

# Determination of $\alpha_s$ from static potential with subtracting infrared contribution

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1. Introduction
2. Theoretical framework
3.  $\alpha_s$  determination
4. Summary

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1. Introduction

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4. Summary

# Current status of $\alpha_s$ determination

Particle Data Group

$$\alpha_s(M_Z^2) = 0.1181 \pm 0.0011$$

Error is 0.9%

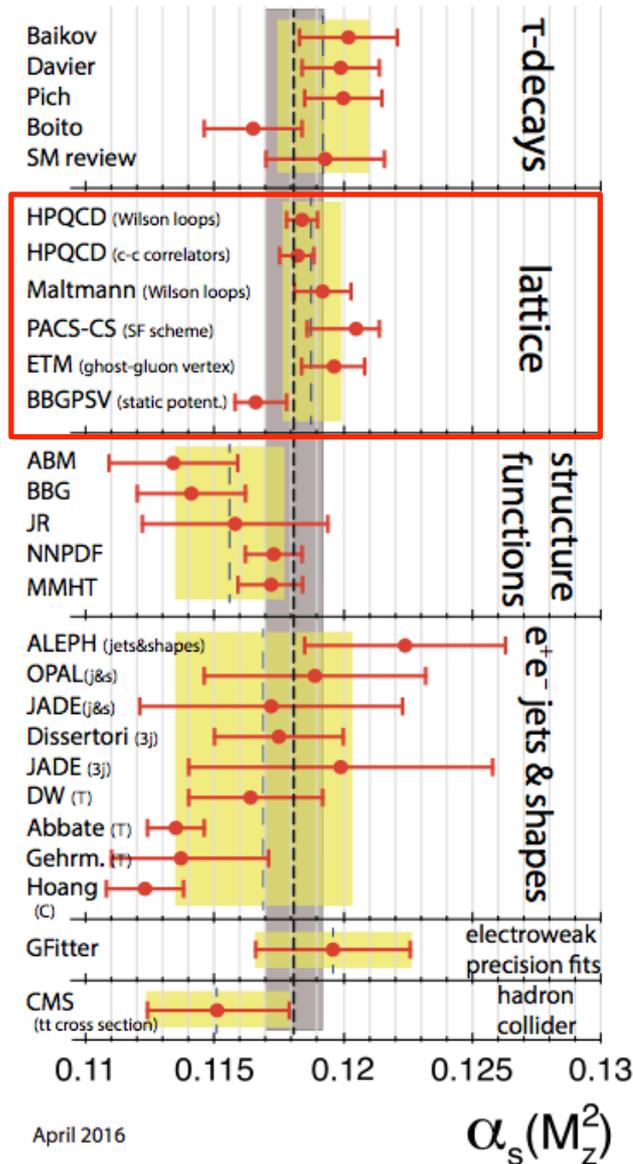
Not sufficiently small in some studies

Top pole mass measurement, Running of higgs quartic coupling, etc.

More accurate  $\alpha_s$  is demanded.

# Various determinations

Observation v.s.  
Theoretical prediction



PDG

Small errors!

# Window problem

Based on Flavor Lattice Averaging Group (FLAG)2016

**Lattice data** Typical lattice spacing  $a$  is  $a^{-1} = 2 - 5\text{GeV}$

Reliable lattice data  $Q \ll a^{-1} \longrightarrow Q \lesssim 1 - 2\text{GeV}$

Perturbation theory

Reliable for  $Q \gg \Lambda_{\text{QCD}} \sim 0.3\text{GeV} \longrightarrow Q \gtrsim 1\text{GeV}$

Very narrow window

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Very narrow window

Step scaling method

Luscher et al.

This method **enlarges the window of lattice simulation.**

Measures an effective coupling at  $Q=10-100\text{GeV}$ .

**ALPHA Collaboration 2017**  $\alpha_s(M_Z^2) = 0.11852 \pm 0.00084$  **0.7% error**

# Window problem

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Very narrow window

Our new approach

Enlarges the window of theoretical prediction

→ Framework beyond perturbation theory

Multipole expansion+Infrared subtraction  $Q \gtrsim 0.5\text{GeV}$

# Contents

✓ 1. Introduction

2. Theoretical framework

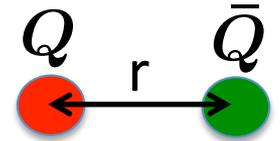
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# Multipole expansion for static potential

Expansion of  $V_{\text{QCD}}(r)$  in  $r$  1999 Brambilla, Pineda, Soto, Vairo

$$V_{\text{QCD}}(r) = \underbrace{V_S(r)}_{\mathcal{O}(1/r)} + \underbrace{\delta E_{\text{US}}(r)}_{\mathcal{O}(r^2)} + \dots$$



$$\delta E_{\text{US}} \propto \int_0^\infty dt e^{-i\Delta V(r)t} \langle \vec{r} \cdot \vec{E}^a(t) \varphi_{\text{adj}}(t, 0)^{ab} \vec{r} \cdot \vec{E}^b(0) \rangle$$

Perturbative evaluation of  $V_S$ :  $\mathcal{O}(\Lambda_{\text{QCD}}^3 r^2)$  uncertainty

Originates from IR contribution

This uncertainty is canceled against  $\delta E_{\text{US}}$ .

→ One can go beyond perturbation theory.

Each term is ambiguous due to renormalons.

# Way to calculate $V_S$

We subtract IR contributions to  $V_S$  to remove renormalon uncertainties in advance. 2000's Y. Sumino

1. Define  $V_S$  as UV quantity by introducing IR cutoff

$$V_S(r; \mu_f) = -4\pi C_F \int_{|\vec{q}| > \mu_f} \frac{d^3 q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \frac{\alpha_V(q)}{q^2} \leftarrow N^3LL$$

2009 Anzai, Kiyoo, Sumino  
Sumirnov, Sumironov, Steinhauser

2. Separate it into cutoff independent part and dependent part using complex function analysis

We obtain

$$V_S(r; \mu_f) = V_S^{\text{RF}}(r) + \mathcal{O}(\mu_f^3 r^2)$$

# Characteristics of this method

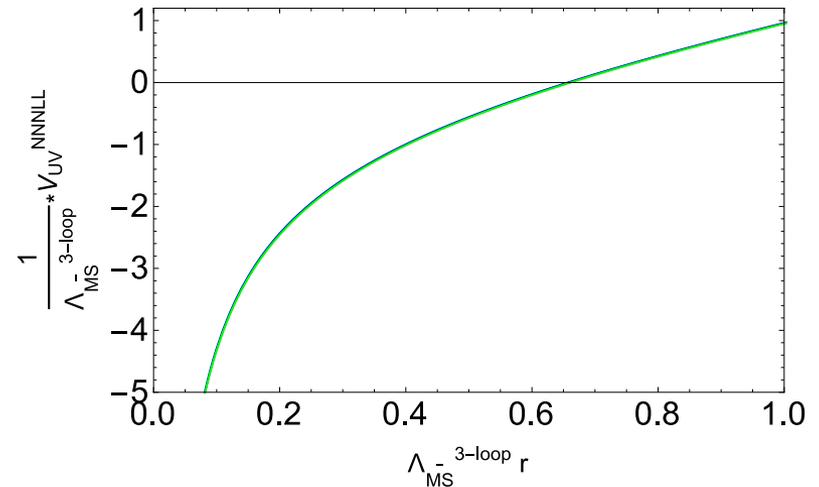
$$V_S(r; \mu_f) = \boxed{V_S^{\text{RF}}(r)} + \mathcal{O}(\mu_f^3 r^2)$$

