Study of Higgs→invisible at ILC using kinematic fit applied jet energy resolution of ILD

Yu Kato
The Univ. of Tokyo

7th Linear Collider School @ Frauenchiemsee 8, May, 2018

katou@icepp.s.u-tokyo.ac.jp

Outline

Motivation

- BSM search using Higgs→invisible at ILC
- Kinematic fit

Evaluate jet energy resolution

ILD model: large/small

- check jet energy & cos θ dependence

also evaluate jet angle resolution → apply to kinematic fit

Kinematic fit

fit variables : $E_{j1}, \theta_{j1}, \phi_{j1}, E_{j2}, \theta_{j2}, \phi_{j2}$

constraint : $m_{jj} = m_Z = 91.2\,\mathrm{GeV}$

use MarlinKinfit - fitter engine : OPALFitter

apply jet energy/angle resolution

- check effect & accuracy of fit

Improve analysis performance

Outline

Motivation

- BSM search using Higgs→invisible at ILC
- Kinematic fit

Evaluate jet energy resolution

ILD model: large/small

- check jet energy & cos θ dependence

also evaluate jet angle resolution → apply to kinematic fit

Kinematic fit

fit variables : $E_{j1}, \theta_{j1}, \phi_{j1}, E_{j2}, \theta_{j2}, \phi_{j2}$

constraint : $m_{jj} = m_Z = 91.2\,\mathrm{GeV}$

use MarlinKinfit - fitter engine : OPALFitter

apply jet energy/angle resolution

- check effect & accuracy of fit

Improve analysis performance

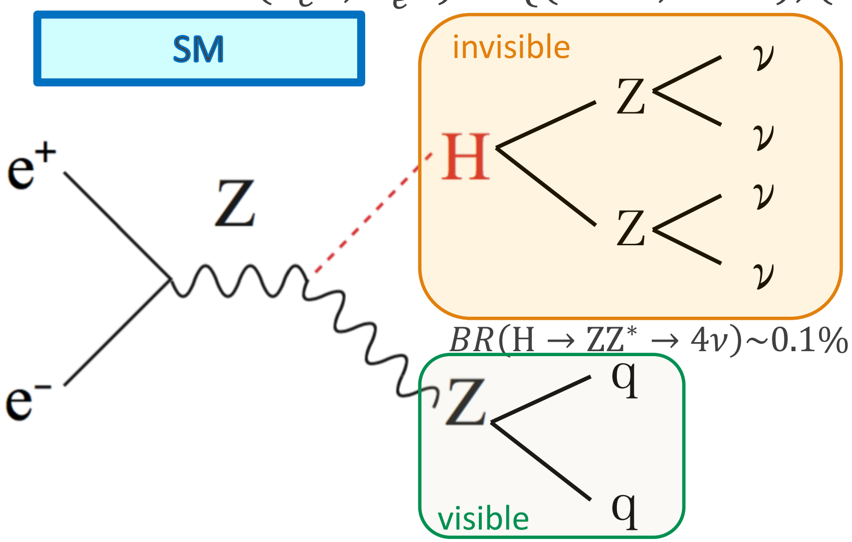
Motivation

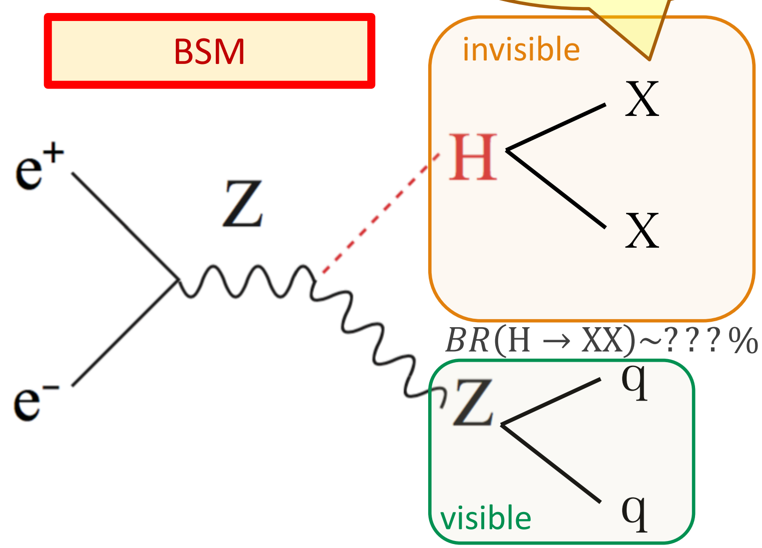
- In SM, Higgs decays invisibly through
  $H \to ZZ^* \to 4\nu$ $(BR(H \to inv.) \sim 0.1\%)$
- If $BR(H \to inv.)$ exceeds SM prediction ,
  it signifies new physics beyond SM (BSM)
- We estimate upper limit of $BR(H \to inv.)$ in SM
- Compare results in ILC between

$$(P_{e^-}, P_{e^+}) = \{(-0.8, +0.3), (+0.8, -0.3)\}$$

Previous study(A. Ishikawa)
(95% CL, 250fb^{-1})
left pol. : right pol.
0.95% : 0.69%

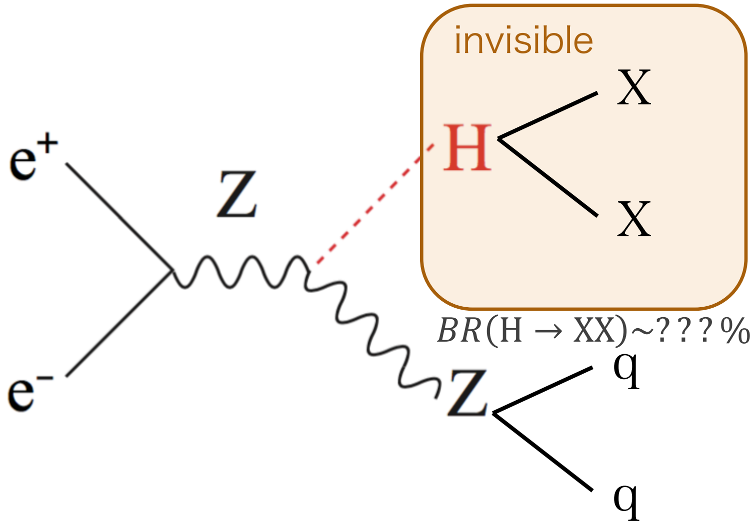
Dark Matter…
SUSY…





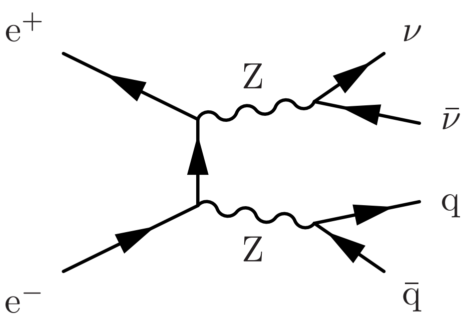
➢ A. Ishikawa (Tohoku Univ.),
"Search for Invisible Higgs Decays at the ILC" LCWS2014@Belgrade

