Higher-order QED corrections to lepton $g-2$

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This talk is based on collaboration w/
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lepton $g-2$

- Intrinsic magnetic property of a single lepton particle is characterized by a dimensionless number, called g-factor.

$$H = -\vec{\mu} \cdot \vec{B}, \qquad \vec{\mu} = g \frac{e}{2m} \vec{s}$$

- Anomaly, $a \equiv (g-2)/2$, is a consequence of quantum nature of elementary particles. R. Kusch and H. M. Foley 1948, J. Schwinger 1948
- Electron $g-2$ is measured by using a Penning trap:
  University of Washington: H. Dehmelt et al. (1987)
  Harvard University: G. Gabrielse et al. (2006, 2008)
  positron $g-2$ measurement is in preparation.
- Muon $g-2$ is measured by using a muon storage ring:
  Old experiments: CERN(1959-1979), BNL(1984-2006)
  On-going experiments: J-PARC(2009–),Fermilab(2011–)

Both are the state-of-the-art measurements in precision physics.

Electron $g-2$ measurement

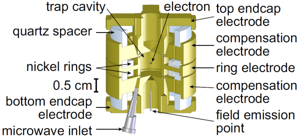


FIG. 4. Cylindrical Penning trap cavity used to confine a single electron and inhibit spontaneous emission.

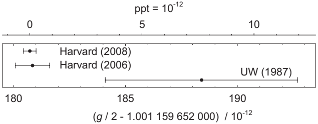


FIG. 1. Measurements [1, 2, 4] of the dimensionless magnetic moment of the electron, $g/2$, which is the electron magnetic moment in Bohr magnetons.

D. Hanneke, S. Fogwell Hoogerheide, G. Gabrielse, PRL100(2008)120801;PRA83(2010)052122

Harvard 2008 measurement

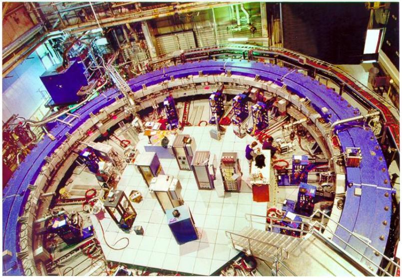
$$a_e \equiv (g_e - 2)/2 = (1\ 159\ 652\ 180.73 \pm 0.28) \times 10^{-12} \qquad [0.24\text{ppb}]$$

Theory needs QED up to 5 loop + hadronic $\mathcal{O}(10^{-12})$ + weak $\mathcal{O}(10^{-14})$:

$$(\alpha/\pi)^5 \sim 0.068 \times 10^{-12}, \qquad \alpha \equiv e^2/(4\pi\epsilon_0\hbar c) = 1/137.03\cdots,$$

where α is the fine-structure constant.

Muon $g-2$ at BNL



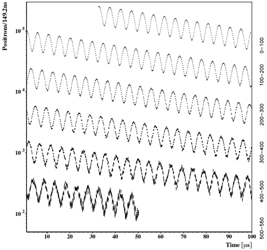


FIG. 2. The positron time spectrum obtained with muon injection for $E > 1.8$ GeV. These data represent 84 million positrons.

BNL final result 2006 G. W. Bennett et al. (Muon g-2), PRD73(2006)072003

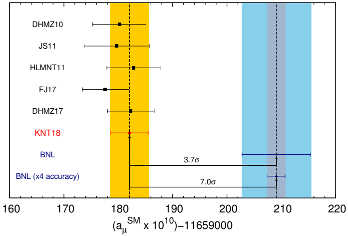
$$a_\mu \equiv (g_\mu - 2)/2 = (116\ 592\ 089 \pm 63) \times 10^{-11} \qquad [0.5\text{ppm}]$$

Theory needs QED up to 5 loop + hadronic $\mathcal{O}(10^{-7})$ + weak $\mathcal{O}(10^{-9})$:

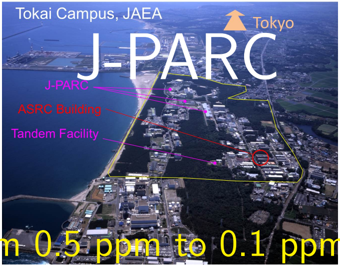
$$(\alpha/\pi)^5 \underline{\pi^2 \ln^2(m_\mu/m_e)} \sim 1.9 \times 10^{-11}$$

because of enhancement due to the electron loop, $m_e \ll m_\mu$.

