ECFA ILC Workshop 2005, Vienna, 15 November 2005

Evaluation of electroweak two-loop corrections in the high energy limit

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- evolution equations and explicit loop calculations
- results

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- expansion by regions
- Mellin-Barnes representation

IV Summary

J.H. Kühn, A.A. Penin, *hep-ph/9906545*

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J.H. Kühn, S. Moch, A.A. Penin, V.A. Smirnov, *Nucl. Phys. B 616 (2001) 286*

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I Motivation

Electroweak precision physics

- experimentally measured by now at energy scales up to $\sim M_{W,Z}$
- future generation of accelerators (LHC, ILC) $\rightarrow$ TeV region
- new energy domain $\sqrt{s} \gg M_{W,Z}$ becomes accessible

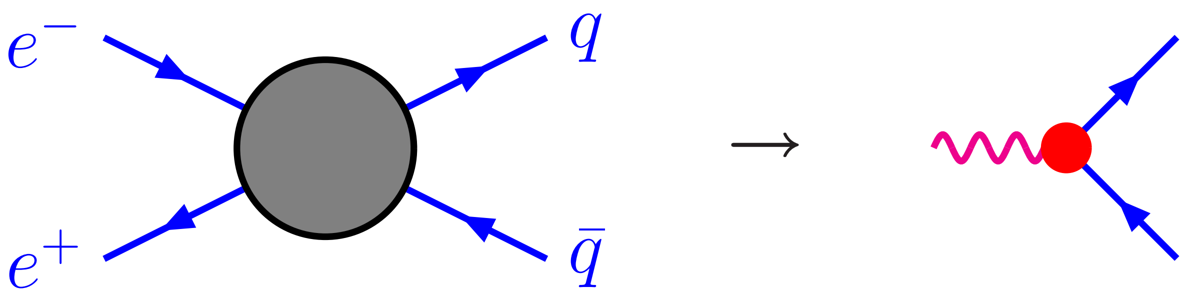
Electroweak radiative corrections at high energies $\sqrt{s} \sim$ TeV $\gg M_{W,Z}$

Fadin et al. '00; Kühn et al. '00, '01, '05; Denner et al. '01, '03, '04, '05; Pozzorini '04; B.J. et al. '03, '04, '05; . . .

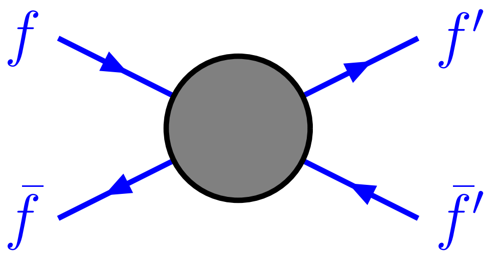
large corrections in *exclusive* cross sections

- electroweak corrections: leading Sudakov logarithms $\alpha^n \ln^j(s/M_{W,Z}^2)$, $j = 2n$, but large coefficients in front of subleading logarithms $(1 \leq j < 2n)$
- 1-loop corrections $\sim 10\%$
- 2-loop corrections $\sim 1\%$, need to be under control for ILC
- individual logarithmic contributions even larger, but strong cancellations

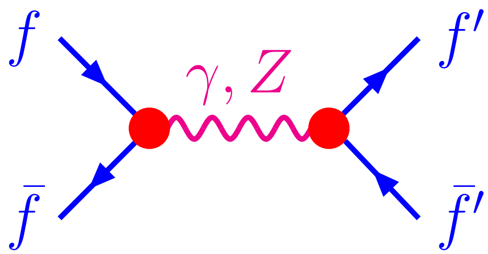
II Four-fermion scattering



Four-fermion scattering: $f\bar{f} \to f'\bar{f}'$, important class of processes

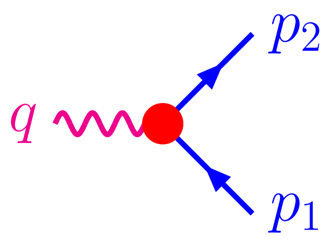


$$A = \frac{ig^2}{s} F^2 \tilde{A}$$



"$\times \tilde{A}$"

Form factor F of vector current:



$$= F \cdot \bar{u}(p_2)\,\gamma^\mu\, u(p_1) + \underbrace{F' \cdot \bar{u}(p_2)\,\sigma^{\mu\nu}\, u(p_1)\, q_\nu}_{\to 0,\ m_f \to 0}$$