Signal

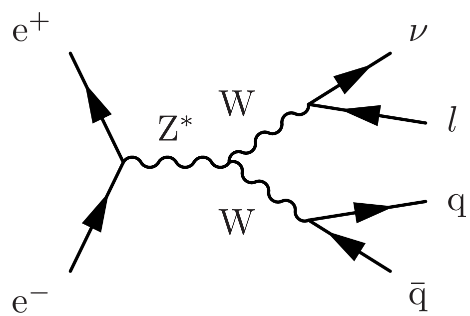


- ✓ 2 jet & missing E
- ✓ $M_{qq} \approx M_Z$: $BR(Z \rightarrow qq)\sim 70\%$
- ✓ $M_{recoil} \approx M_{Higgs}$
- ✓ s channel process

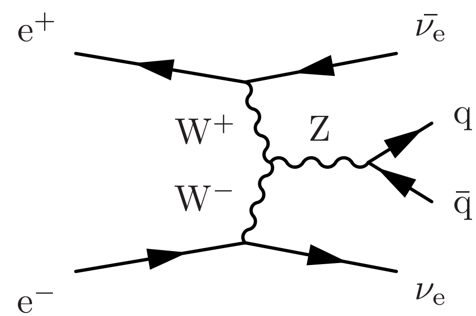
Main background



ZZ semi-leptonic



WW semi-leptonic



ν ν Z semi-leptonic

Analysis Setup

- Simulation
  - ILCSoft: v01-19-05
  - Samples: DBD sample + Signal sample ($e^+e^- \to qqH, H \to ZZ^* \to 4\nu$)
  - Detector: ILD full simulation (ILD_o1_v05)
  - $\sqrt{s} = 250$ GeV, $\int Ldt = 250$ fb^{-1} , $(P_{e^-}, P_{e^+}) = (-0.8, +0.3), (+0.8, -0.3)$ "Left" "Right"

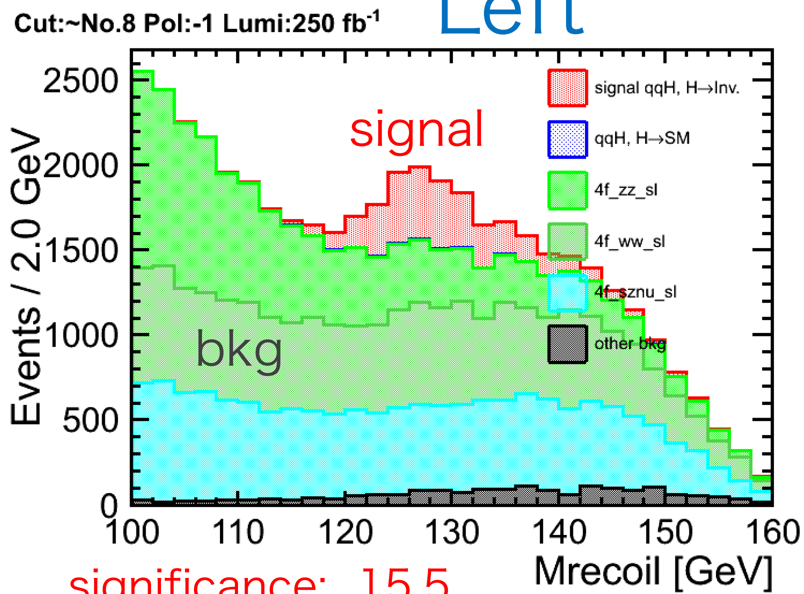
- Flow of analysis
  1. Particle flow reconstruction (PandoraPFA)
  2. Isolated lepton finder (veto)
  3. Durham jet finder (forced 2 jets)
  4. Kinematic fit with MarlinKinfit (OPALFitter)
  5. Event selection
     - Optimized assuming signal BR(H→invisible) = 10%
  6. Estimate upper limit of BR (95% CL)

1st. result

$$significance = \frac{N_s}{\sqrt{N_s + N_b}}$$

Recoil mass dist. [Ecm = 250 GeV, 250 fb^{-1},BR(H->inv.)=10%]

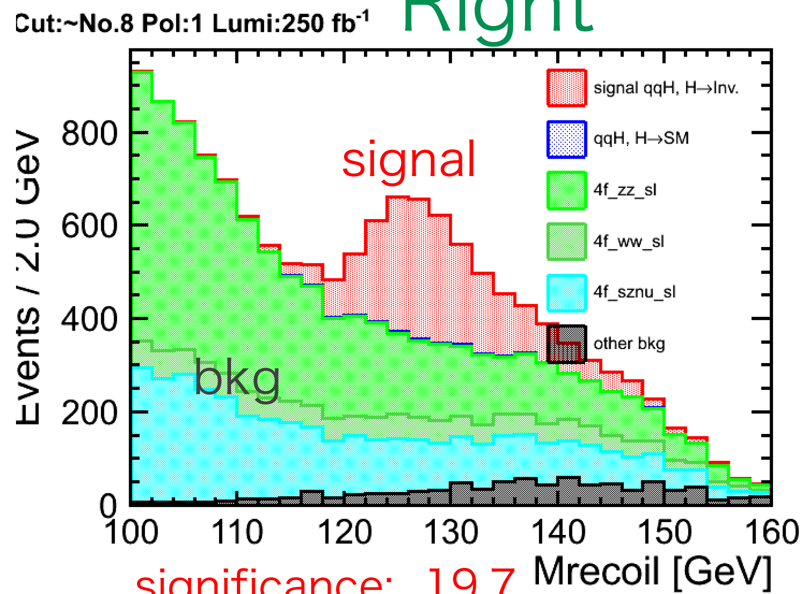
Left



significance: 15.5

UL = 0.73%

Right



significance: 19.7

UL = 0.52%

MVA input variables	
di-jet inv. mass	one jet polar angle
di-jet polar angle	another jet polar angle