New Muon $g-2$ experiments



A. Keshavarzi, D. Nomura, and T. Teubner, arXiv:1802.02995



Precision will be reduced from 0.5 ppm to 0.1 ppm.

theory of lepton $g-2$

The Standard Model contribution to the lepton $g-2$:

$$a_l = \underbrace{a_l(\mathrm{QED})}_{\gamma,\ e,\ \mu,\ \tau} + \underbrace{a_l(\mathrm{weak})}_{W^{\pm},\ Z^0} + a_l(\mathrm{hadron})$$

The QED contribution depending on lepton masses involved.
For the electron $g-2$, the dimensionless a_e is divided into

$$a_e(\mathrm{QED}) = \underbrace{A_1}_{\gamma,\ e} + \underbrace{A_2(m_e/m_\mu)}_{\gamma,\ e,\ \mu} + \underbrace{A_2(m_e/m_\tau)}_{\gamma,\ e,\ \tau} + \underbrace{A_3(m_e/m_\mu, m_e/m_\tau)}_{\gamma,\ e,\ \mu,\ \tau}.$$

A_1 is the same for any lepton, mass-independent and universal.
Perturbation expansion of QED:

$$A_i = \left(\frac{\alpha}{\pi}\right) A_i^{(2)} + \left(\frac{\alpha}{\pi}\right)^2 A_i^{(4)} + \left(\frac{\alpha}{\pi}\right)^3 A_i^{(6)} + \left(\frac{\alpha}{\pi}\right)^4 A_i^{(8)} + \left(\frac{\alpha}{\pi}\right)^5 A_i^{(10)} + \cdots$$

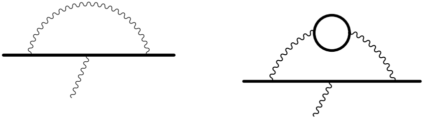
The 10th-order mass-independent $A_1^{(10)}$ is hardest to evaluate.
Here after, we discuss the electron $g-2(a_e)$ only.

QED electron $g-2$

One electron scattering by an external photon:

$$e\bar{u}(p+q/2)\left[\gamma^{\mu}F_1(q^2)+\frac{i}{2m}\sigma^{\mu\nu}q_{\nu}F_2(q^2)\right]u(p-q/2)A_{\mu}(q)$$

$$a_e \equiv F_2(q^2=0), \quad F_1(q^2=0)=1$$



The muon and tau-lepton contribute to a_e very little:

$$a_e(\text{QED:mass-dependent}) = 2.747\ 5719\ (13) \times 10^{-12}$$

from 4th, 6th, 8th and 10th-order graphs involving fermion loops.

QED mass-independent term

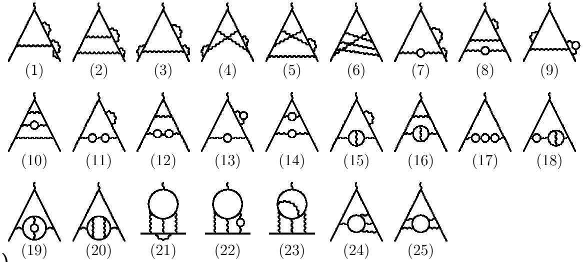
Focus on the mass-independent A_1:

n loops	# of F diagrams	w/ fermion loops	w/o fermion loops
1	1	0	1
2	7	1	6
3	72	22	50
4	891	373	518
5	12,672	6,318	6,354

n loops	$A_1^{(2n)}$	who & when
1	$A_1^{(2)} = 0.5$	Schwinger 1948
2	$A_1^{(4)} = -0.328\ 478\ 965\ \cdots$	Petermann 1957, Sommerfield 1958
3	$A_1^{(6)} = 1.181\ 241\ 456\ \cdots$	Laporta and Remiddi 1996
4	$A_1^{(8)} = -1.912\ 245\ 764\ \cdots$	Laporta 2017
5	$A_1^{(10)} = 6.675\ (192)$	Aoyama et al. (AHKN) 2018