High energy behaviour $s \sim |t| \sim |u| \gg M_{W,Z}^2$ references: see Kühn et al. '01

- all *collinear* logarithms of amplitude $A \rightsquigarrow$ form factors F^2
- *reduced amplitude* $\tilde{A} \to$ only *soft* logarithms
- $\tilde{A}$ satisfies an *evolution equation* (known from massless QCD calculations):

$$\frac{\partial \tilde{A}}{\partial \ln s} = \chi\big(\alpha(s)\big)\, \tilde{A}\,, \quad \chi = \text{matrix of soft anomalous dimensions}$$

$\Rightarrow \tilde{A}$ known; needed: form factor F

Simplified models

Standard Model

SU(2)$_\mathrm{L}$ × U(1)$_\mathrm{Y}$

mixing $(\sin\theta_W \neq 0)$

$M_W \neq M_Z$, $M_\gamma = 0$

$\longleftrightarrow$

SU(2)$_M$ × U(1)$_\lambda$ model

no mixing $(\sin\theta_W \approx 0)$

$\hookrightarrow M \equiv M_W \approx M_Z$

mass gap $M \gg \lambda \to 0$

$\longleftrightarrow$

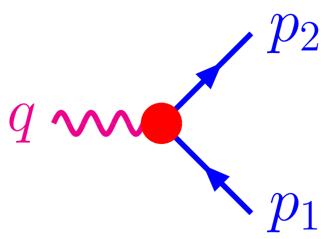
massive SU(2) model

only weak interaction

1 mass $M \neq 0$

High energy behaviour of the form factor

$\hookrightarrow$ Sudakov limit:



$$= F(Q^2) \cdot \bar{u}(p_2)\,\gamma^\mu\,u(p_1)$$

- momentum transfer $-q^2 \equiv Q^2 \gg M^2 \equiv M^2_{W,Z}$
- neglect fermion masses
- *logarithmic approximation*: neglect terms $\propto M^2/Q^2$, keep only logs $\ln^j(Q^2/M^2)$

Evolution equation for the form factor

Sen '81; Collins '89; Korchemsky '89; ...

$$\frac{\partial F(Q^2)}{\partial \ln Q^2} = \left[\int_{M^2}^{Q^2} \frac{dx}{x} \gamma(\alpha(x)) + \zeta(\alpha(Q^2)) + \xi(\alpha(M^2)) \right] F(Q^2)$$

Expand γ, ζ and ξ in the coupling $\alpha(M^2)$ and solve evolution equation
$\hookrightarrow$ obtain correspondence between perturbative results & γ, ζ, ξ

Previously known: massive 1-loop result & massless 2-loop result
$\Rightarrow$ NNLL approximation of massive 2-loop form factor: $\alpha^2 (\ln^4 + \ln^3 + \ln^2)$

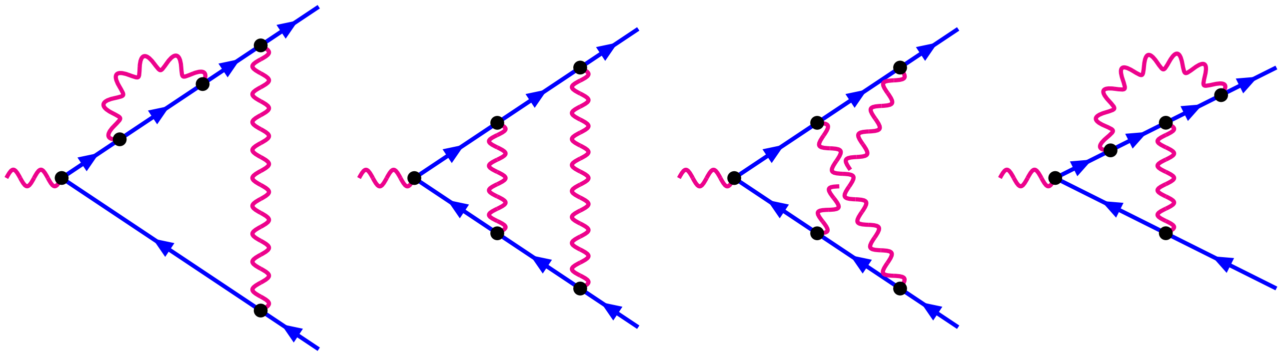
But: growing coefficients with alternating signs $\rightarrow$ cancellations between logs
$\hookrightarrow$ need N^{3}LL approximation including $\alpha^2 \ln^1$