TMVA v-4.2.0

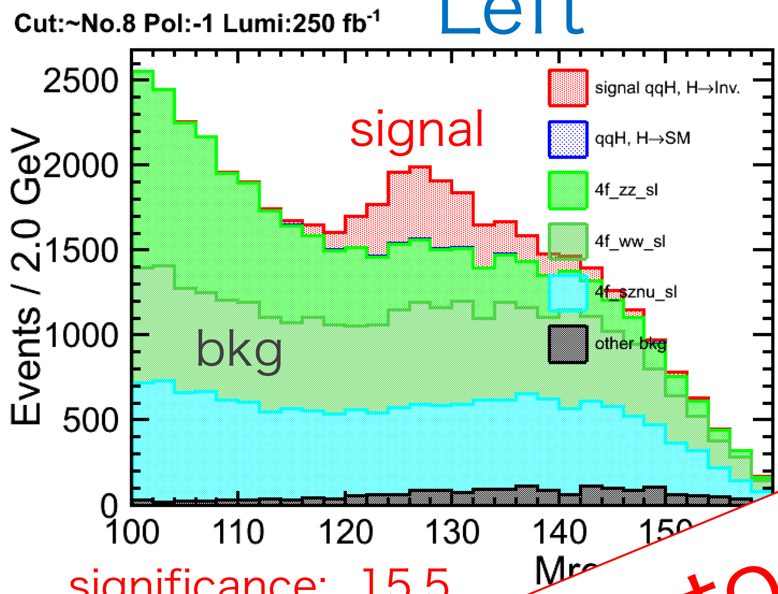
No.	Cut	No.	Cut
1	Isolated lepton veto	5	80 < di-jet invariant mass < 100
2	Loose Cut (Ptz,Mz,Mrecoil)	6	| di-jet polar angle |< 0.9
3	#pfo >15 & #all_track > 6 & # track_in_one_jet > 1	7	100 < recoil mass < 160
4	20 GeV < di-jet Pt < 80 GeV	8	BDT cut

1st. result

$$significance = \frac{N_s}{\sqrt{N_s + N_b}}$$

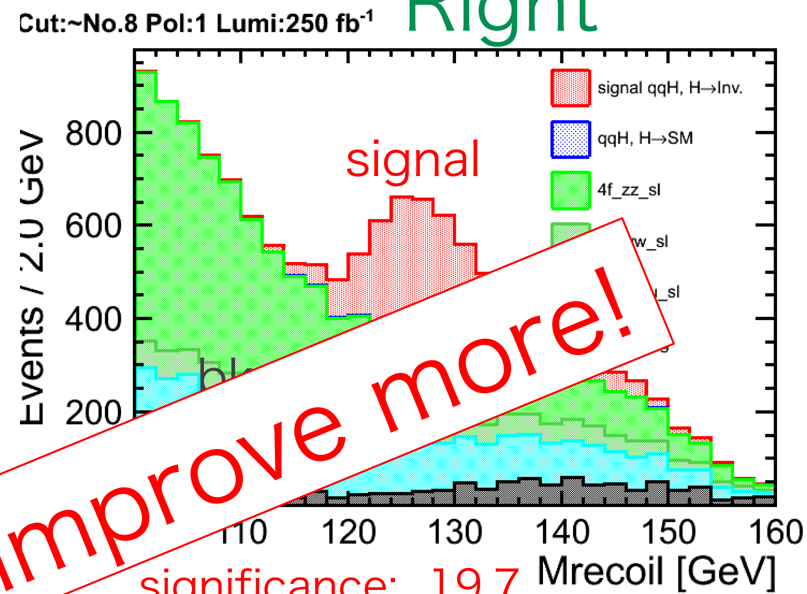
Recoil mass dist. [Ecm = 250 GeV, 250 fb^{-1},BR(H->inv.)=10%]

Left



significance: 15.5
UL = 0.73%

Right



significance: 19.7
UL = 0.52%

We want to improve more!

MVA inp variable	
di-jet inv. mass	on jet polar angle
di-jet polar angle	another jet polar angle

TMVA v-4.2.0

	Cut	No.	Cut
1	Isolated lepton veto	5	80 < di-jet invariant mass < 100
2	Loose Cut (Ptz,Mz,Mrecoil)	6	| di-jet polar angle |< 0.9
3	#pfo >15 & #all_track > 6 & # track_in_one_jet > 1	7	100 < recoil mass < 160
4	20 GeV < di-jet Pt < 80 GeV	8	BDT cut

Principle of kinematic fit

seek minimum of $\chi^2_T(\vec{\eta},\vec{\xi},\vec{\lambda})$
under kinematic constraints

$$\chi^2_T(\vec{\eta},\vec{\xi},\vec{\lambda}) = \chi^2(\vec{\eta}) + F_C(\vec{\eta},\vec{\xi},\vec{\lambda})$$

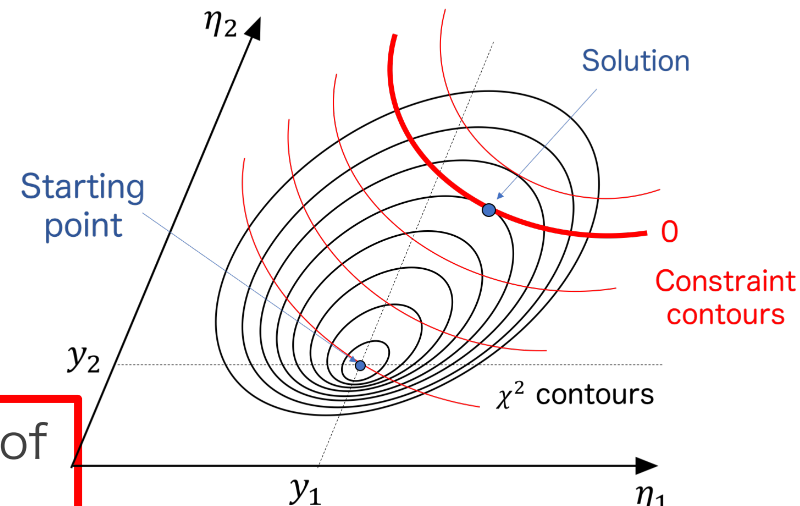
$$\chi^2(\vec{\eta}) = (\vec{\eta}-\vec{y})^T V^{-1}(\vec{\eta}-\vec{y})$$

$$F_C(\vec{\eta},\vec{\xi},\vec{\lambda}) = 2\vec{\lambda}^T \cdot \vec{f}(\vec{\eta},\vec{\xi})$$