QED 8th-order $A_1^{(8)}$

891 Feynman vertex diagrams: S. Laporta, PLB772(2017)232



History of $A_1^{(8)}$:

year	who	$A_1^{(8)}$	comment
2017	Laporta	$-1.912245764 \cdots$	almost analytic, 1100 digits
2015	AHKN	$-1.91298\ (84)$	latest numerical
2008	AHKN	$-1.9144\ (35)$	two integrals revised
2005	KN	$-1.7283\ (35)$	light-by-light revised
1990	Kinoshita	$-1.43\ (14)$	improved
1981	K & Lindquist	$-0.8\ (2.5)$	1st result

More on 8th-order terms

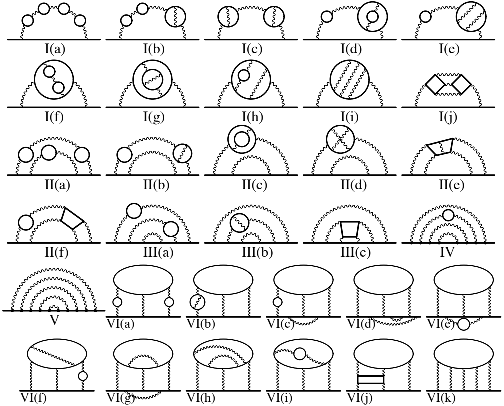
- More on the mass-independent term $A_1^{(8)}$: P. Marquard,et al.
  1) Alternative analytic result: $A_1^{(8)} = -1.87\ (12)$ arXiv:1708.07138
     Consistent with Laporta's -1.912 and AHKN's -1.913.
  2) Alternative numerical work on the contribution from 518 diagrams w/o fermion loops:

Laporta	$-2.176\ 866\ 02\cdots$	S. Laporta, PLB772(2017)232
AHKN	$-2.177\ 33(82)$	AKHN, PRD91(2015)033006
Volkov	$-2.34\ (17)$	S. Volkov, PRD96(2017)096018

- Mass-dependent terms $A_2^{(8)}$ and $A_3^{(8)}$:
  1) Numerical calculation: AKHN, PRL109(2012)111807
     Change the loop fermion mass from m_e to $m_\mu (m_\tau)$. Easy.
  2) Analytic calculation: A. Kurz et al. PRD93(2016)053017, NPB879(2014)1
     An additional small expansion parameter $m_e/m_\mu\ (m_\tau) \ll 1$.
     Difficult but easier than the mass-independent term A_1.

Don't worry about the 8th order any more. It's **CORRECT**.

QED 10th-order vertex diagrams



12,672 Feynman vertex diagrams divided into 32 subsets:

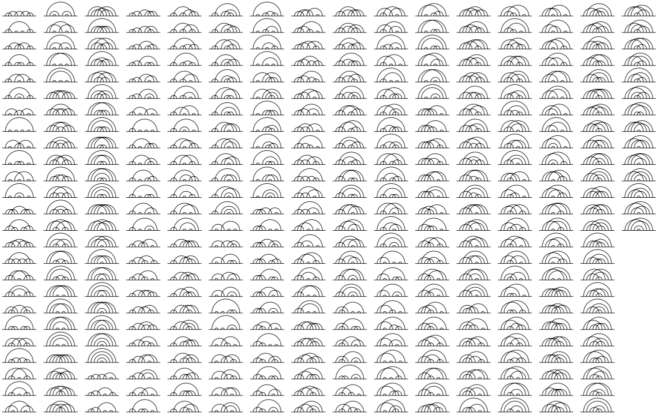
· 6,354 vertex diagrams w/o a fermion loop, Set V. difficult

· 6,318 diagrams w/ closed fermion loops, Set I-IV, IV. easier

10th-order Set V

The hardest diagrams to evaluate belong to Set V.
Ward-Takahashi concatenation:

$6354/9 = 706 \rightarrow 389$, because of time-reversal symmetry.

