On the angular distributions of the Z boson and the two Higgs bosons produced in e^+e^- collisions.

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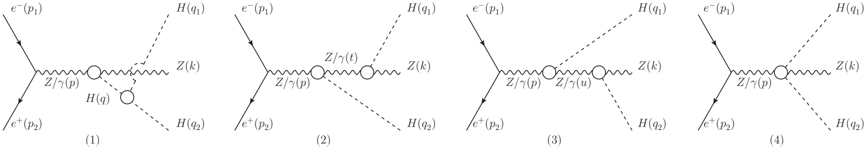
Introduction

1. One of the main targets of experiments at the future LCs is the measurement of the trilinear self-coupling λ_H of the Higgs boson.
2. The process $e^+e^- \to ZHH$ is expected to be the best reaction to measure λ_H in the earlier stage of experiments (i.e. $\sqrt{s} \simeq 500$ GeV).
3. The process is sensitive to the couplings $HHZZ$ and $HHZ\gamma$, too, which cannot be accessed through single Higgs production processes such as $e^+e^- \to ZH$.



$\to$ The Feynman diagrams, which represent the leading contributions to the scattering amplitudes. The small circles denote the Higgs couplings.

Introduction; about the process $e^+e^- \to ZHH$.

The process $e^+e^- \to ZHH$ has been investigated over the years:

- the total cross section in the SM: Gounaris et al.(1979)
- the total cross section in the MSSM: Djouadi et al.(1996), in composite Higgs models: Grber et al.(2013), and in other several new physics models: Asakawa et al.(2010).
- the one-loop radiative corrections: Zhang et al.(2004).
- the accuracy of measuring λ_H: Ilyin et al.(1996) and more.
- the expected constraints on parameters in an effective Lagrangian: Barger et al.(2003) and more.
- the analytic form of the 2 Higgs energy distributions: Djouadi et al.(1996).
- the analytic form of the invariant mass distribution of the 2 Higgs bosons (m_{HH}) and that of the Z and Higgs bosons (m_{ZH}): Contino et al.(2014).
- numerical studies of the final-state distributions: Miller et al.(2000) and more.

Introduction; about the process $e^+e^- \to ZHH$.

The process $e^+e^- \to ZHH$ has been investigated over the years:

- the accuracy of measuring λ_H: Ilyin et al.(1996) and more.
- the expected constraints on parameters in an effective Lagrangian: Barger et al.(2003) and more.
  - Most of the above studies restricted themselves to the total cross section as inputs from experiments.
  - If one intends to determine more than one parameters at the same time, only the total cross section is not enough and one needs to consider other observables such as m_{HH}.
- the analytic form of the 2 Higgs energy distributions: Djouadi et al.(1996).
- the analytic form of the invariant mass distribution of the 2 Higgs bosons (m_{HH}) and that of the Z and Higgs bosons (m_{ZH}): Contino et al.(2014).
- numerical studies of the final-state distributions: Miller et al.(2000) and more.

The purpose of this work is to introduce such observables. Our findings can be summarized as follows:

1. We find an analytic expression for the differential cross section, which has in the most general case 9 non-zero functions.
2. We clarify the relations between our 9 functions and the observables which exist in the literature.
3. The 9 functions are divided into 4 categories under CP and $\mathrm{CP}\widetilde{\mathrm{T}}$.
4. We show that our new observables can provide us different information about most of the parameters in an effective Lagrangian than the total cross section.

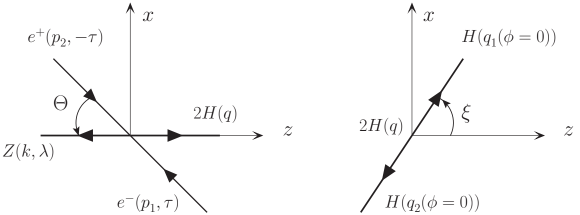
Outline

- Introduction.
- Kinematics of $e^+e^- \rightarrow ZHH$.
- Differential cross section.
- Symmetry properties.
- Numerical studies.