Pure UV contribution

Coulomb+Linear form

N<sup>3</sup>LL accuracy

No unphysical singularities



# Characteristics of this method

$$V_S(r; \mu_f) = V_S^{\text{RF}}(r) + \mathcal{O}(\mu_f^3 r^2)$$

IR related part

$$V_{\text{QCD}}(r) = V_S(r; \mu_f) + \delta E_{\text{US}}(r; \mu_f)$$

with  $k < \mu_f$

# Characteristics of this method

$$V_S(r; \mu_f) = V_S^{\text{RF}}(r) + \mathcal{O}(\mu_f^3 r^2)$$

2004 Sumino  
2017 Takaura

The leading  $\mu_f$  dependence  
of  $\delta E_{\text{US}}$  is canceled against this.

$$V_{\text{QCD}}(r) = V_S(r; \mu_f) + \delta E_{\text{US}}(r; \mu_f)$$

with  $k < \mu_f$

We reach the expression

$$V_{\text{QCD}}(r) = V_S^{\text{RF}}(r) + \frac{\delta E_{\text{US}}^{\text{RF}}(r)}{= C r^2}$$

Each term is free from renormalons.

# Contents

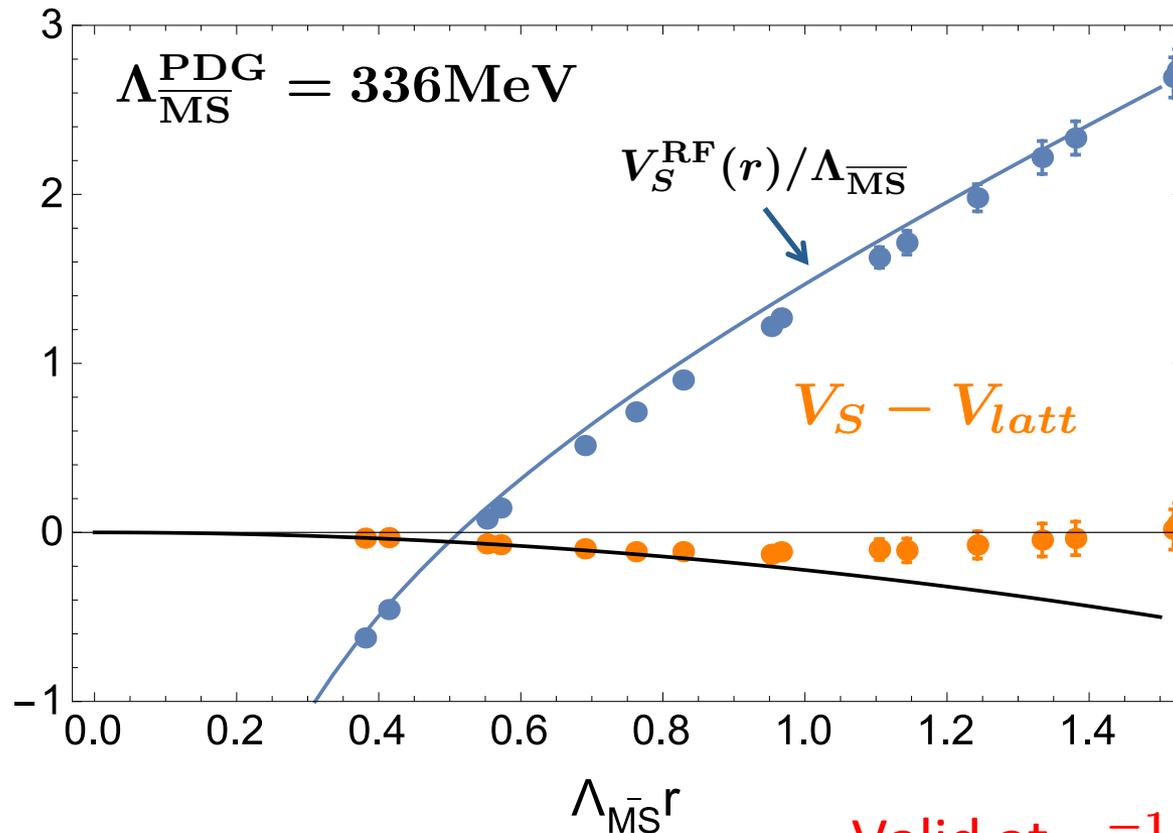
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# Continuum limit v.s. Multipole exp.

According to multipole expansion

$$V_{latt}(r) - V_S(r) \propto r^2$$



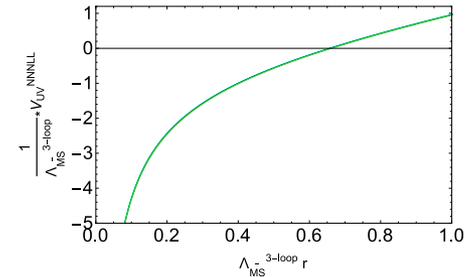
Valid at  $r^{-1} \gtrsim 0.5 \text{ GeV}$

# Global fit

Multipole expansion [ $\Lambda_{\overline{MS}}$  unit]

$$V_{th.}(r) = V_S^{\text{RF}}(r) + A_0 D_{,i} + A_2 r^2$$

**V.S.**



Lattice data with removing discretization error [GeV unit]

$i=1,2,3$  (latt1,latt2,latt3),  $D=1,2$

$$V_{latt,D,i}(r) \rightarrow V_{latt,D,i}(r) - C_F \alpha_{D,i} \left( \frac{1}{r} - \left[ \frac{1}{r} \right]_{D,i} \right) + f_D \frac{a_i^2}{r^3}$$

Tree-level correction

Extrapolation to  $a \rightarrow 0$

**Determine**  $\{ \Lambda_{\overline{MS}}, A_2, A_0 D_{,i}, \alpha_{D,i}, f_D \}$

# Result

$$\alpha_s(M_Z^2) = 0.1180 \pm 0.0007(\text{stat.})_{-0.0012}^{+0.0013}(\text{system.})$$

(a) Finite  $a$  effect ( $\pm 0.0002$ )

(b) Mass ( $\pm 0.0002$ )