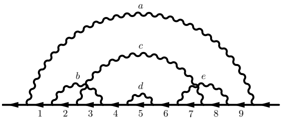


Numerical approach to QED Feynman diagrams

Uniqueness of Kinoshita's approach to QED $g-2$:

- Ward-Takahashi sum of vertex diagrams
- Feynman parameter space

Momentum space, 20 dim. v.s. Feynman parameter space, 13 dim.



· $\Lambda^{\nu}(p,q)$ ... sum of 9 vertex diagrams
· $\Sigma(p)$ .... a self-energy diagram
· q............. momentum of an external photon
· $p \pm q/2$ ... momenta of external on-shell electrons

$$\Lambda^{\nu}(p,q) \approx q_{\mu}\left[\frac{\partial \Lambda^{\mu}(p,q)}{\partial q_{\nu}}\right]_{q=0} - \frac{\partial \Sigma(p)}{\partial p_{\nu}}$$

The r.h.s. is to be calculated instead of the l.h.s.

Feynman parametric amplitude

Loop momenta are exactly and analytically integrated out.
The bare amplitude of a n-loop self-energy like diagram $\mathcal{G}$ is

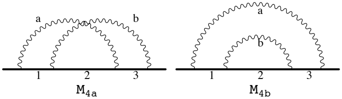
$$M_{\mathcal{G}}^{(2n)} = \left(\frac{-1}{4}\right)^n (n-1)! \int (dz)_{\mathcal{G}}$$
$$\left[\frac{1}{n-1}\left(\frac{E_0+C_0}{U^2V^{n-1}} + \frac{E_1+C_1}{U^3V^{n-2}} + \cdots + \frac{E_{n-2}+C_{n-2}}{U^nV}\right)\right.$$
$$\left. + \left(\frac{N_0+Z_0}{U^2V^n} + \frac{N_1+Z_1}{U^3V^{n-1}} + \cdots + \frac{N_{n-1}+Z_{n-1}}{U^{n+1}V}\right)\right] .$$

E_i and C_i are from $\partial\Lambda/\partial q$. N_i and Z_i are from $\partial\Sigma/\partial p$.
All are expressed by the building blocks:

- z_i... Feynman parameter of line i.
- B_{ij}... "correlation function" of lines i and line j, determined by and only by the topology of a graph.
- A_i... scalar current of the external momentum p on line i.

UV and IR counter terms: 4th-order example

Divergence structure of the WT-sum is same as that of the self-energy diagram.

On-shell renormalization: 

$$a_{4a} \equiv M_{4a} - 2L_2 M_2, \quad a_{4b} \equiv M_{4b} - dm_2 M_{2*} - B_2 M_2$$

UV in M_{4a}, M_{4b}, L_2, dm_2, B_2 and IR in M_{4b}, L_2, B_2.
Both a_{4a} and a_{4b} are IR divergent, but the sum $a_{4a} + a_{4b}$ is finite.
Thanks the Kinoshita-Lee-Nauenberg IR cancellation theorem.