Kinematics of $e^+e^- \to ZHH$; 5 independent kinematic variables.

$$e^-(p_1) + e^+(p_2) \to Z(k) + H(q_1) + H(q_2).$$

$$\xrightarrow[\text{decompose}]{} e^-(p_1) + e^+(p_2) \to Z(k) + 2H(q) \text{ and } 2H(q) \to H(q_1) + H(q_2).$$



- Θ : The polar angle of the Z boson from the e^- direction. (left)
- ξ : The polar angle of the H boson from the z-axis. (right)
- ϕ : The azimuthal angle of the the H boson from the x-axis. (right)

- Q : The mass of $2H$ (i.e. m_{HH}). It determines the magnitude of the Higgs bosons' momenta.
- E : The e^+e^- c.m. energy. $E \geq m_Z + 2m_H \simeq 342$ GeV.

These 5 are the only independent kinematic variables:

$\{\Theta, \xi, \phi, Q, E\} \longleftrightarrow$ all the four-momenta of the Z, H and H bosons.

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- Introduction.
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- <u>Differential cross section.</u>
- Symmetry properties.
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Differential cross section; 9 non-zero functions in the most general case.

The complete differential cross section is given by

$$\frac{d\sigma(\tau)}{d\Omega} \equiv \frac{d\sigma(\tau)}{d\cos\Theta dQ^2 d\cos\xi d\phi} = \frac{1}{1024\pi^4}\frac{l}{E^3}\sqrt{1-\frac{4m_H^2}{Q^2}}\sum_\lambda \left|\mathcal{M}_\tau^\lambda\right|^2,$$

where the scattering amplitudes $\mathcal{M}_\tau^\lambda$ are the functions of Θ, ξ, ϕ, and Q: $\mathcal{M}_\tau^\lambda = \mathcal{M}_\tau^\lambda(\Theta, \xi, \phi, Q)$.

We find that the amplitude-squared summed over λ has the following form in the most general case:

$$\begin{aligned}\sum_\lambda |\mathcal{M}_\tau^\lambda|^2 =& F_1\left(1+\cos^2\Theta\right) + F_2\left(1-3\cos^2\Theta\right) + F_3\cos\Theta \\ &+ F_4\sin\Theta\cos\phi + F_5\sin 2\Theta\cos\phi + F_6\sin^2\Theta\cos 2\phi \\ &+ F_7\sin\Theta\sin\phi + F_8\sin 2\Theta\sin\phi + F_9\sin^2\Theta\sin 2\phi,\end{aligned}$$

where the 9 functions F_i are $F_i = F_i(\tau, \xi, Q)$, depend on the Higgs couplings, and can be experimentally determined by measuring the angles Θ and ϕ.

Differential cross section; comparison with the other observables.

The integration over ϕ leads to the $\cos\Theta$ distribution (Miller et al.(2000) and many):

$$\int_0^{2\pi} d\phi \frac{d\sigma(\tau)}{d\Omega} = 2\pi\mathcal{N}\Big[F_1(1+\cos^2\Theta) + F_2(1-3\cos^2\Theta) + F_3\cos\Theta\Big],$$

where $\mathcal{N} = \frac{1}{1024\pi^4}\frac{l}{E^3}\sqrt{1-\frac{4m_H^2}{Q^2}}$.

The further integration over $\cos\Theta$ leads to

$$\frac{d\sigma(\tau)}{dQ^2 d\cos\xi} = \int_{-1}^{1} d\cos\Theta \int_0^{2\pi} d\phi \frac{d\sigma(\tau)}{d\Omega} = \frac{16}{3}\pi\mathcal{N}F_1$$

Variables conversions with variables $x_1 = 2q_1/E$, $x_2 = 2q_2/E$ lead to

$$\frac{d\sigma(\tau)}{dx_1 dx_2} = \frac{1}{192\pi^3}F_1 \text{ (Djouadi et al.1996)}, \qquad \frac{d\sigma(\tau)}{dQ^2 dm_{ZH_1}^2} = \frac{1}{192\pi^3 E^4}F_1 \text{ (Contino et al.2014)}.$$

The total cross section is

$$\sigma(\tau) = \frac{16}{3}\pi\int dQ^2 \int d\cos\xi\ \mathcal{N}F_1.$$

These observables are directly related to F_1, which is just one of our 9 functions. Most of the 9 functions have not been studied.