(c) Higher order ( $+0.0012$   
 $-0.0010$ )

$$V_S^{\text{N}^3\text{LL}}(r) \rightarrow V_S^{\text{N}^3\text{LL}}(r) \pm \delta V_S(r)$$

$$\text{with } \delta V_S(r) = V_S^{\text{N}^3\text{LL}}(r) - V_S^{\text{N}^2\text{LL}}(r)$$

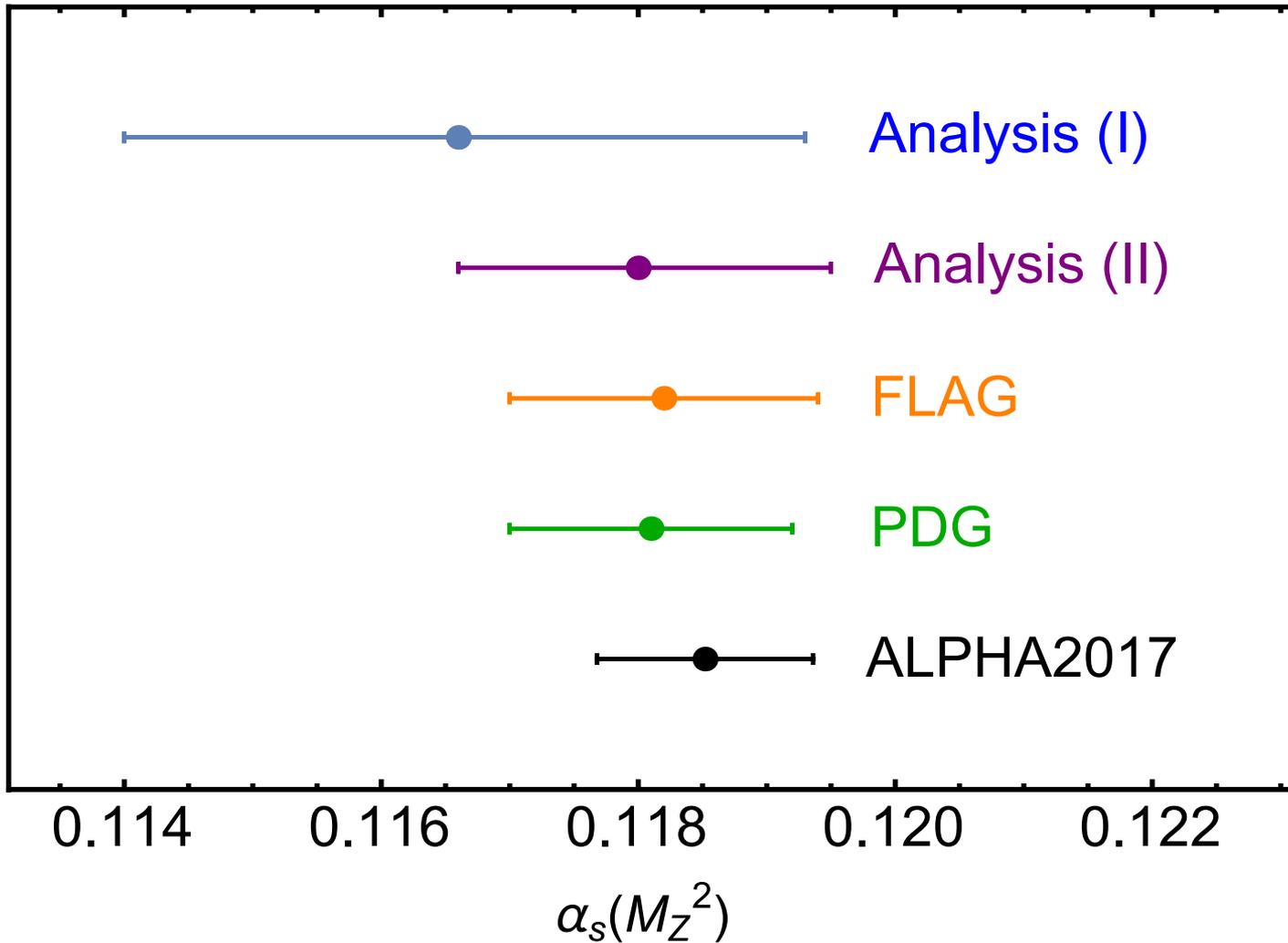
(d) OPE ( $\pm 0.0004$ )

(e) Ultrasoft scale ( $\pm 0.0002$ )

(f) Factorization scheme ( $\pm 0.0003$ )

# Result

Preliminary



# Summary

- $\alpha_s$  determinations with lattice QCD give small error, but most of them have the window problem.

## Window problem

Lattice data  $Q \lesssim 1 - 2\text{GeV}$

Perturbation theory  $Q \gtrsim 1\text{GeV}$

- We used a new framework based on the multipole expansion, where renormalons are subtracted from the leading term.