Express the finite contribution in terms of the finite quantities:

$$a_{4a} + a_{4b} = \Delta M_{4a} + \Delta M_{4b} - \Delta LB_2$$

$$\Delta M_{4a} \equiv M_{4a} - 2L_2^{\mathrm{UV}} M_2, \quad L_2 = L_2^{\mathrm{UV}} + L_2^{\mathrm{R}},$$

$$\Delta M_{4b} \equiv M_{4b} - dm_2^{\mathrm{UV}} M_{2*} - B_2^{\mathrm{UV}} M_2 - L_2^{\mathrm{R}} M_2 - dm_2^{\mathrm{R}} M_{2*}$$

$$B_2 = B_2^{\mathrm{UV}} + B_2^{\mathrm{R}}, \ dm_2 = dm_2^{\mathrm{UV}} + dm_2^{\mathrm{R}}, \ \Delta LB_2 = L_2^{\mathrm{R}} + B_2^{\mathrm{R}}$$

Contribution from the 10th order Set V

Do the same separation for 389 Set V self-energy-like diagrams.
The integrands of ΔM_{X001}–ΔM_{X389} are automatically generated.
The residual finite renormalization term is obtained as

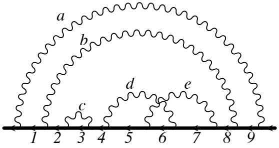
$$
\begin{aligned}
A_1^{(10)}[\text{Set V}] = {} & \Delta M_{10} + \Delta M_8 \,(-7\Delta LB_2) + \Delta M_6 \,\{-5\Delta LB_4 + 20(\Delta LB_2)^2\} \\
& + \Delta M_4 \,\{-3\Delta LB_6 + 24\Delta LB_4 \,\Delta LB_2 - 28(\Delta LB_2)^3\} \\
& + \Delta M_4 \,(2\Delta dm_4 \,\Delta L_{2^*}) \\
& + M_2 \,\{-\Delta LB_8 + 8\Delta LB_6 \,\Delta LB_2 - 28\Delta LB_4 \,(\Delta LB_2)^2 + 4(\Delta LB_4)^2 + 14(\Delta LB_2)^4\} \\
& + M_2 \,\Delta dm_6 \,(2\Delta L_{2^*}) \\
& + M_2 \,\Delta dm_4 \,(-16\Delta LB_2 \,\Delta L_{2^*} - 2\Delta dm_{2^*} \,\Delta L_{2^*} + \Delta L_{4^*}),
\end{aligned}
$$

where

$$\Delta M_{10} = \sum_{\mathcal{G}=X001}^{X389} \Delta M_{\mathcal{G}} \ .$$

Each $\Delta M_{\mathcal{G}}$ is to be numerically evaluated.

Automatic code generation for Set V



X_{253}

X253 represents 18 vertex diagrams
6354 vertex diagrams → 389 integrals

Diagram information
X253: "abccdedeba"

automation → GencodeN →

About 72,000 lines

Fortran Programs
ΔM(X253)

1. Amplitude M(X253)
2. UV subtraction terms
   M(X253)^{R} = M(X253)—(23 UV terms)
3. IR subtraction terms
   ΔM(X253)=M(X253)^{R}—(91 IR terms)

When they are numerically integrated by VEAGS, quadruple precision of real numbers is used.

HOKUSAI-BigWaterfall 2017-, 2.5 Pflops
HOKUSAI-GreatWave 2015-, 1 Pflops
RICC 2009-2017, 96Tflops
RSCC 2004-2009, 12Tflops RIKEN Wako

New computers HOKUSAI GW & BW

神奈川沖浪裏

北斎

下野黒髪山きりふりの滝

1 PFLOPS Fujitsu PRIMEHPC FX100
(34560 cores)
April 2015-
Cutting edge supercomputer
Compatibility with the K computer
Availability for highly parallelized programs

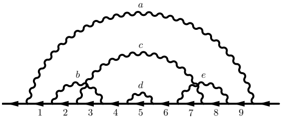
HOKUSAI GREAT WAVE

HOKUSAI BIG WATERFALL

2.58 PFLOPS IA Cluster of Xeon Gold 6148 (33600cores)
October 2017-
Raising HPC environment of RIKEN
Popular architecture
High versatility

A typical numerical integral of Set V

Example: X024



- 14 (# of lines) - 1 - 2 (# of s.e. diag.) = 11 dimensional integral.
- 31 UV and 44 IR counter terms.
- the integrand consists of 86,850 FORTRAN lines,
  occupying 24.4 MB as an executable.
- it takes 20 hours to evaluate 4×10^9 sampling points
  using double-double precision
  with 25 node (1,000 core) Intel Xeon machine (HOKUSAI-BigWaterfall).