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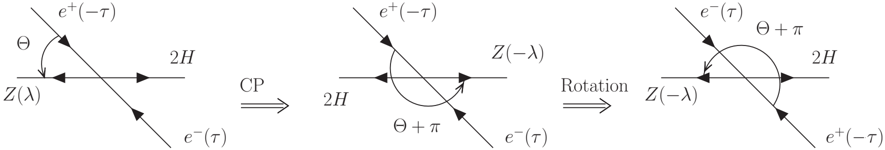
Symmetry properties; charge-conjugation (C) and parity (P).

CP, and a rotation around the y-axis by π; it does not affect final results:



- $\Theta \to \Theta + \pi$: $\cos\Theta \to -\cos\Theta$, $\sin\Theta \to -\sin\Theta$.
- $\lambda \to -\lambda$.

While the momentum of 2H is unchanged, the momentum of each Higgs boson is changed by CP and the rotation as

$$\begin{aligned}
\vec{q}_1 &= \Big(r\sin\xi\cos\phi,\ r\sin\xi\sin\phi,\ l/2 + (E-w)/Q \times r\cos\xi\Big),\\
&\xrightarrow[\text{CP}]{} \Big(-r\sin\xi\cos\phi,\ -r\sin\xi\sin\phi,\ -l/2 - (E-w)/Q \times r\cos\xi\Big),\\
&\xrightarrow[\text{Rotation}]{} \Big(+r\sin\xi\cos\phi,\ -r\sin\xi\sin\phi,\ +l/2 + (E-w)/Q \times r\cos\xi\Big).
\end{aligned}$$

The consequence of CP *and* the rotation is, therefore,

$\cos\Theta \to -\cos\Theta$, $\sin\Theta \to -\sin\Theta$, $\lambda \to -\lambda$, $\sin\phi \to -\sin\phi$ (or $\phi \to 2\pi - \phi$).

Symmetry properties; a consequence of CP invariance.

We observe that 4 of the 9 terms change sign:

$$\begin{aligned}
\sum_\lambda |\mathcal{M}_\tau^\lambda|^2 = {} & F_1\left(1+\cos^2\Theta\right) + F_2\left(1-3\cos^2\Theta\right) + F_3\cos\Theta + F_4\sin\Theta\cos\phi \\
& + F_5\sin 2\Theta\cos\phi + F_6\sin^2\Theta\cos 2\phi + F_7\sin\Theta\sin\phi + F_8\sin 2\Theta\sin\phi \\
& + F_9\sin^2\Theta\sin 2\phi \\
\xrightarrow[\text{CP+Rot.}]{} {} & F_1\left(1+\cos^2\Theta\right) + F_2\left(1-3\cos^2\Theta\right) - F_3\cos\Theta - F_4\sin\Theta\cos\phi \\
& + F_5\sin 2\Theta\cos\phi + F_6\sin^2\Theta\cos 2\phi + F_7\sin\Theta\sin\phi - F_8\sin 2\Theta\sin\phi \\
& - F_9\sin^2\Theta\sin 2\phi.
\end{aligned}$$

- The F_3, F_4, F_8 and F_9 terms are CP-odd.
- CP invariance of the differential cross section requires these 4 functions to be identically zero.

Note that

- all the functions F_i are invariant under CP *and* the rotation, because (1) F_i are independent on Θ, ϕ and (2) all the other kinematic variables on which F_i depend (for instance ξ) are invariant under CP *and* the rotation.
- the helicity flip $\lambda \to -\lambda$ does nothing, since the summation over the helicity λ has been done.

Symmetry properties; CP and $\mathrm{CP\widetilde{T}}$.

Functions	Symm. properties		beam pol.
	CP	$\mathrm{CP\widetilde{T}}$	
F_1	$+$	$+$	-
F_2	$+$	$+$	-
F_3	$-$	$-$	$\circ$
F_4	$-$	$-$	$\circ$
F_5	$+$	$+$	-
F_6	$+$	$+$	-
F_7	$+$	$-$	$\circ$
F_8	$-$	$+$	-
F_9	$-$	$+$	-

- The symbol $+$ $(-)$ means that the function is even (odd) under CP or $\mathrm{CP\widetilde{T}}$.
- $\mathrm{CP\widetilde{T}}$ violation indicates the existence of re-scattering effects (Hagiwara et al.(1987)).
- (CP,$\mathrm{CP\widetilde{T}}$)=$(\times,\times)$: 4 non-zero, (CP,$\mathrm{CP\widetilde{T}}$)=$(-,\times)$: 6 non-zero, (CP,$\mathrm{CP\widetilde{T}}$)=$(\times,-)$: 5 non-zero, and (CP,$\mathrm{CP\widetilde{T}}$)=$(-,-)$: 9 non-zero.