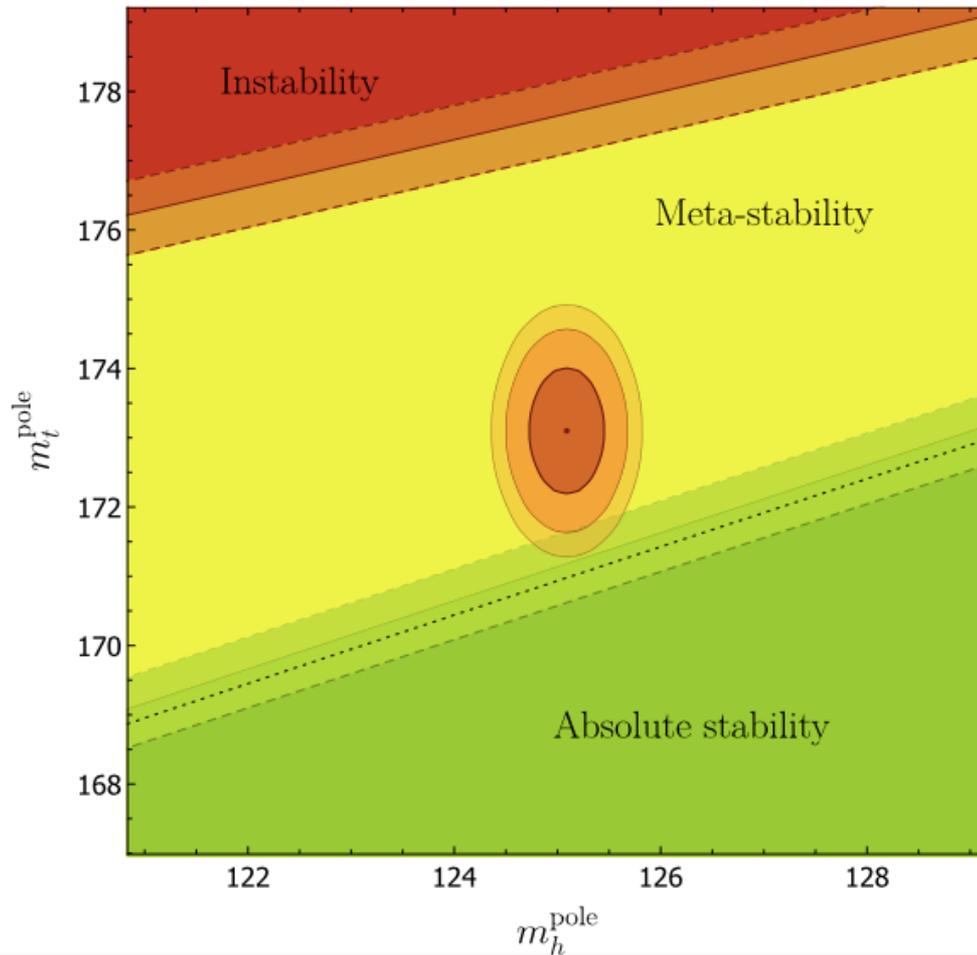
$$r^{-1} \gtrsim 0.5\text{GeV}$$

Error can be reduced with simulations in finer lattices.

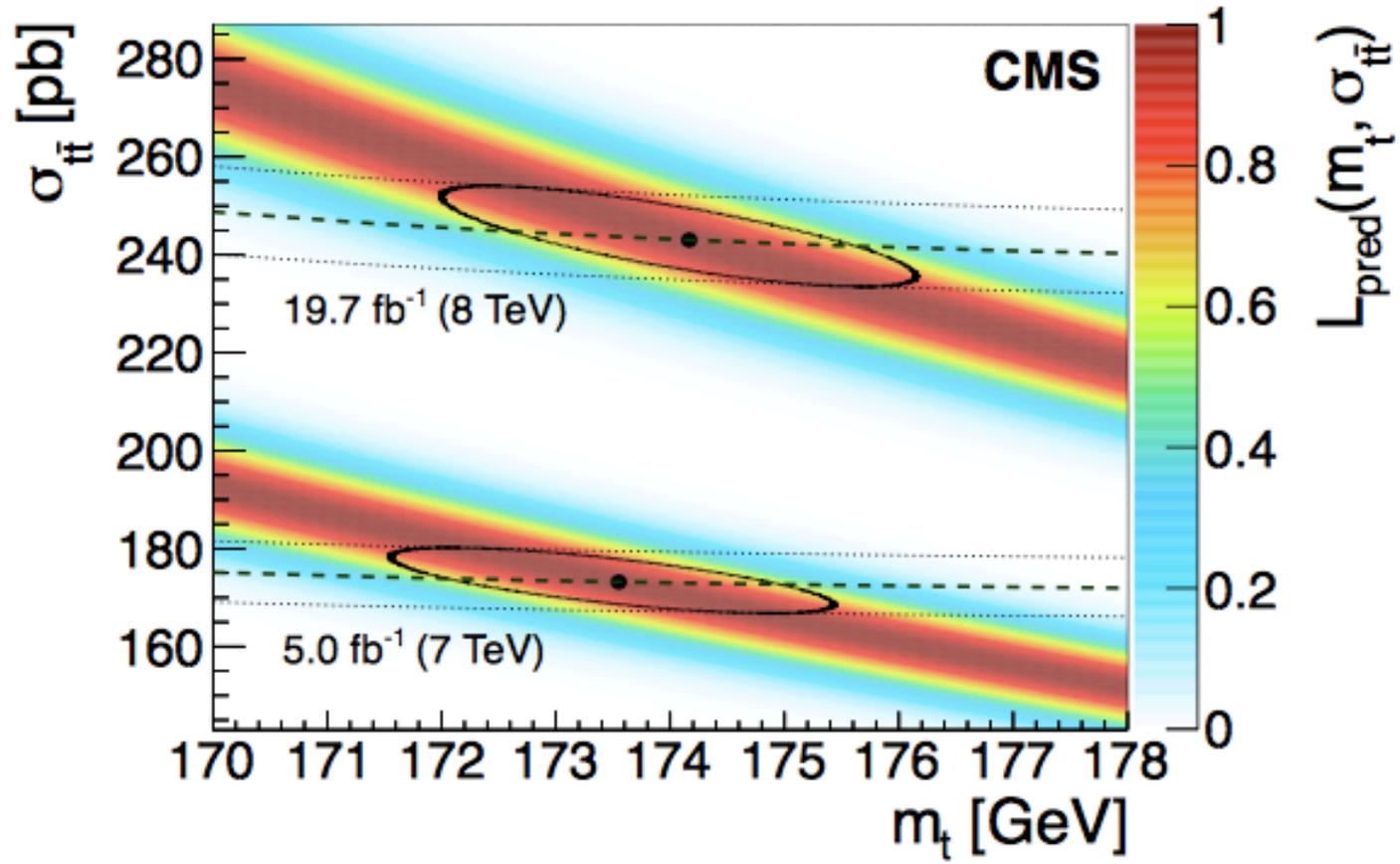
Backup

# Why is $\alpha_s$ important?

2017 Andreassen, Frost and Schwartz



# Why is $\alpha_s$ important?

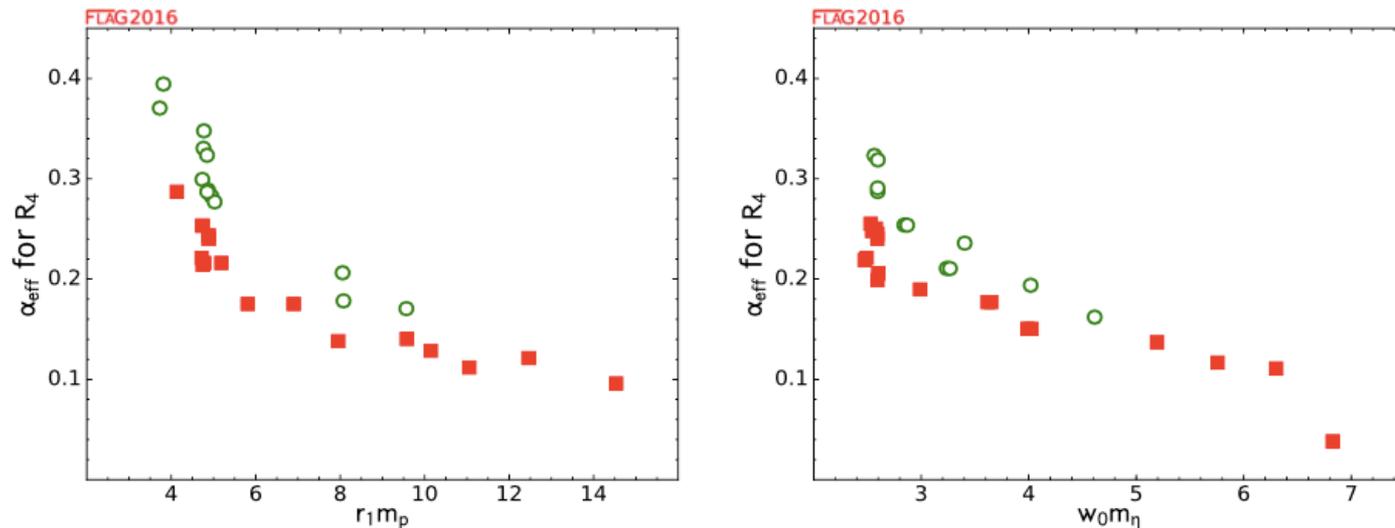


# Problem in $\alpha_s$ determination using lattice QCD

Window problem

Based on FLAG2016

$\alpha_s$  determination by HPQCD (0.5% error)