$\Rightarrow$ **Calculation of massive 2-loop diagrams necessary!**

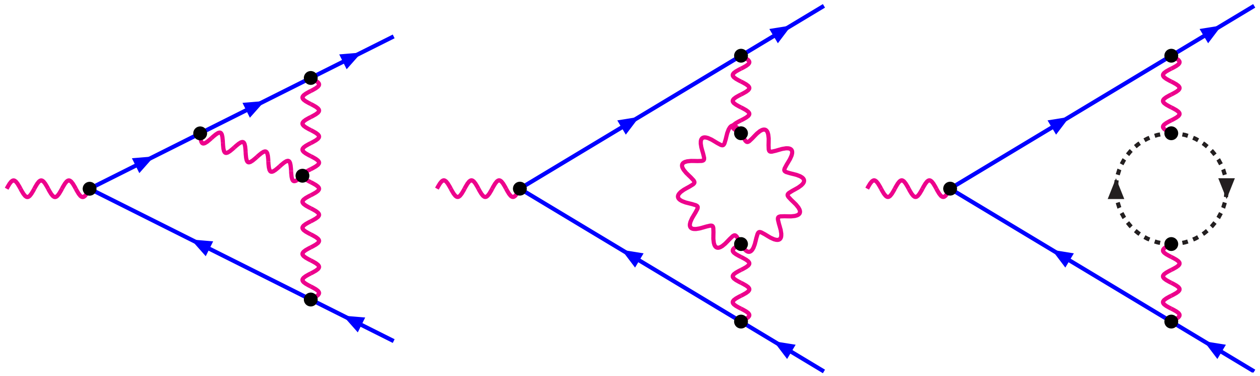
Massive SU(2) form factor in two loops: diagrams

2-loop vertex diagrams (massless fermions, massive bosons, $M_{\text{Higgs}} = M$):

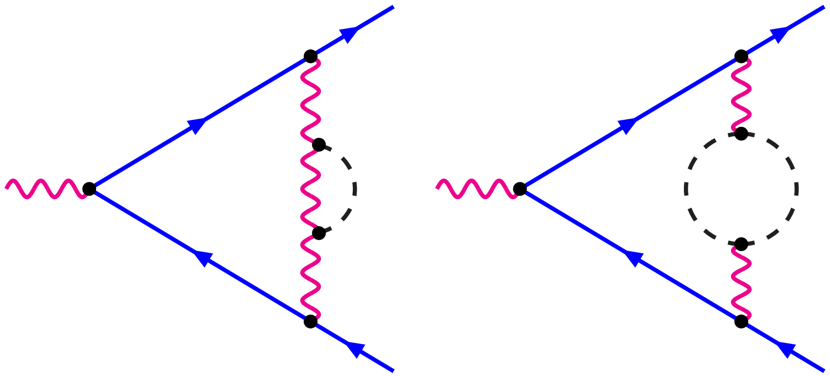
Abelian:



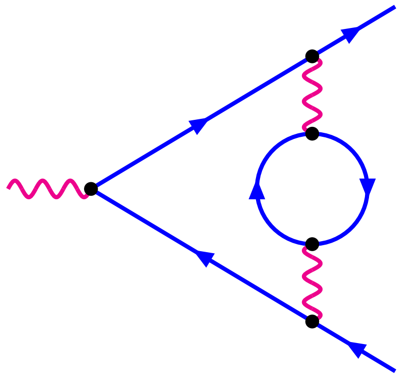
non-Abelian:



Higgs:



fermion:



$+$ 1-loop$\times$1-loop corrections $+$ renormalization

Massive SU(2) form factor in two loops: result

$$\left(\frac{\alpha}{4\pi}\right)^2 \Bigg[\frac{9}{32}\ln^4\left(\frac{Q^2}{M^2}\right) - \frac{19}{48}\ln^3\left(\frac{Q^2}{M^2}\right) - \left(-\frac{7}{8}\pi^2 + \frac{463}{48}\right)\ln^2\left(\frac{Q^2}{M^2}\right)$$

confirmed ✓

$$+ \left(\frac{13}{2}\sqrt{3}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{15}{4}\sqrt{3}\pi - \frac{61}{2}\zeta_3 - \frac{11}{24}\pi^2 + 29\right)\ln\left(\frac{Q^2}{M^2}\right) \Bigg]$$

new!