- $\vec{y}$: measured values (N)
- $\vec{\eta}$: fit parameters (N)
- $\vec{\xi}$: unmeasured parameters (J)
- $\vec{\lambda}$: Lagrange multipliers (K)
- $\vec{f}$: constraint functions (K)
- V : covariance matrix (N × N)

method of Lagrange multipliers

$$\begin{cases} \frac{1}{2}\frac{\partial \chi^2_T}{\partial \eta_i} = V^{-1}_{ij}(\eta_j - y_j) + \frac{\partial f_k}{\partial \eta_i}\lambda_k = 0 & (i = 1, ..., N) \\ \frac{1}{2}\frac{\partial \chi^2_T}{\partial \xi_i} = \frac{\partial f_k}{\partial \xi_i}\lambda_k = 0 & (i = 1, ..., J) \\ \frac{1}{2}\frac{\partial \chi^2_T}{\partial \lambda_i} = f_i = 0 & (i = 1, ..., K) \end{cases}$$



In this study, we need error information of
$E_{j1}, \theta_{j1}, \phi_{j1}, E_{j2}, \theta_{j2}, \phi_{j2}$

Outline

Motivation

- BSM search using Higgs→invisible at ILC
- Kinematic fit

Evaluate jet energy resolution

ILD model: large/small

- check jet energy & cos θ dependence

also evaluate jet angle resolution → apply to kinematic fit

Kinematic fit

fit variables : $E_{j1}, \theta_{j1}, \phi_{j1}, E_{j2}, \theta_{j2}, \phi_{j2}$

constraint : $m_{jj} = m_Z = 91.2\ \mathrm{GeV}$

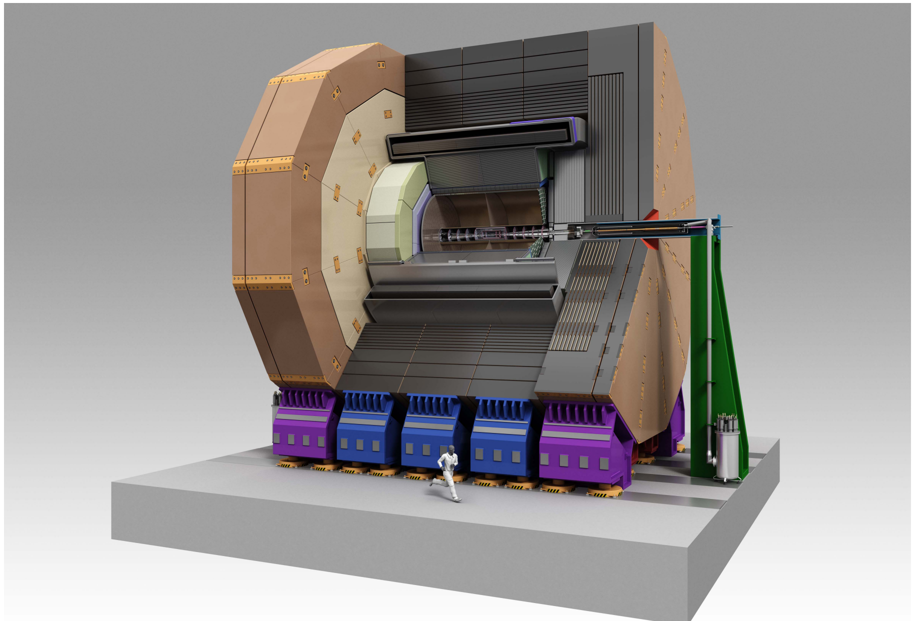
use MarlinKinfit - fitter engine : OPALFitter

apply jet energy/angle resolution

- check effect & accuracy of fit

Improve analysis performance

International Large Detector



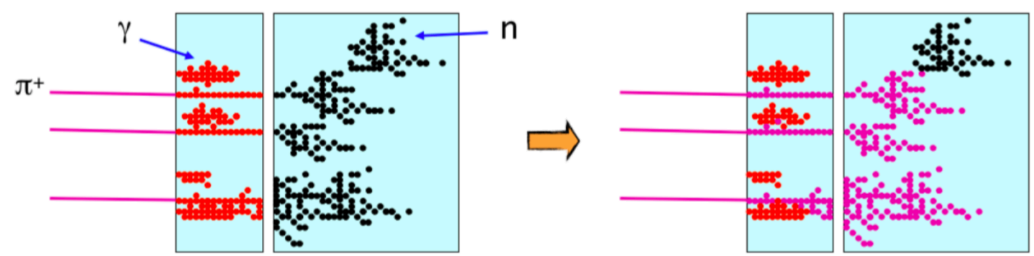
- Concept of detector system at ILC
  - high-performance vertex detector
  - high-resolution trackers
  - high-granularity calorimeters etc.
- Main request
  - Reconstruct all the particles, especially hadron jets
- To achive high energy resolution,
  - Particle Flow Algorithm

Particle Flow Algorithm

The energy(momentum) for each particle is extracted from the subdetector system in which we expect the measurement to be most accurate.

Charged particles → Tracker
Photon → E Cal.
Neutral particles → H Cal.