**Fig. 31**  $\alpha_{\text{eff}}$  for  $R_4$  from HPQCD 10 data (left) and from HPQCD 14A (right). A similar graph for  $R_6/R_8$  is shown in FLAG 13. Symbols correspond to  $\circ$  for data with  $1 \leq a\mu \leq 1.5$  and  $\blacksquare$  for  $a\mu > 1.5$ , while  $\star$  ( $a\mu < 1/2$ ) is not present. This corresponds exactly to the  $a\mu$  part of

our continuum limit criterion, but does not consider how many lattice spacings are present. Note that mistunings in the quark masses have not been accounted for, but, estimated as in HPQCD 14A [5], they are smaller than the size of the symbols in the graphs

# Step scaling method

## Step scaling method

The only method evaluated to be sufficiently free from window problem.

This method **enlarges the window of lattice simulation.**

### Algorithm

measures an effective coupling ( $Q=L^{-1}$ )

measures it in smaller volume lattice ( $Q=2L^{-1}$ )

⋮

measures an effective coupling ( $Q=10-100\text{GeV}$ )



Running to  
high energy

Safely perform matching with perturbation theory

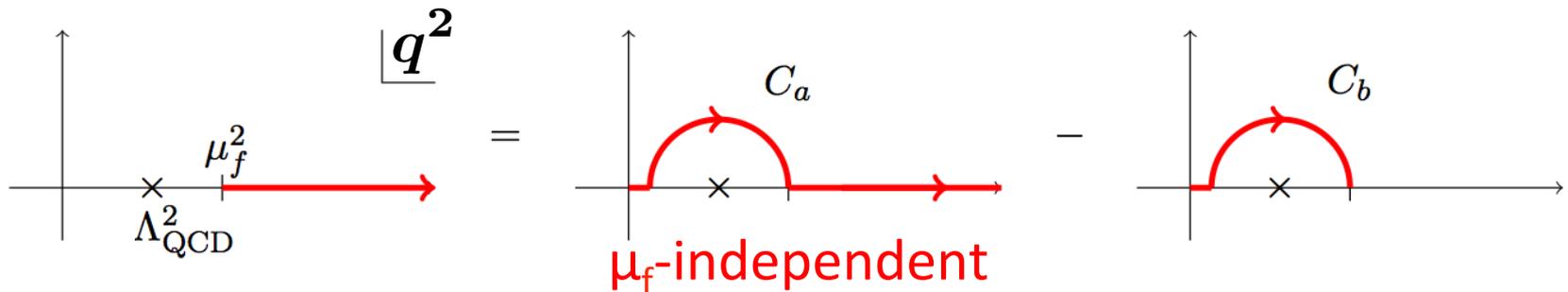
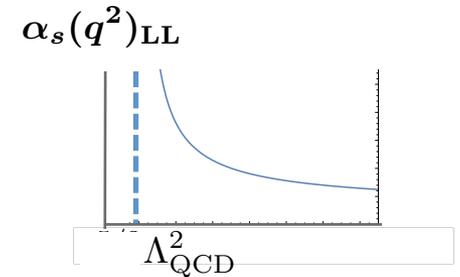
**ALPHA Collaboration 2017**  $\alpha_s(M_Z^2) = 0.11852 \pm 0.00084$  **0.7% error**

# Formulation

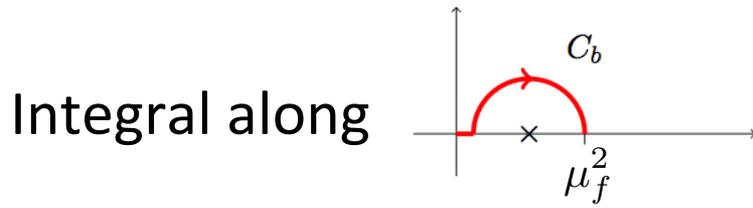
2000's Y. Sumino

At leading log

$$\begin{aligned}
 V_S(r; \mu_f) &= -\frac{2C_F}{\pi} \int_{\mu_f}^{\infty} dq \frac{\sin(qr)}{qr} \alpha_s(q^2)_{\text{LL}} \\
 &= -\frac{2C_F}{\pi} \text{Im} \int_{\mu_f}^{\infty} dq \frac{e^{iqr}}{qr} \alpha_s(q^2)_{\text{LL}}
 \end{aligned}$$

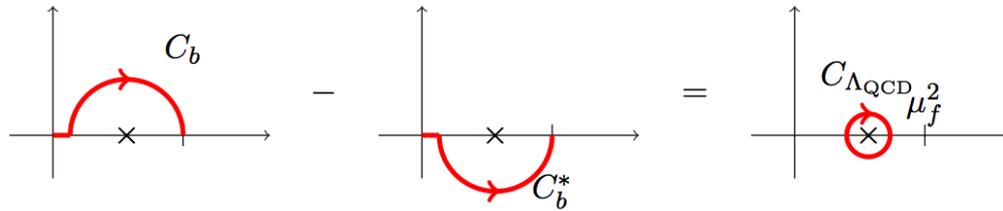


# Formulation



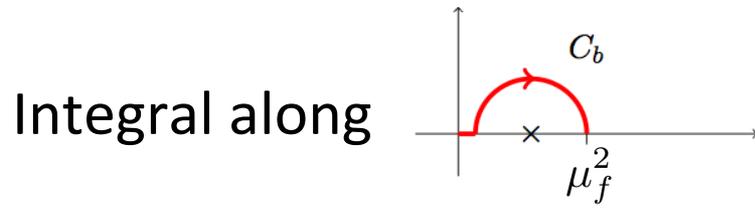
$$-\frac{2C_F}{\pi r} \text{Im} \int_{C_b} \frac{dq}{q} \boxed{e^{iqr}} \alpha_s(q)_{LL} \sim 1 + iqr - \frac{1}{2}(qr)^2 + \dots$$

If  $\{\text{Integrand}(z)\}^* = \text{Integrand}(z^*)$



$\mu_f$ -independent!