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Numerical studies; an effective Lagrangian for the Higgs couplings.

We obtain non-standard Higgs couplings to the Higgs boson itself, the Z boson and the photon from the effective Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\rm eff} &= (1+\delta_1) m_Z^2 \frac{H}{v} Z_\mu Z^\mu + \delta_2^{AA} \frac{H}{v} A_{\mu\nu} A^{\mu\nu} + \tilde{\delta}_4^{AA} \frac{H}{v} A_{\mu\nu} \widetilde{A}^{\mu\nu} \\
&+ \sum_{V=Z,A} \left\{ \delta_2^V \frac{H}{v} Z_{\mu\nu} V^{\mu\nu} + \delta_3^V \frac{1}{v} \left[(\partial^\mu H) Z^\nu - (\partial^\nu H) Z^\mu \right] V_{\mu\nu} + \tilde{\delta}_4^V \frac{H}{v} Z_{\mu\nu} \widetilde{V}^{\mu\nu} \right\} \\
&+ (1+\delta_5) m_Z^2 \frac{H^2}{2v^2} Z_\mu Z^\mu - \frac{m_H^2}{2v}(1+\delta_9) H^3 + \delta_{10} \frac{H}{v} (\partial^\mu H)^2 \\
&+ \sum_{V=Z,A} \left\{ \delta_6^V \frac{H^2}{2v^2} Z_{\mu\nu} V^{\mu\nu} + \delta_7^V \frac{H}{v^2} \left[(\partial^\mu H) Z^\nu - (\partial^\nu H) Z^\mu \right] V_{\mu\nu} + \tilde{\delta}_8^V \frac{H^2}{2v^2} Z_{\mu\nu} \widetilde{V}^{\mu\nu} \right\}
\end{aligned}
$$

- All of the 18 parameters δ_i are zero at the tree level in the SM.
- The 5 operators with $\tilde{\delta}_4^V$, $\tilde{\delta}_8^V$ and $\tilde{\delta}_4^{AA}$ are CP-odd and the other 13 operators are CP-even.
- We focus on the dependence on the parameters δ_5, δ_6^V, δ_7^V, $\tilde{\delta}_8^V$, δ_9^V, δ_{10}.
- We choose as a benchmark point $\delta_2^Z = \delta_2^A = \delta_2^{AA} = -\delta_3^Z = -\delta_3^A = -0.05$.

Numerical studies; new functions sensitive to the difference from the total cross section.

We integrate the differential cross section over Q^2 and $\cos\xi$:

$$\frac{d\sigma(\tau)}{d\cos\Theta d\phi} = \mathcal{C}_1\left(1+\cos^2\Theta\right) + \mathcal{C}_2\left(1-3\cos^2\Theta\right) + \mathcal{C}_3\cos\Theta + \mathcal{C}_4\sin\Theta\cos\phi + \mathcal{C}_5\sin 2\Theta\cos\phi$$
$$+\mathcal{C}_6\sin^2\Theta\cos 2\phi + \mathcal{C}_7\sin\Theta\sin\phi + \mathcal{C}_8\sin 2\Theta\sin\phi + \mathcal{C}_9\sin^2\Theta\sin 2\phi,$$

where the 9 coefficients are

$$\mathcal{C}_i(\tau) = \int_{4m_H^2}^{(E-m_Z)^2} dQ^2 \int_0^1 d\cos\xi \, \frac{1}{1024\pi^4}\frac{l}{E^3}\sqrt{1-\frac{4m_H^2}{Q^2}}F_i(\tau, Q, \xi).$$

Recall $\sigma(\tau) = \frac{16}{3}\pi\mathcal{C}_1(\tau)$.