Cross-check for integrals of Set V

- Reshuffle integration variables of the 389 integrals.
- 2017 calculation is therefore independent from 2015 calculation.
- An error was found in X024 and its value was revised.
- No error has been found in other 388 integrals.
- Numerical results with different mappings are in good agreement.

Integrals showing relatively large discrepancies:

integral	2017 result	2015 result	difference
X100	−15.232(17)	−15.292(33)	0.060
X141	−12.496(17)	−12.557(35)	0.060
X113	−4.443(17)	−4.385(32)	−0.058
X325	11.539(17)	11.596(34)	−0.056
X146	−2.246(17)	−2.299(34)	0.053
X236	2.107(21)	2.056(18)	0.051
X153	14.845(17)	14.894(34)	−0.048
X251	−1.343(20)	−1.391(08)	0.047
X044	4.365(16)	4.412(28)	−0.047
X144	23.677(17)	23.724(37)	−0.047
X252	−10.865(17)	−10.909(34)	0.044
X256	−13.996(17)	−14.041(34)	0.044

New value of $A_1^{(10)}$

New and massive evaluation of Set V leads to

the mass-independent 10th-order $A_1^{(10)}$

$$A_1^{(10)} = 6.675\ (192) \longleftarrow 7.795\ (336)$$

AKN,PRD97(2018)036001[arXiv:1712.06060]

The shift -1.120 comes from

- X024 revision -1.257.
- statistics improvement $+0.137$
  within the statistical uncertainty.

Numerical calculation was conducted on computers at RIKEN:

RICC(Intel Xeon, 2009–2017), HOKUSAI-GreatWaves(Fujitsu FX100, 2015–), and HOKUSAI-BigWaterfall(Intel Xeon, 2017–).

The fine-structure constant α(Rb)

Need the fine-structure constant α from outside QED.
The Rb atom measurement provides

CODATA2014,RMP88(2016)035009[arXiv:1507.07956]

$$\alpha^{-1}(\mathrm{Rb}:16) = 137.035\,998\,995\,(85)$$

$$\alpha^{-1}(\mathrm{Rb}:10) = 137.035\,999\,049\,(90)$$

$$\alpha(\mathrm{Rb}) = \left[\frac{2R_\infty}{c} \times \frac{A_r(\mathrm{Rb})}{A_r(e)} \times \frac{h}{m(\mathrm{Rb})}\right]^{1/2}$$

· R_∞ ... Rydberg constant from the H-atom spectroscopy and QED.
· $A_r(\mathrm{Rb}), A_r(e)$ are the relative atomic masses of the Rb atom and the electron.
  Both are determined by measuring frequency in a Penning trap.

· $h/m(\mathrm{Rb})$ is measured by using interference of matter wave.

The uncertainty is dominated by the last one $h/m(\mathrm{Rb})$.
The shift $-0.000\,000\,054$ and improvement from (90) to (85) come from the first two terms.

Electron $g-2$, theory v.s. experiment

With the fine-structure constant α(Rb),

the SM prediction of electron $g-2$

$$a_e(\text{theory} : 17) = 1\ 159\ 652\ 182.032\ (00)(13)(12)(720) \times 10^{-12}$$

$$a_e(\text{theory} : 15) = 1\ 159\ 652\ 181.643\ (25)(23)(16)(763) \times 10^{-12}$$

Uncertainties are due to QED 8th, QED 10th, hadron, and α(Rb).

Hadronic effects are given in F. Jegerlehner, arXiv:1705.00263.

The shift $+0.39 \times 10^{-12} = (+0.02 - 0.07 - 0.01 + 0.45) \times 10^{-12}$.