# Formulation



$$-\frac{2C_F}{\pi r} \text{Im} \int_{C_b} \frac{dq}{q} \boxed{e^{iqr}} \alpha_s(q)_{LL} \sim 1 + iqr - \frac{1}{2}(qr)^2 + \dots$$

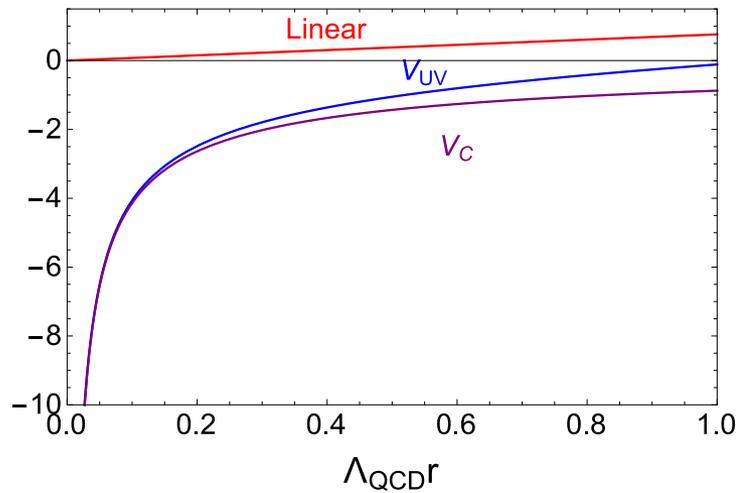
If  $\{\text{Integrand}(z)\}^* \neq \text{Integrand}(z^*)$

$\mu_f$  dependent

# $\mu_f$ independent part

$$V_S(r; \mu_f) = V_C(r) + \frac{2\pi C_F}{\beta_0} \Lambda_{\text{QCD}}^2 r + \mathcal{O}(\mu_f^3 r^2)$$

Pure UV contribution



Coulomb+Linear pot.

# $\mu_f$ dependent part

$$V_S(r; \mu_f) = V_C(r) + \frac{2\pi C_F}{\beta_0} \Lambda_{\text{QCD}}^2 r + \mathcal{O}(\mu_f^3 r^2)$$

$\mu_f$  dependent : IR related part

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$\mu_f$  dependent : IR related part

Multipole expansion

$$V_{\text{QCD}}(r) = \underbrace{V_S(r; \mu_f)}_{\text{UV}} + \underbrace{\delta E_{\text{US}}(r; \mu_f)}_{\text{IR}} + \dots$$

UV

IR EFT  $k < \mu_f$

$\delta E_{\text{US}}(r; \mu_f) \sim \mathcal{O}(\mu_f^3 r^2)$  : UV related part

1999 Brambilla et al, 2004 Sumino, 2017 Takaura

# $\mu_f$ dependent part

$$V_S(r; \mu_f) = V_C(r) + \frac{2\pi C_F}{\beta_0} \Lambda_{\text{QCD}}^2 r + \mathcal{O}(\mu_f^3 r^2)$$

$\mu_f$  dependent : IR related part

Multipole expansion

$$V_{\text{QCD}}(r) = \underbrace{V_S(r; \mu_f)}_{\text{UV}} + \underbrace{\delta E_{\text{US}}(r; \mu_f)}_{\text{IR EFT } k < \mu_f} + \dots$$

Cancel

$\delta E_{\text{US}}(r; \mu_f) \sim \mathcal{O}(\mu_f^3 r^2)$  : UV related part

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# $\mu_f$ dependent part

$$V_S(r; \mu_f) = V_C(r) + \frac{2\pi C_F}{\beta_0} \Lambda_{\text{QCD}}^2 r + \mathcal{O}(\mu_f^3 r^2)$$

$\mu_f$  dependent : IR related part

Multipole expansion

$$V_{\text{QCD}}(r) = \underbrace{V_S(r; \mu_f)}_{\text{UV}} + \underbrace{\delta E_{\text{US}}(r; \mu_f)}_{\text{IR EFT } k < \mu_f} + \dots$$

Cancel

$\delta E_{\text{US}}(r; \mu_f) \sim \mathcal{O}(\mu_f^3 r^2)$  : UV related part

1999 Brambilla et al, 2004 Sumino, 2017 Takaura

$$V_{\text{QCD}}(r) = V_S^{UV}(r) + \delta E_{\text{US}}^{IR}(r) + \dots \quad \text{No mixing}$$

# Outline of lattice analysis

$$\{r/a, aV, \delta(aV)\} : \text{latt1}(a=a_1), \text{latt2}(a=a_2), \text{latt3}(a=a_3) \quad (a_1 > a_2 > a_3)$$

→ Extract the continuum limit of  $r_1[V(r) - V(r_1)] \equiv X(r)$

$$\{X(r; a)\}_{a=a_1, a_2, a_3} \rightarrow X(r; a = 0) \quad \text{Def. of } r_1 \quad r_1^2 \frac{dV}{dr}(r_1) = 1$$

1. To construct the above sequence at a certain  $r$ , the lattice data should be interpolated.
2. From the above sequence, we extrapolate the data to  $a \rightarrow 0$ .