Previous → PFA



Previous: $E_{JET} = E_{ECAL} + E_{HCAL}$

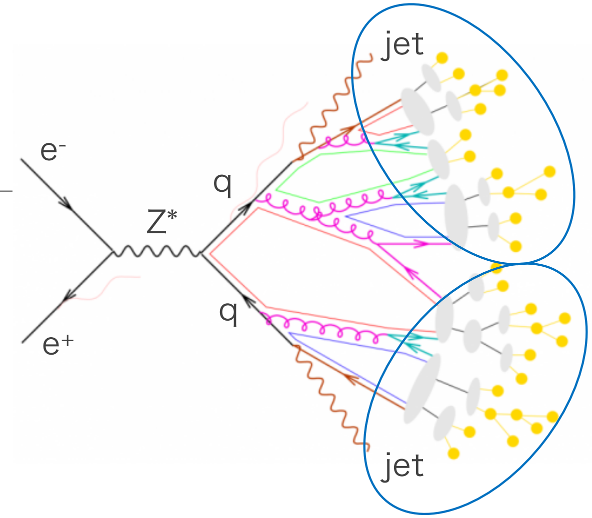
$\gtrsim 55\%/\sqrt{E(\mathrm{GeV})}$

PFA: $E_{JET} = E_{TRACK} + E_{\gamma} + E_{n}$

$\lesssim 30\%/\sqrt{E(\mathrm{GeV})}$

Setting of Evaluation JER

- ILCSoft : v01-19-05 (gcc49)
- ILDConfig : v01-19-05-p01
- ILD models : ILD_l5_o1_v02, (ILD_s5_o1_v02)
- Samples : Z→uds (w/o overlay)



√s [GeV]	30	40	60	91	120	160	200	240	300	350	400	500
l5 [events]	10k	10k	10k	10k	10k	10k	10k	10k	9k	10k	9k	10k
s5 [events]	10k	10k	10k	10k	9k	10k	10k	9k	10k	10k	10k	10k

- Jet resolution definition
  - use RMS_{90} method
  - Energy

$$\frac{\sigma_E}{E} = \frac{\mathrm{RMS}_{90}(E_j)}{mean_{90}(E_j)} = \sqrt{2}\frac{\mathrm{RMS}_{90}(E_{jj})}{mean_{90}(E_{jj})}$$

(J. S. Marshall and M. A. Thomson, "Pandora Particle Flow Algorithm", arXiv:1308.4537 [physics.ins-det])

  - Angle

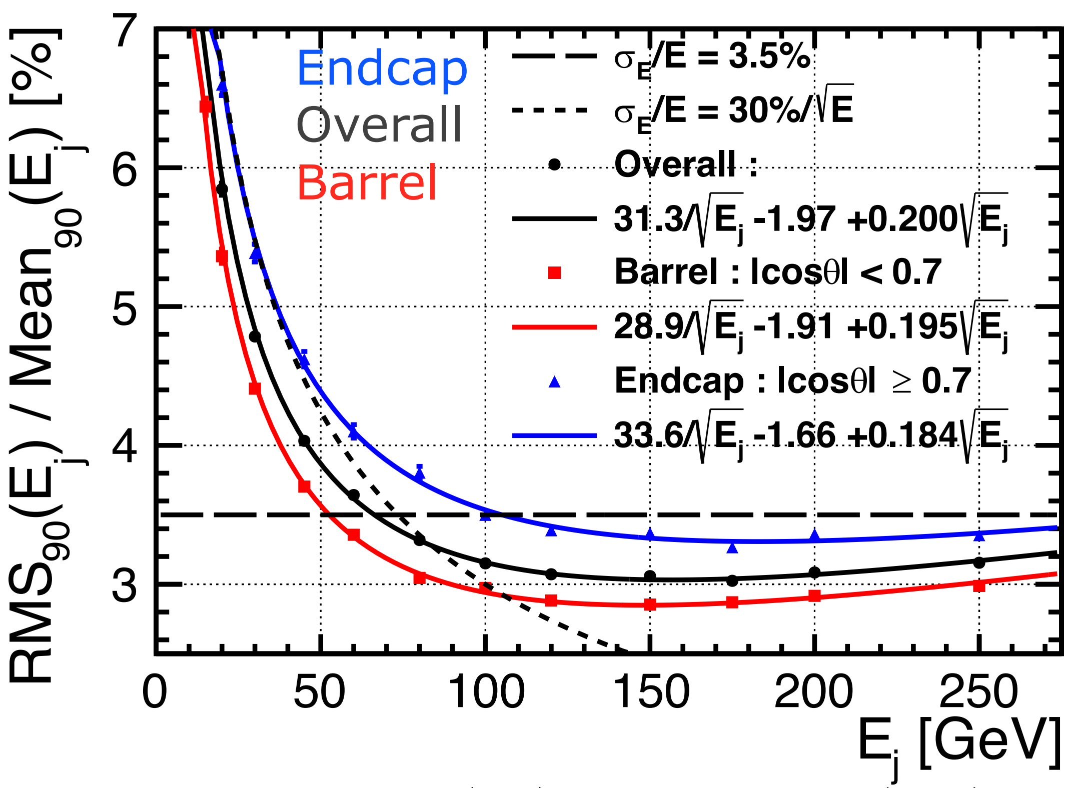
use jet clustering: Durham

$$\delta\phi = \mathrm{RMS}_{90}(\phi_{rec} - \phi_{mc})$$
$$\delta\theta = \mathrm{RMS}_{90}(\theta_{rec} - \theta_{mc})$$

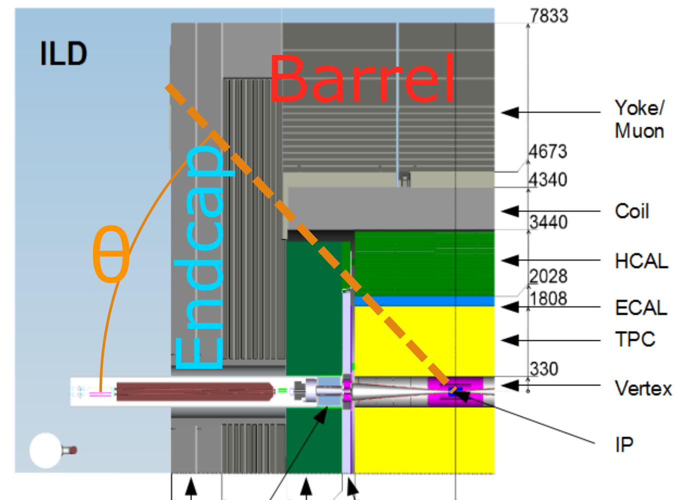
JER: Comparison Barrel/Endcap

JER was evaluated separately for barrel and endcap regions.