B.J., Kühn, Moch '03; B.J., Kühn, Penin, Smirnov '04, '05

Electroweak corrections to four-fermion scattering

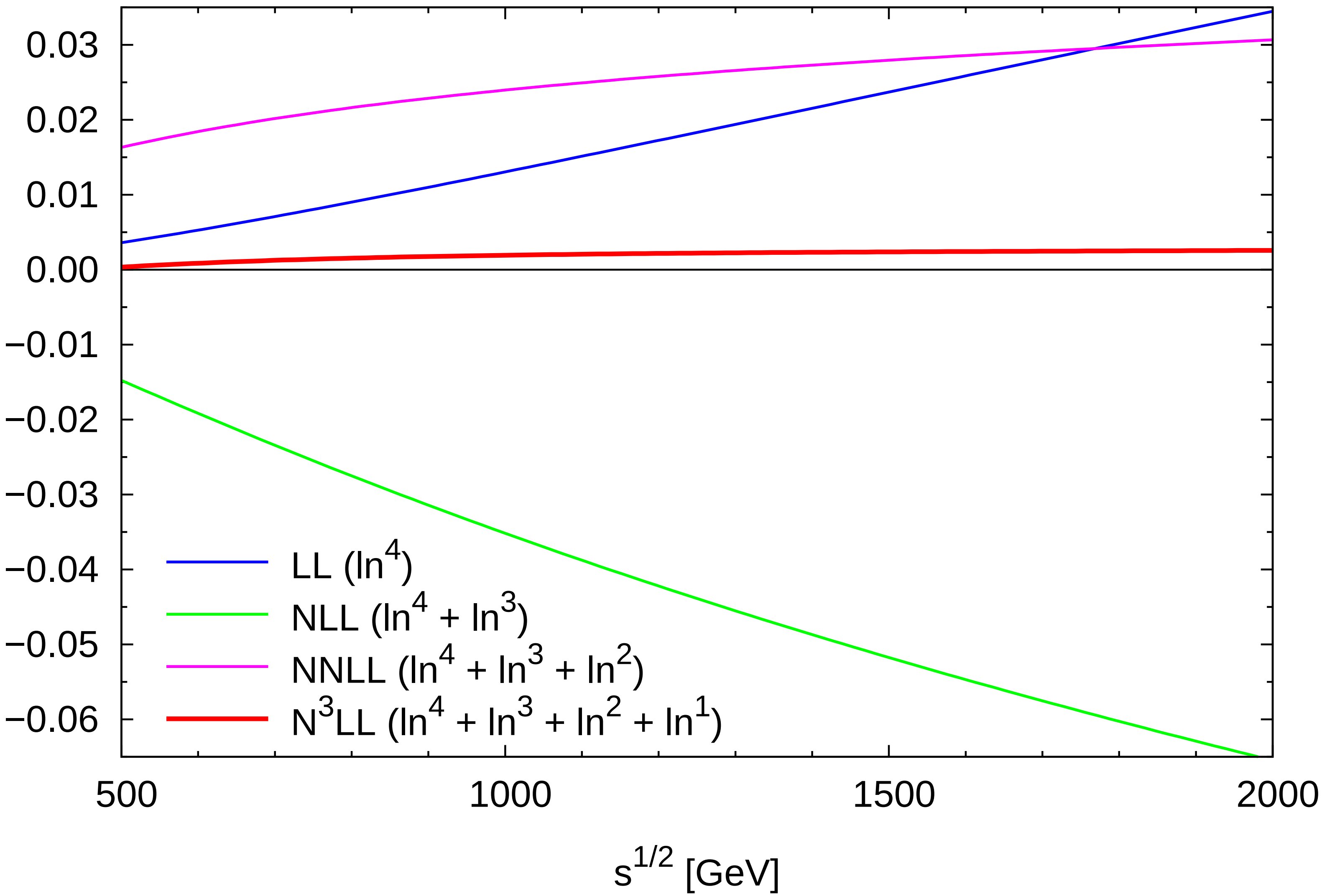
- include reduced amplitude $\tilde{A}$
- take into account electroweak quantum numbers
- SU(2)$_M\times$ U(1)$_\lambda$: calculate in the equal mass case $M_W = M_Z = \lambda = M$, then factorize off IR-singular QED contribution — B.J., Kühn, Penin, Smirnov '04, '05
- mass difference $M_Z \neq M_W$: expand around $M_Z \approx M_W$
- obtain electroweak 2-loop corrections to $f\bar{f} \to f'\bar{f}'$ cross sections, forward-backward asymmetries & left-right asymmetries

⇒ predictions with $\lesssim$1% accuracy (except for the production of t & b quarks)