# Interpolation

$$V_f(r) = \frac{cc}{r} + \sigma r + c + \frac{c_1}{r^3} + d_1 r^2$$

Correction terms arise from finite lattice spacing and volume effect

Rotational symmetry ✘

Dirac.1  $\vec{r}/a = n(1, 0, 0)$

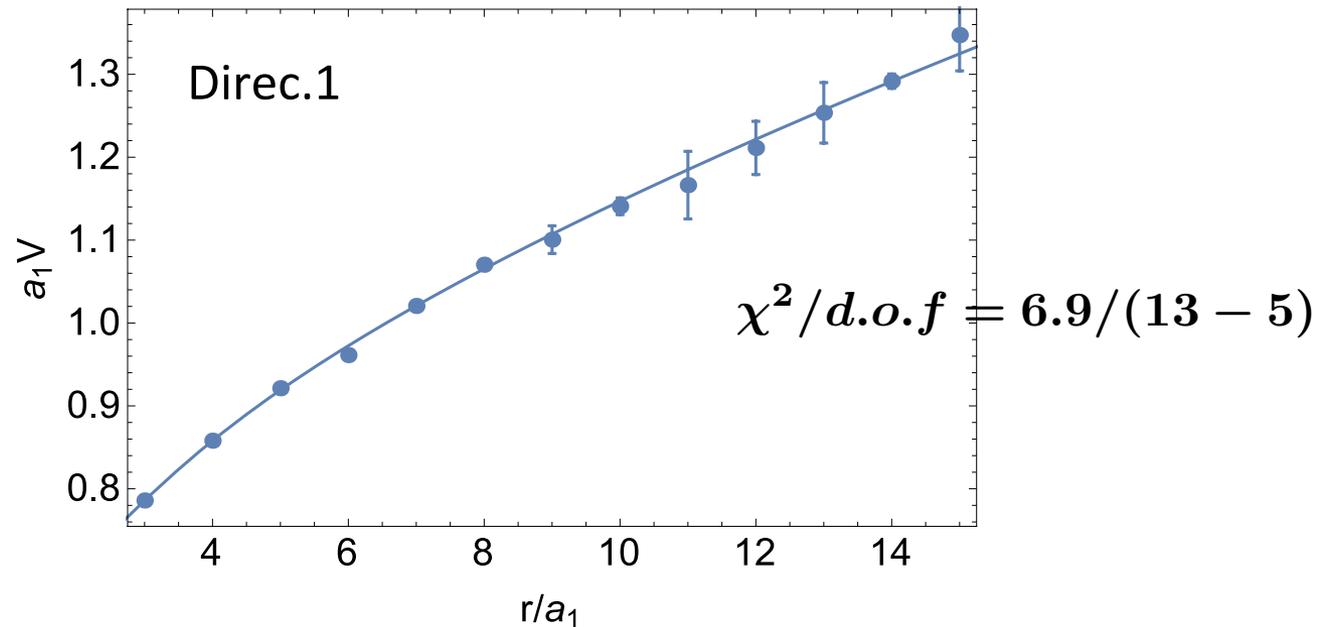
Dirac.2  $\vec{r}/a = n(1, 1, 0)$

We interpolate the data separately according to a direction.

# Interpolation

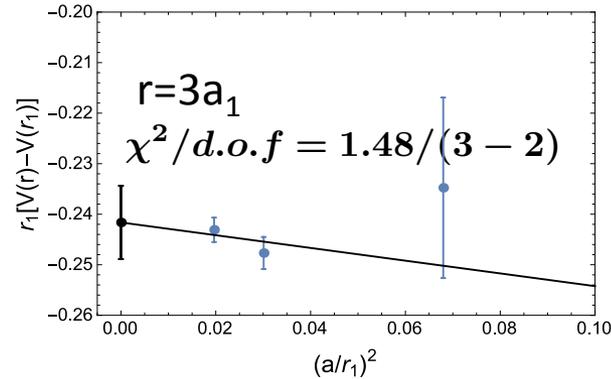
$$V_f(r) = \frac{cc}{r} + \sigma r + c + \frac{c_1}{r^3} + d_1 r^2$$