QED 8th, QED 10th, hadron, and α(Rb)

the Harvard measurement of electron g-2

$$a_e(\text{HV} : 08) = 1\ 159\ 652\ 180.72\ (28) \times 10^{-12}$$

D. Hanneke, S. Fogwell, and G. Gabrielse, PRL100(2008)120801 [arXiv:1801.1134]

Difference between measurement and theory:

$$a_e(\text{HV} : 08) - a_e(\text{theory} : 17) = (-1.31 \pm 0.77) \times 10^{-12}.$$

the fine-structure constant $\alpha(a_e)$

Solve $\alpha(a_e)$ from $a_e(\mathrm{HV}:08) = a_e(\mathrm{theory})$:

α from a_e

$$\alpha^{-1}(a_e:17) = 137.035\,999\,1491\,(00)(15)(13)(330)$$

$$\alpha^{-1}(a_e:15) = 137.035\,999\,1570\,(29)(27)(18)(330)$$

Uncertainties are due to QED 8th, QED 10th, hadron, and measurement.

The shift comes from $A_1^{(10)}$ and then $A_1^{(8)}$.

Difference between two determinations of α:

$$\alpha^{-1}(a_e:17) - \alpha^{-1}(\mathrm{Rb}:16) = (0.155 \pm 0.091) \times 10^{-6}$$

CODATA2017 and the Planck constant

$\alpha(a_e)$ and $\alpha(\mathrm{Rb})$ are used to determine **exact** values of some fundamental constants.

In the new SI, the Planck constant h, the elementary charge e, the Boltzmann constant k, and the Avogadro number N_A become defined numbers like the speed of light c:

$$\begin{aligned} h &= 6.626\ 070\ 15 \times 10^{-34}\ \mathrm{Js}, \\ e &= 1.602\ 176\ 634 \times 10^{-19}\ \mathrm{C}, \\ k &= 1.380\ 649 \times 10^{-23}\ \mathrm{JK}^{-1}, \\ N_A &= 6.022\ 140\ 76 \times 10^{23}\ \mathrm{mol}^{-1}. \end{aligned}$$

Definition of kilogram is based on the Planck constant h after the new SI launches in Fall of 2018.

Good bye, the International Prototype Kilogram after 2018.

P. J. Mohr, D. B. Newell, B. N. Taylor and E. Tiesinga, Metrologia 55(2018)125

Summary

- QED $g-2$ up to the 8th-order contribution has been firmly established.
- QED $g-2$ of the 10th order has been extensively checked.
- QED $g-2$ is ready for the on-going new experiments of electron-positron $g-2$ and of muon $g-2$.
- QED $g-2$ was served for the new SI. After the new SI launches, the fine-structure constant α is the unique source of uncertainties of other fundamental physical constants.
- QED $g-2$ shows that we are able to compute many and complex Feynman diagrams using analytic/numerical methods with help of powerful computers.

backup

QED contributions to muon $g-2$

Changes made so far:

- exact result for the mass-independent 8th-order $A_1^{(8)}$.
- revised result for the mass-independent 10th-order $A_1^{(10)}$.
- values of the fine-structure constant $\alpha(Rb)$ and $\alpha(a_e)$.

Dominant contributions w/ electron loops $A_2^{(8)}(m_\mu/m_e)$ has NOT been modified.

QED contributions to muon $g-2$

$$a_\mu(\mathrm{QED}:\alpha(a_e)) = 1\ 165\ 847\ 188.41\ (7)(17)(6)(28)[34] \times 10^{-12}$$

$$a_\mu(\mathrm{QED}:\alpha(\mathrm{Rb})) = 1\ 165\ 847\ 189.71\ (7)(17)(6)(72)[75] \times 10^{-12}$$

Uncertainties are due to lepton-mass ratios, QED 8th, QED 10th, α and combined.

Further numerical improvement on QED 8th and 10th is possible.

Targets are diagrams involving a light-by-light scattering subdiagram.