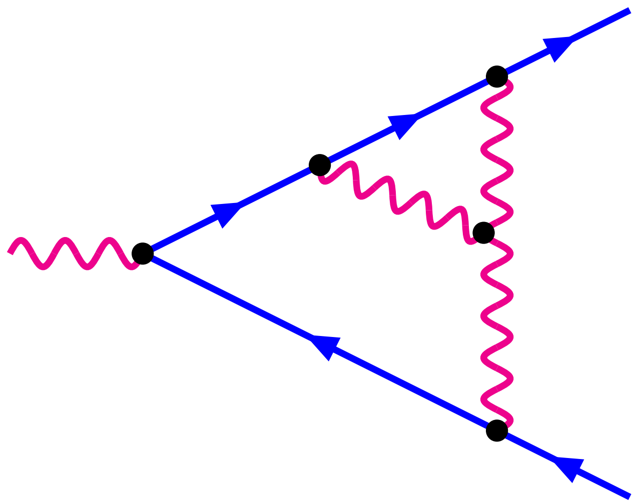
Electroweak results: example $\sigma(e^+e^- \to q\bar{q})$ $(q = d, s, b)$

numerical 2-loop result:

$$\left(\frac{\alpha_{\mathrm{ew}}}{4\pi}\right)^2 \left[+2.79 \ln^4\left(\frac{s}{M_W^2}\right) - 51.98 \ln^3\left(\frac{s}{M_W^2}\right) + 321.34 \ln^2\left(\frac{s}{M_W^2}\right) - 603.43 \ln\left(\frac{s}{M_W^2}\right) \right]$$



III Evaluating Feynman diagrams in the high energy limit



$$= \; ? + \mathcal{O}\left(\frac{M^2}{Q^2}\right)$$

Reduction to scalar diagrams

- **given** from Feynman rules: $\mathcal{F}^\mu = \bar{u}(p_2)\, \Gamma^\mu(p_1, p_2)\, u(p_1)$
- **wanted:** form factor $F(Q^2)$ with $\mathcal{F}^\mu = F(Q^2) \cdot \bar{u}(p_2)\, \gamma^\mu\, u(p_1)$
- can be calculated using the properties of Dirac matrices and spinors, combined with tensor reduction
- more elegantly with a *projector* on the form factor:

$$F(Q^2) = \frac{\mathrm{Tr}\left[\gamma_\mu\, \not{p}_2\, \Gamma^\mu(p_1, p_2)\, \not{p}_1\right]}{2(d-2)\, q^2}$$

- **output:** form factor $F(Q^2)$ in terms of *scalar Feynman integrals*

$$\int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2\, \frac{\prod_{j=1}^{N} (\ell_j \cdot \ell'_j)^{\nu_j}}{\prod_{i=1}^{L} ({k'_i}^2 - M_i^2)^{n_i}}$$

with propagators and irreducible scalar products in the numerator

Expansion by regions

Beneke, Smirnov '98

a powerful method for the asymptotic expansion of Feynman diagrams

- **given:** scalar Feynman integral & limit like $Q^2 \gg M^2$
- **wanted:** expansion of the *integral* in M^2/Q^2
- **problem:** direct expansion of the *integrand* leads to (new) IR/UV singularities

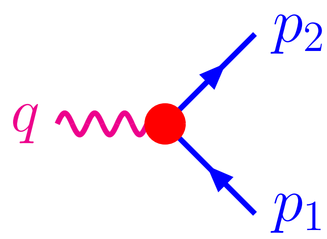
Recipe for the method of expansion by regions:

1. *divide* the integration domain into *regions* for the loop momenta
2. in every region, *expand* the integrand in a *Taylor series* with respect to the parameters that are considered small *there*
3. *integrate* the expanded integrands over the *whole integration domain*
4. put to zero any *integral without scale* (like with dimensional regularization)

- usually only a few regions give non-vanishing contributions
- for logarithmic approximation: only leading order of the expansion needed
  $\hookrightarrow$ in step 2. all small parameters in the integrand are simply set to zero

Expansion by regions: example

Vertex form factor in the Sudakov limit $Q^2 \gg M^2$



- typical regions for each loop momentum k:

hard	(h):	all components of $k \sim Q$
soft	(s):	all components of $k \sim M$
ultrasoft	(us):	all components of $k \sim M^2/Q$
1-collinear	(1c):	$k^2 \sim 2p_1 \cdot k \sim M^2, \quad 2p_2 \cdot k \sim Q^2$
2-collinear	(2c):	$k^2 \sim 2p_2 \cdot k \sim M^2, \quad 2p_1 \cdot k \sim Q^2$