sv01-19-05.mILD_l5_o1_v02_nobg



$$\frac{\sigma_{E_j}}{E_j} = \frac{\mathrm{RMS}_{90}(E_j)}{\mathrm{mean}_{90}(E_j)} = \sqrt{2}\frac{\mathrm{RMS}_{90}(E_{jj})}{\mathrm{mean}_{90}(E_{jj})}$$



$$\theta = \cos^{-1}(0.7)$$

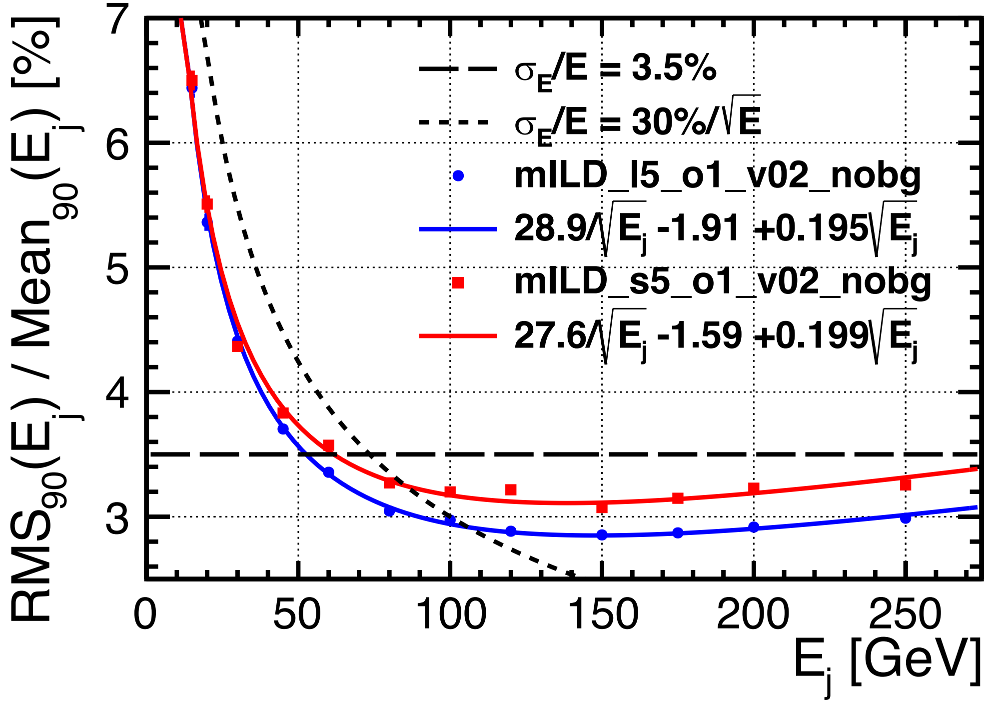
$$\frac{\sigma_E}{E} = \frac{\alpha}{\sqrt{E}} \oplus \beta(E)$$

	α	β
Overall	31.3	-1.97+0.200√E
Barrel	28.9	-1.91+0.195√E
Endcap	33.6	-1.66+0.184√E

JER: Comparison Large/Small

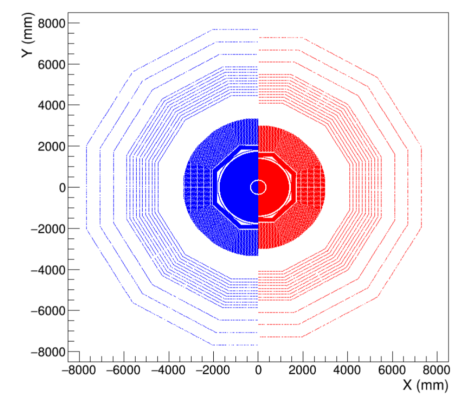
The two detector models (large/small) were evaluated for comparison.

sv01-19-05 |cosθ|<0.7



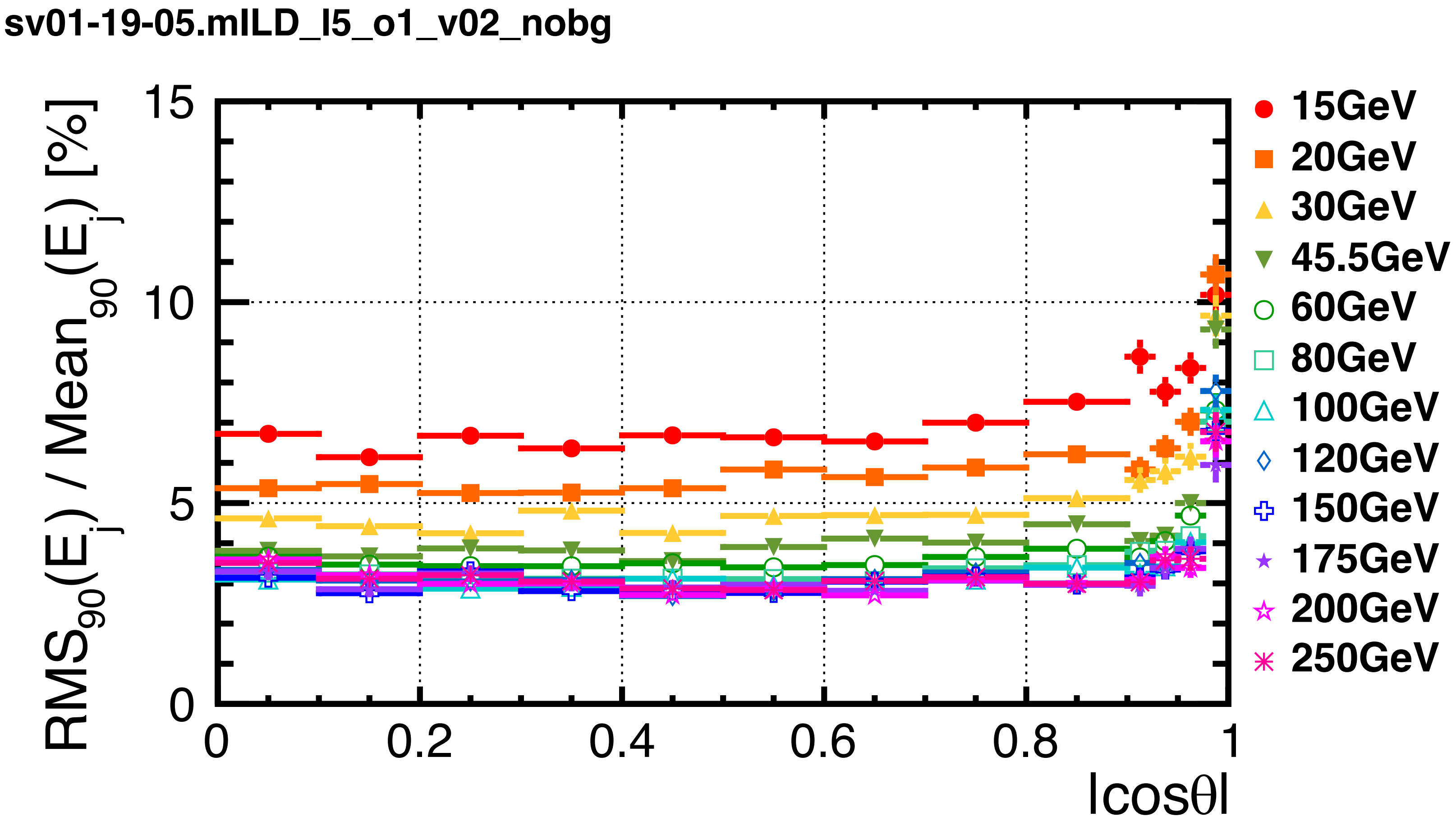
$$\frac{\sigma_E}{E} = \frac{\alpha}{\sqrt{E}} \oplus \beta(E)$$