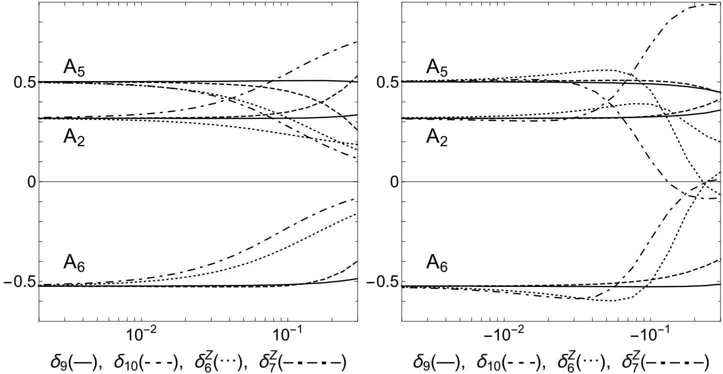
Assuming unpolarized e^+e^- beams, we define 8 functions by means of $\mathcal{C}_i(\tau)$ as

$$A_i \equiv \frac{\sum_\tau \mathcal{C}_i(\tau)}{\sum_\tau \mathcal{C}_1(\tau)} \quad (i = 2, 3, \cdots, 9).$$

- A large dependence of A_i on a parameter
  $\Longrightarrow \mathcal{C}_i$ $(i = 2, 3, \cdots, 9)$ has the different dependence on the parameter than $\mathcal{C}_1$ (i.e. the total cross section) (^^).

Numerical studies; CP-even and $\mathrm{CP\widetilde{T}}$-even functions: A_2, A_5 and A_6.

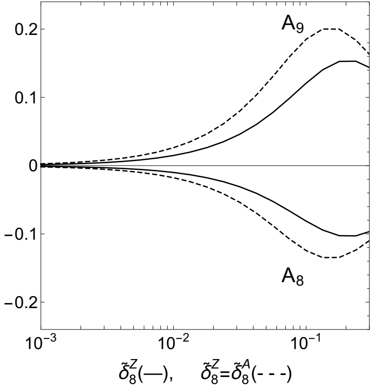
CP-even and $\mathrm{CP\widetilde{T}}$-even functions A_2, A_5 and A_6 are shown as deviations caused by adding non-zero parameters δ_9 (solid curve), δ_{10} (dashed curve), δ_6^Z (dotted curve) and δ_7^Z (broken curve):



- Small dependence on $\delta_9 \Longrightarrow \mathcal{C}_2$, $\mathcal{C}_5$ and $\mathcal{C}_6$ have the similar dependence on δ_9 as $\mathcal{C}_1$ (- -).
- Large dependences on δ_{10}, δ_6^Z and $\delta_7^Z \Longrightarrow \mathcal{C}_2$, $\mathcal{C}_5$ and $\mathcal{C}_6$ have different dependences on these parameters than $\mathcal{C}_1$ (^^).

Numerical studies; CP-odd and $\mathrm{CP}\widetilde{\mathrm{T}}$-even functions: A_8 and A_9.

CP-odd and $\mathrm{CP}\widetilde{\mathrm{T}}$-even functions A_8 and A_9 are shown as deviations caused by adding non-zero CP-odd parameters $\tilde{\delta}_8^Z$ (solid curve) and $\tilde{\delta}_8^Z = \tilde{\delta}_8^A$ (dashed curve):



- $\mathcal{C}_8$ and $\mathcal{C}_9$ obviously have advantages over the total cross section in determining CP-odd parameters, because the total cross section is both CP-even and $\mathrm{CP}\widetilde{\mathrm{T}}$-even (^^).

Summary.

1. We have found an analytic expression for the differential cross section, which has in the most general case 9 non-zero functions F_i.
2. We have clarified the relations between our 9 functions and the observables which exist in the literature.
3. We have studied the symmetry properties of the 9 functions, and divided them into 4 categories under CP and $\mathrm{CP}\widetilde{\mathrm{T}}$.
4. We have numerically shown that our new observables can provide us different information about most of the parameters in the effective Lagrangian than the total cross section.

Thank you so much for your attention.