- scalar 1-loop correction: $f = \frac{e^{\varepsilon \gamma_E}}{i\pi^{d/2}} \int \frac{\mathrm{d}^d k}{(k^2 - M^2)(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)}$

$$f^{(h)} = \frac{1}{Q^2} \left[-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \ln(Q^2) - \frac{1}{2} \ln^2(Q^2) + \frac{\pi^2}{12} + \mathcal{O}\left(\frac{M^2}{Q^2}\right) \right]$$

$$f^{(1c)} + f^{(2c)} = \frac{1}{Q^2} \left[\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \ln(Q^2) - \frac{1}{2} \ln^2(M^2) + \ln(M^2) \ln(Q^2) - \frac{5}{12}\pi^2 + \mathcal{O}\left(\frac{M^2}{Q^2}\right) \right]$$

$$\Rightarrow f = f^{(h)} + f^{(1c)} + f^{(2c)} = \frac{1}{Q^2} \left[-\frac{1}{2} \ln^2\left(\frac{Q^2}{M^2}\right) - \frac{\pi^2}{3} + \mathcal{O}\left(\frac{M^2}{Q^2}\right) \right]$$

Parameterization of Feynman integrals

- Feynman parameters:

$$\prod_i \frac{1}{A_i^{n_i}} = \frac{\Gamma(\sum_i n_i)}{\prod_i \Gamma(n_i)} \left(\prod_i \int_0^1 \mathrm{d}x_i \, x_i^{n_i-1} \right) \frac{\delta(\sum_i x_i - 1)}{(\sum_i x_i A_i)^{\sum_i n_i}}$$

- Schwinger parameters:

$$\frac{1}{A^n} = \frac{1}{i^n \, \Gamma(n)} \int_0^\infty \mathrm{d}\alpha \, \alpha^{n-1} \, e^{i\alpha A} , \quad \text{numerator } A^n = \left(\frac{1}{i} \frac{\partial}{\partial \alpha} \right)^n e^{i\alpha A} \Big|_{\alpha=0}$$

⇒ more convenient with non-standard propagators (→ expansion by regions)

⇒ any number of propagators and numerators may be combined

⇒ can always be transformed to Feynman parameters

↪ evaluation:

$$\int \mathrm{d}^d k \, e^{i(\alpha k^2 + 2p\cdot k)} = i\pi^{d/2} \, (i\alpha)^{-d/2} \, e^{-ip^2/\alpha}$$

$$\int_0^\infty \frac{\mathrm{d}\alpha \, \alpha^{n-1}}{(A + \alpha B)^r} = \frac{\Gamma(n) \, \Gamma(r-n)}{\Gamma(r) \, A^{r-n} \, B^n}$$

Mellin-Barnes representation

Boos, Davydychev '91

Feynman integrals with many scales are hard to evaluate

$\hookrightarrow$ separate scales by Mellin-Barnes representation:

$$\frac{1}{(A+B)^n} = \frac{1}{\Gamma(n)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i}\, \Gamma(-z)\, \Gamma(n+z)\, \frac{B^z}{A^{n+z}}$$

- Mellin-Barnes integrals go along the imaginary axis, leaving poles of $\Gamma(-z+\ldots)$ to the right and poles of $\Gamma(z+\ldots)$ to the left of the integration contour
- applicable to massive propagators ($A=k^2$, $B=-M^2$) or to any complicated parameter integral
- evaluation:
  close the integration contour to the right ($|B| \le |A|$) or to the left ($|B| \ge |A|$) and pick up the residues within the contour using $\operatorname{Res}\Gamma(z)\big|_{z=-i} = (-1)^i/i!$
- close link to *expansion by regions*:
  Mellin-Barnes representation of the full integral
  $\hookrightarrow$ asymptotic expansion with contributions corresponding to the regions

IV Summary

Four-fermion scattering

- calculation in the high energy limit reduced to the SU(2) form factor
- IR-singular QED contributions factorized
- mass difference $M_Z \neq M_W$ taken into account by expanding around $M_Z \approx M_W$
- electroweak 2-loop corrections obtained in N^3LL approximation
  $\hookrightarrow$ error $\lesssim$1%

Evaluating Feynman diagrams in the high energy limit

- expansion by regions
- Mellin-Barnes representation

$\Rightarrow$ promising methods for more calculations to come . . .