	α	β
Small	28.9	-1.91+0.195√E
Large	27.6	-1.59+0.199√E



- Impact of small detector seen for large jet energy
- JER goal (ILC TDR) satisfied for both models

Result : energy & angle dependence



apply this result to kinematic fit

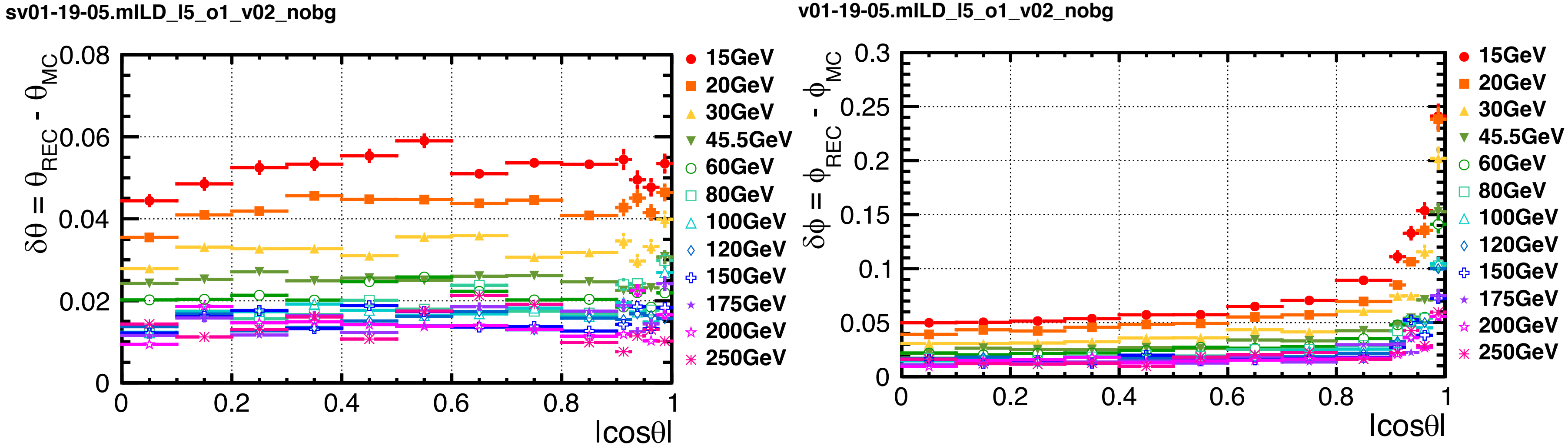
Angular resolution

polar angle

$$\delta\theta = RMS_{90}(\theta_{rec} - \theta_{mc})$$

azimuth angle

$$\delta\phi = RMS_{90}(\phi_{rec} - \phi_{mc})$$



For evaluation of angular resolution, use jet clustering.

Durham algorithm

$$y_{ij} = \frac{2\min(E_i^2, E_j^2)(1 - \cos\theta_{ij})}{Q^2}$$

apply this result to kinematic fit

Outline

Motivation

- BSM search using Higgs→invisible at ILC
- Kinematic fit

Evaluate jet energy resolution

ILD model: large/small

- check jet energy & $\cos\theta$ dependence

also evaluate jet angle resolution → apply to kinematic fit

Kinematic fit

fit variables : $E_{j1}, \theta_{j1}, \phi_{j1}, E_{j2}, \theta_{j2}, \phi_{j2}$

constraint : $m_{jj} = m_Z = 91.2\,\mathrm{GeV}$

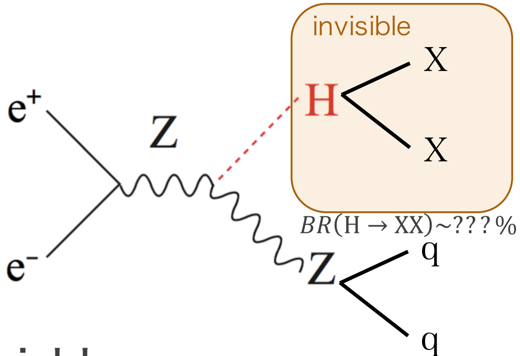
use MarlinKinfit - fitter engine : OPALFitter