Back up

Momentum parametrization.

$$p_1^\mu = \frac{E}{2}\big(1, \sin\Theta, 0, -\cos\Theta\big), \qquad p_2^\mu = \frac{E}{2}\big(1, -\sin\Theta, 0, \cos\Theta\big),$$
$$k^\mu = (w, 0, 0, -l), \qquad q^\mu = (q_1 + q_2)^\mu = (E - w, 0, 0, l),$$

w is the energy of the Z boson: $w = (E^2 + m_Z^2 - Q^2)/(2E)$ where $Q^2 = q \cdot q$, $-l$ is the momentum of the Z boson: $l = \sqrt{w^2 - m_Z^2}$, and $q_{1,2}^\mu$ are the four-momenta of the two Higgs bosons. We parametrize $q_{1,2}^\mu$ in the rest frame of q^μ as

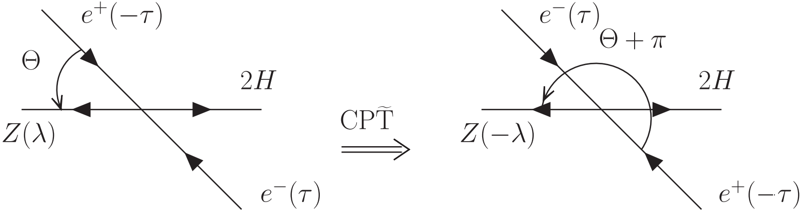
$$\begin{aligned}
q^\mu &= (q_1 + q_2)^\mu = (Q, 0, 0, 0),\\
q_1^\mu &= \big(Q/2, r\sin\xi\cos\phi, r\sin\xi\sin\phi, r\cos\xi\big),\\
q_2^\mu &= \big(Q/2, -r\sin\xi\cos\phi, -r\sin\xi\sin\phi, -r\cos\xi\big),
\end{aligned}$$

where $r = \sqrt{Q^2/4 - m_H^2}$. Since we cannot distinguish the two Higgs bosons, we define the regions of the angles as $0 \le \xi \le \pi/2$ and $0 \le \phi \le 2\pi$ and identify the Higgs boson whose momentum along the z-axis is positive as the Higgs boson that has q_1^μ. The four-momenta $q_{1,2}^\mu$ in our e^+e^- c.m. frame can be easily obtained by a single boost along the positive direction of the z-axis:

$$q_1^\mu = \left(\frac{E-w}{2} + \frac{l}{Q} r\cos\xi, r\sin\xi\cos\phi, r\sin\xi\sin\phi, \frac{l}{2} + \frac{E-w}{Q} r\cos\xi\right),$$
$$q_2^\mu = \left(\frac{E-w}{2} - \frac{l}{Q} r\cos\xi, -r\sin\xi\cos\phi, -r\sin\xi\sin\phi, \frac{l}{2} - \frac{E-w}{Q} r\cos\xi\right).$$

Consequence of $\mathrm{CP\widetilde{T}}$ invariance.

CP and time-reversal without interchanging the initial and final states ($\widetilde{\mathrm{T}}$):



$$\mathrm{CP\widetilde{T}} : \cos\Theta \to -\cos\Theta,\ \sin\Theta \to -\sin\Theta,\ \lambda \to -\lambda.$$

$$\sum_{\lambda=\pm,0} |\mathcal{M}_\tau^\lambda|^2 \xrightarrow[\mathrm{CP\widetilde{T}}]{} F_1\left(1+\cos^2\Theta\right) + F_2\left(1-3\cos^2\Theta\right) - F_3\cos\Theta - F_4\sin\Theta\cos\phi + F_5\sin 2\Theta$$

$$+ F_6\sin^2\Theta\cos 2\phi - F_7\sin\Theta\sin\phi + F_8\sin 2\Theta\sin\phi + F_9\sin^2\Theta\sin 2\phi.$$

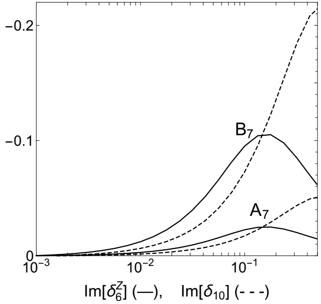
The F_3, F_4, and F_7 terms are $\mathrm{CP\widetilde{T}}$-odd and the $\mathrm{CP\widetilde{T}}$ invariance of the differential cross section requires these 3 functions to be identically zero.

