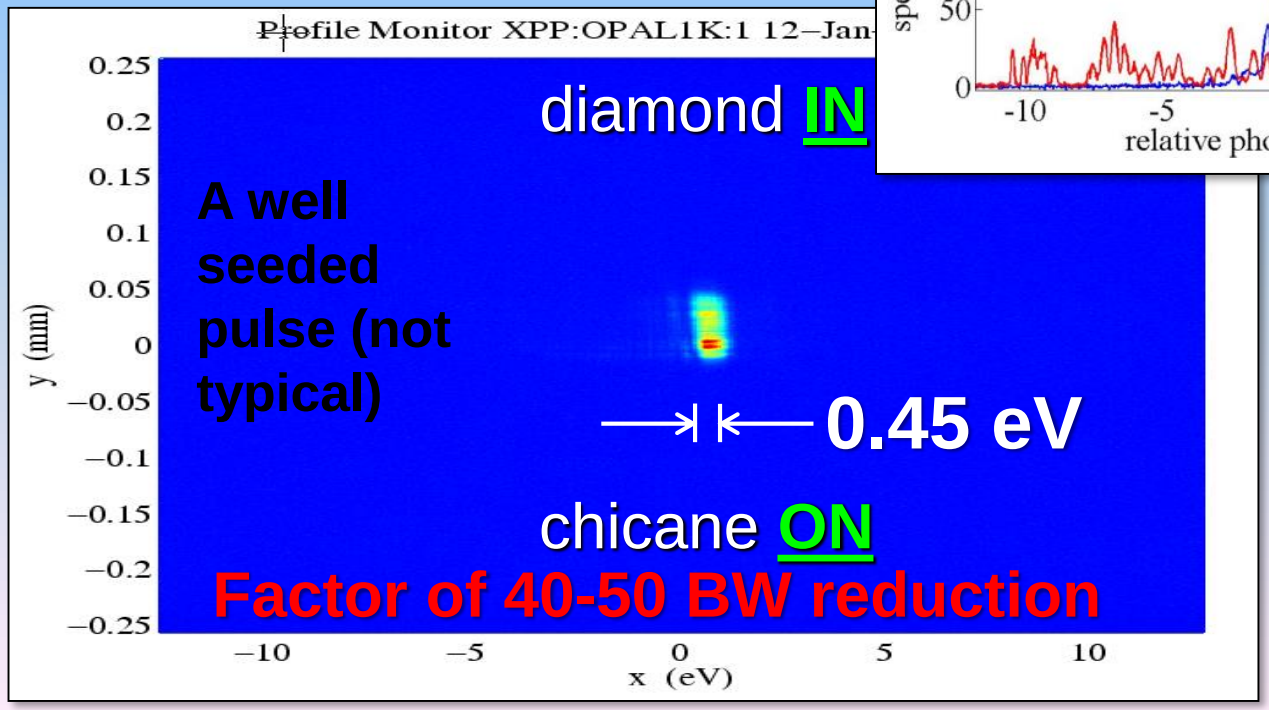
insert diamond & turn on chicane



seeded



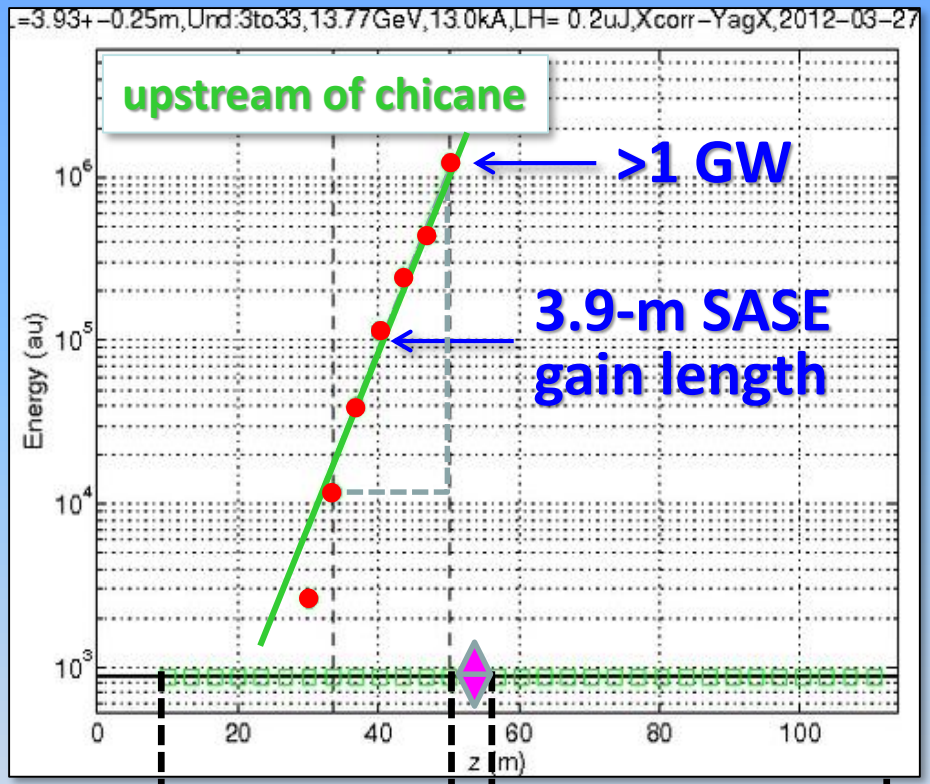




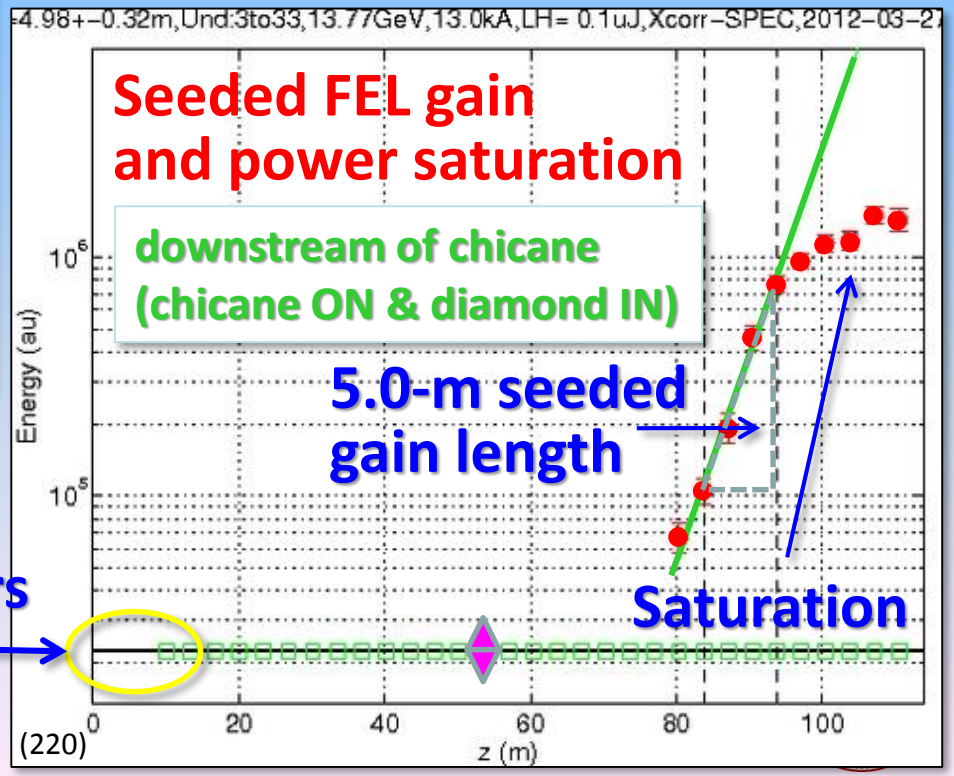
Fourier Transform limit is 5 fs