Correction terms arise from finite lattice spacing and volume effect

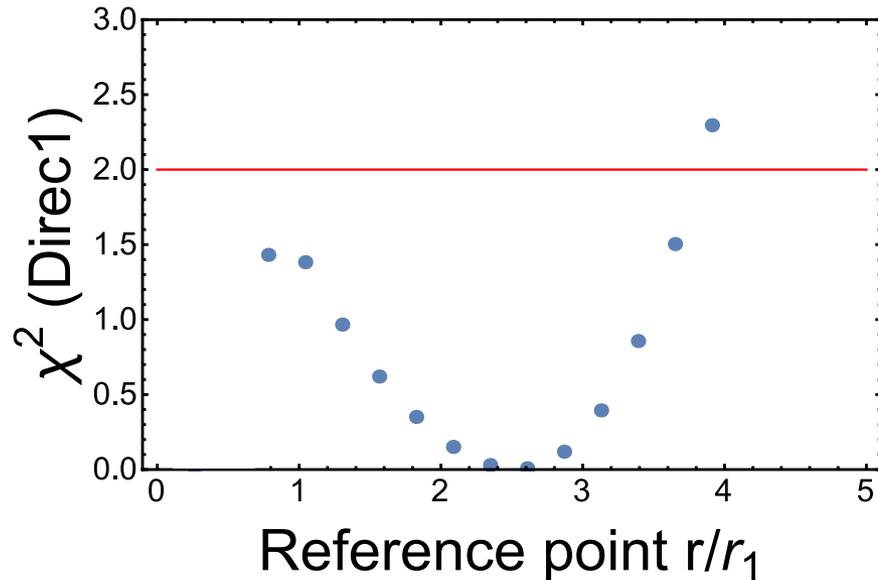


# Extrapolation to continuum limit

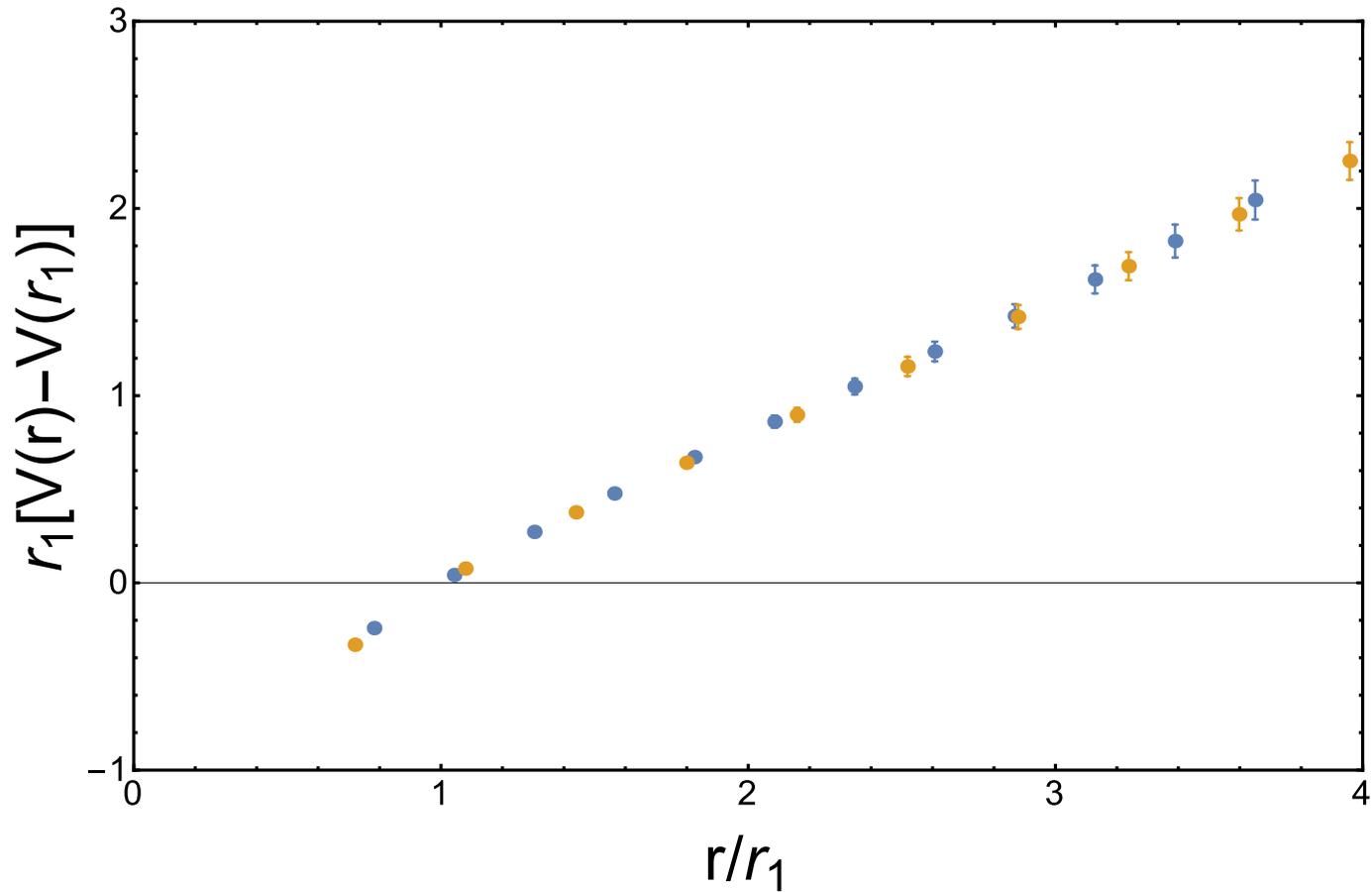
$\{X(r; a)\}_{a=a_1, a_2, a_3}$



Fit with  $c+d a^2$



# Lattice result



Definition of  $r_1$   $\longleftrightarrow r_1^2 \frac{dV(r)}{dr} \Big|_{r=r_1} = 1$

# Systematic errors

(a) Higher order corrections

$$V_S^{\text{N}^3\text{LL}}(r) \rightarrow V_S^{\text{N}^3\text{LL}}(r) \pm \delta V_S(r)$$

$$\text{with } \delta V_S(r) = V_S^{\text{N}^3\text{LL}}(r) - V_S^{\text{N}^2\text{LL}}(r)$$

(b) Finite  $a$  effect

$$\text{Previous analysis: } 2a < r < L/2 \longrightarrow a < r < L/2$$

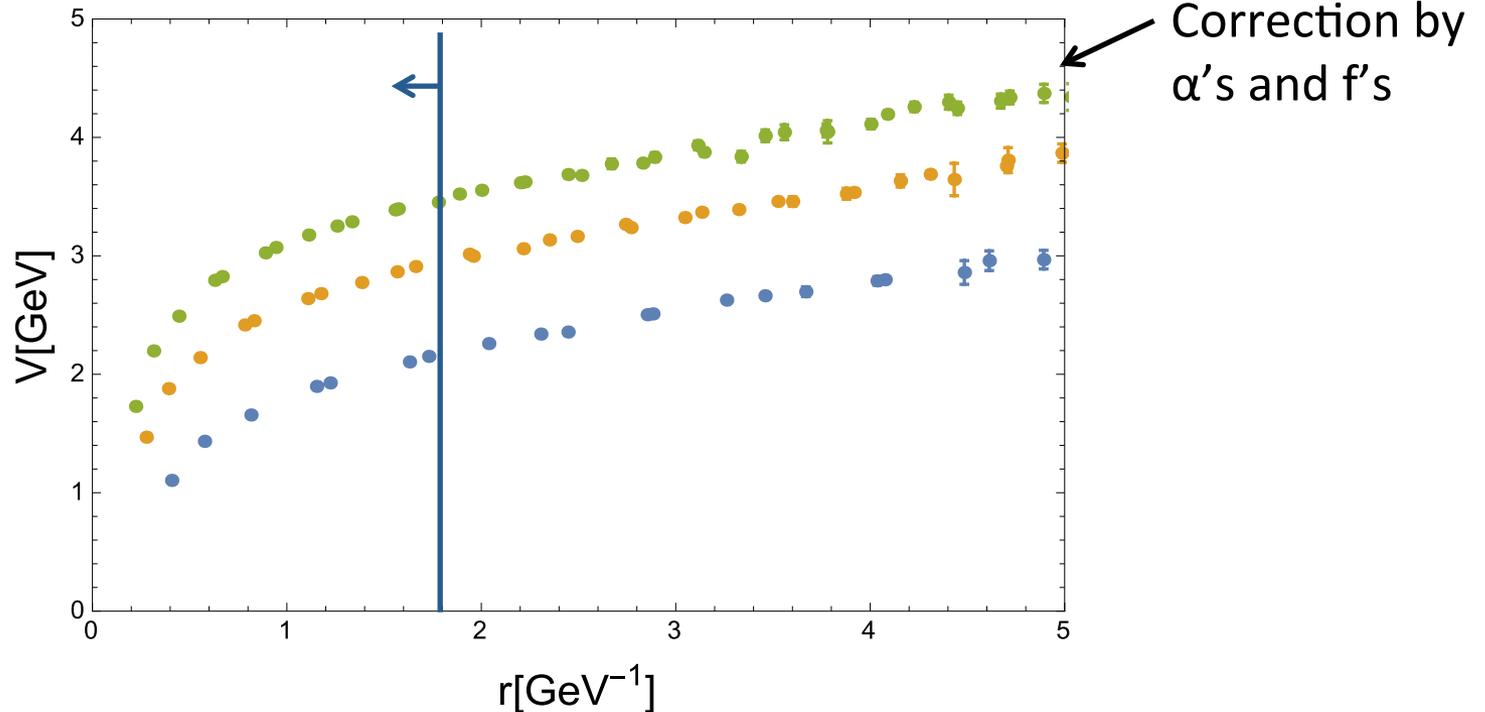
(c) Matching range with OPE

$$\Lambda_{\overline{MS}}^{\text{PDG}} r < 0.8 \longrightarrow \Lambda_{\overline{MS}}^{\text{PDG}} r < 1$$

(d) IR divergence at 3-loop

Way to regularize 3-loop perturbative coefficient

# Global fit



Parameters:  $\{\Lambda_{\overline{MS}}, A_2, A_0, \alpha_{D,i}, f_D\}$

$$\longrightarrow \Lambda_{\overline{MS}} = 334 \pm 10 \text{ MeV}$$

D1:  
 $\alpha_1 = 0.19 \pm 0.15, \alpha_2 = 0.27 \pm 0.12, \alpha_3 = 0.27 \pm 0.11$

D2:  
 $\alpha_1 = -0.31 \pm 0.86, \alpha_2 = -0.59 \pm 0.90, \alpha_3 = -0.62 \pm 0.93$

# Analysis strategy

## 1. Step-by-step analysis

- (i) extract the continuum limit of lattice data
- (ii) compare it with OPE to determine  $\alpha_s$

### Check if

Lattice data can be smoothly extrapolated to  $a \rightarrow 0$ .

$V_s^{\text{RF}}$  can explain lattice result up to  $O(r^2)$  nonperturbative correction.

## 2. Global fit

Perform (i) and (ii) simultaneously in global fit

We can avoid model dependent analysis, required in 1<sup>st</sup> analysis.

$\alpha_s$  can be determined with higher precision.