A benefit from polarized beams.

Due to the existence of the overall τ in F_3, F_4 and F_7, the corresponding functions A_3, A_4 and A_7 can be suppressed. We define functions as

$$B_i \equiv \frac{(1+P_-)(1-P_+)\mathcal{C}_i(+) + (1-P_-)(1+P_+)\mathcal{C}_i(-)}{(1+P_-)(1-P_+)\mathcal{C}_1(+) + (1-P_-)(1+P_+)\mathcal{C}_1(-)},$$

where P_- $(-1 \leq P_- \leq 1)$ and P_+ $(-1 \leq P_+ \leq 1)$ denote the degrees of longitudinal polarization of the electron and the positron, respectively. We choose $(P_-, P_+) = (-0.8, 0.3)$.



The results show that $B_7 > A_7$, i.e. the sensitivity to re-scattering effects can be significantly increased by means of longitudinally polarized e^+e^- beams

CP: $\Theta \to \Theta + \pi$, $\phi \to 2\pi - \phi$,

$$\int_{-1}^{1} d\cos\Theta \int_{0}^{2\pi} d\phi \to \int_{1}^{-1} d(-\cos\Theta) \int_{2\pi}^{0} d(-\phi)$$
$$= \int_{-1}^{1} d\cos\Theta \int_{0}^{2\pi} d\phi$$

With appropriate integration over $\cos\Theta$ and ϕ, it is possible to isolate the angular distributions:

$$\int_{-1}^{1} d\cos\Theta \frac{d\sigma}{d\cos\Theta d\phi} = \frac{8}{3}\mathcal{C}_1 + \frac{\pi}{2}\mathcal{C}_4\cos\phi + \frac{4}{3}\mathcal{C}_6\cos 2\phi + \frac{\pi}{2}\mathcal{C}_7\sin\phi + \frac{4}{3}\mathcal{C}_9\sin 2\phi, \tag{1a}$$

$$\left(\int_{0}^{1} - \int_{-1}^{0}\right) d\cos\Theta \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_3 + \frac{4}{3}\mathcal{C}_5\cos\phi + \frac{4}{3}\mathcal{C}_8\sin\phi, \tag{1b}$$

and

$$\frac{1}{2\pi}\int_{0}^{2\pi} d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_1(1+\cos^2\Theta) + \mathcal{C}_2(1-3\cos^2\Theta) + \mathcal{C}_3\cos\Theta, \tag{2a}$$

$$\frac{1}{4}\left(\int_{0}^{\pi/2} - \int_{\pi/2}^{\pi} - \int_{\pi}^{3\pi/2} + \int_{3\pi/2}^{2\pi}\right) d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_4\sin\Theta + \mathcal{C}_5\sin 2\Theta, \tag{2b}$$

$$\frac{1}{4}\left(\int_{0}^{\pi/4} - \int_{\pi/4}^{\pi/2} - \int_{\pi/2}^{3\pi/4} + \int_{3\pi/4}^{\pi} + \int_{\pi}^{5\pi/4} - \int_{5\pi/4}^{3\pi/2} - \int_{3\pi/2}^{7\pi/4} + \int_{7\pi/4}^{2\pi}\right) d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_6\sin^2\Theta, \tag{2c}$$

$$\frac{1}{4}\left(\int_{0}^{\pi} - \int_{\pi}^{2\pi}\right) d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_7\sin\Theta + \mathcal{C}_8\sin 2\Theta, \tag{2d}$$

$$\frac{1}{4}\left(\int_{0}^{\pi/2} - \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} - \int_{3\pi/2}^{2\pi}\right) d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_9\sin^2\Theta. \tag{2e}$$

By combining the 2 approaches in eq. (1) and the 5 approaches in eq. (2), we obtain the 10 ($= 2 \times 5$) combinations. The 2 of them simply give zero (i.e. eqs. (1b) and (2c), and eqs. (1b) and (2e)). Each of the remaining 8 combinations gives one of $\mathcal{C}_i$ ($i = 1, 3, 4, 5, 6, 7, 8, 9$). For example, eqs. (1b) and (2b) gives $\mathcal{C}_5$. Only $\mathcal{C}_2$ is not determined in this method.