Gain Curve Shows Seeded Power Saturation

U3-U15 SASE gain length: 3.9 m



U17-U33 Seeded gain length: 5.0 m



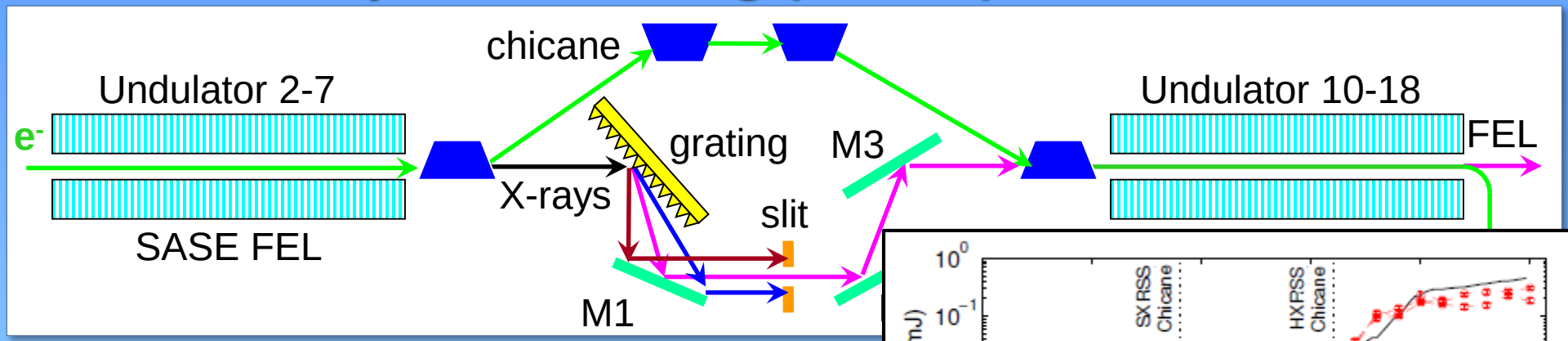
U1,2 out ← U3-U15 SASE → ← U17-U33 Seeded →

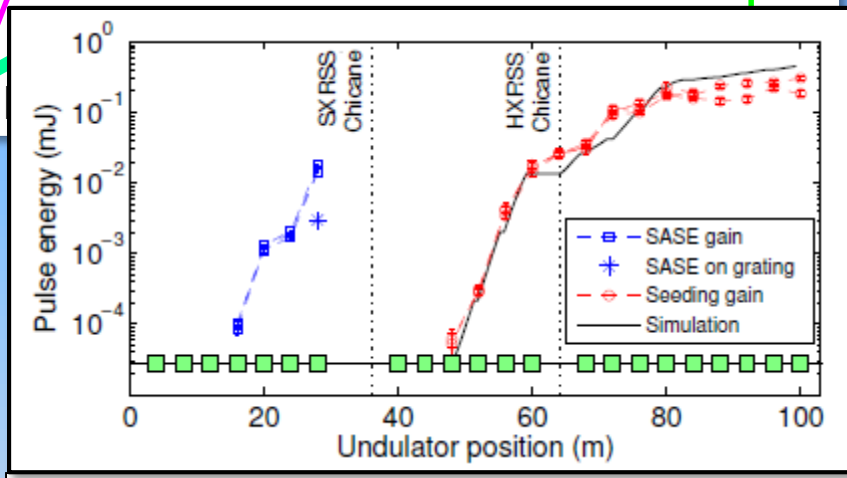
HXRSS mono

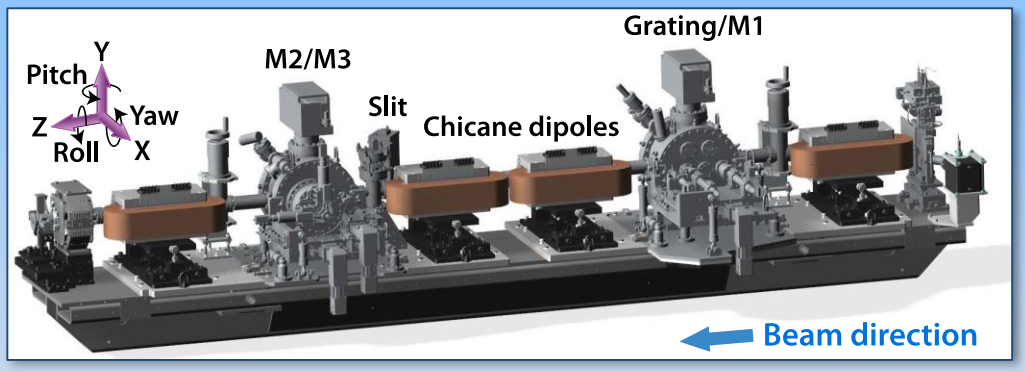
Undulators #1 & #2 extracted

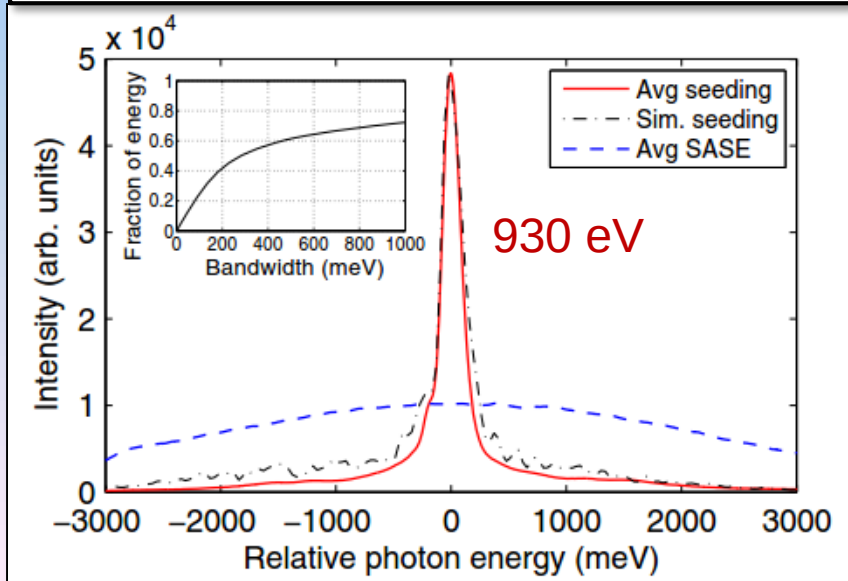
SLAC NATIONAL ACCELERATOR LABORATORY

Soft X-ray Self-Seeding (SXRSS)









- Spectral brightness ~5x higher than SASE

D. Ratner et al., PRL 114, 054801 (2015)

Thank you for your attention!