apply jet energy/angle resolution

- check effect & accuracy of fit

Improve analysis performance

ZH processor



- Fit variables

$$E_{j1}, \theta_{j1}, \phi_{j1}, E_{j2}, \theta_{j2}, \phi_{j2}$$

- Z mass constraint

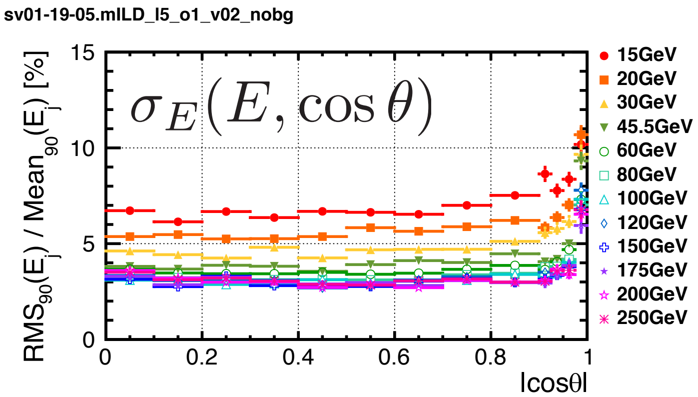
$$m_{jj} = m_Z = 91.2 \text{ GeV}$$

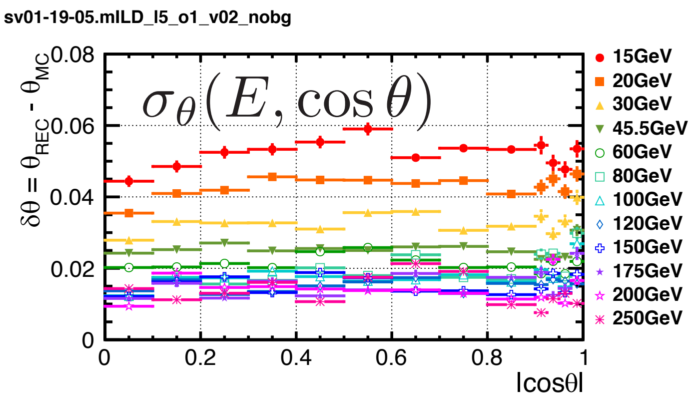
- Jet mass asumption

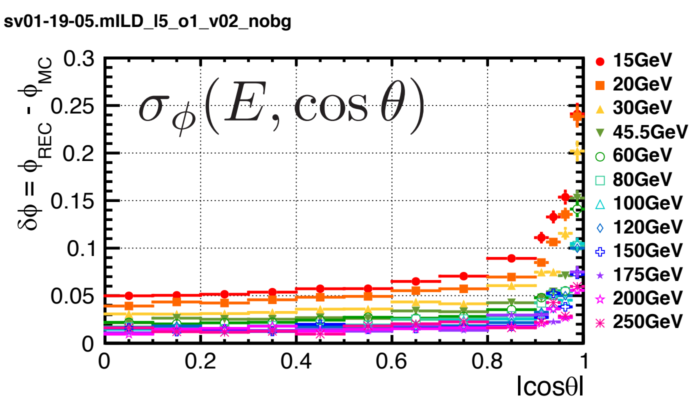
$$m_j^{before} = m_j^{after}$$

- Implement of jet resolution

$$\sigma_E(E, \cos\theta), \sigma_\theta(E, \cos\theta), \sigma_\phi(E, \cos\theta)$$





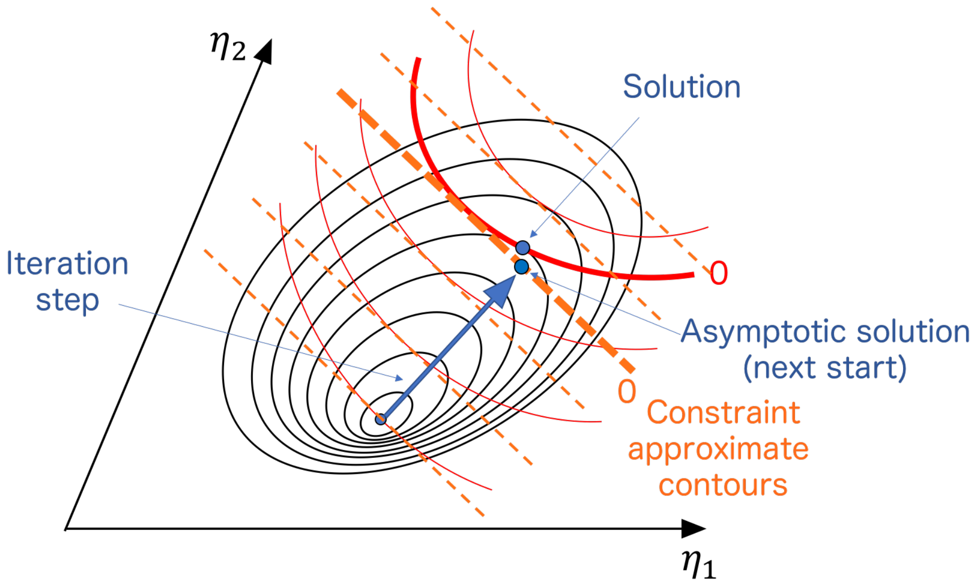


MarlinKinfit : OPALFitter

For iterative solution : Taylor-expansion of the constraints

$$f_i(\vec{\eta}^{\nu+1}, \vec{\xi}^{\nu+1}) = f_i(\vec{\eta}^{\nu}, \vec{\xi}^{\nu}) + \left.\frac{\partial f_i}{\partial \eta_j}\right|_{\eta_j=\eta_j^\nu} (\eta_j^{\nu+1} - \eta_j^\nu) + \left.\frac{\partial f_i}{\partial \xi_k}\right|_{\xi_k=\xi_k^\nu} (\xi_k^{\nu+1} - \xi_k^\nu)$$

$$\begin{cases} V_{ij}^{-1}(\eta_j^{\nu+1} - y_j) + \left.\dfrac{\partial f_k}{\partial \eta_i}\right|_{\eta_i=\eta_i^\nu} \lambda_k^{\nu+1} = 0 & (i = 1, ..., N) \\ \left.\dfrac{\partial f_k}{\partial \xi_i}\right|_{\xi_i=\xi_i^\nu} \lambda_k^{\nu+1} = 0 & (i = 1, ..., J) \\ f_i(\vec{\eta}^{\nu+1}, \vec{\xi}^{\nu+1}) = 0 & (i = 1, ..., K) \end{cases}$$



Convergence condition

- ✓ $\delta\chi^2 < 0.01\% \cap \delta F_C < 10^{-3} \cap F_C < 10^{-2} \cdot \chi^2$

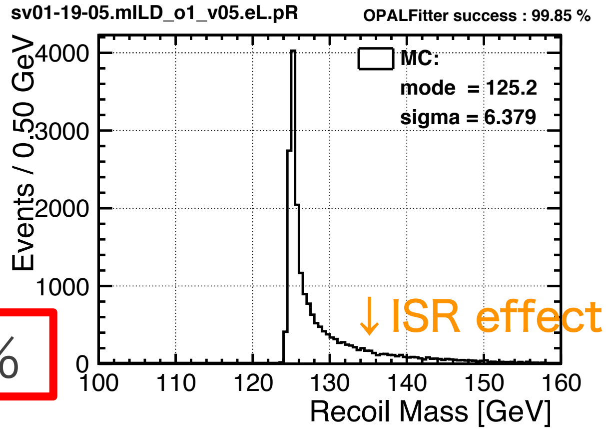
or

- ✓ all $f_i < 10^{-6} \cap \delta(\eta, \xi, \lambda) < 10^{-6}$

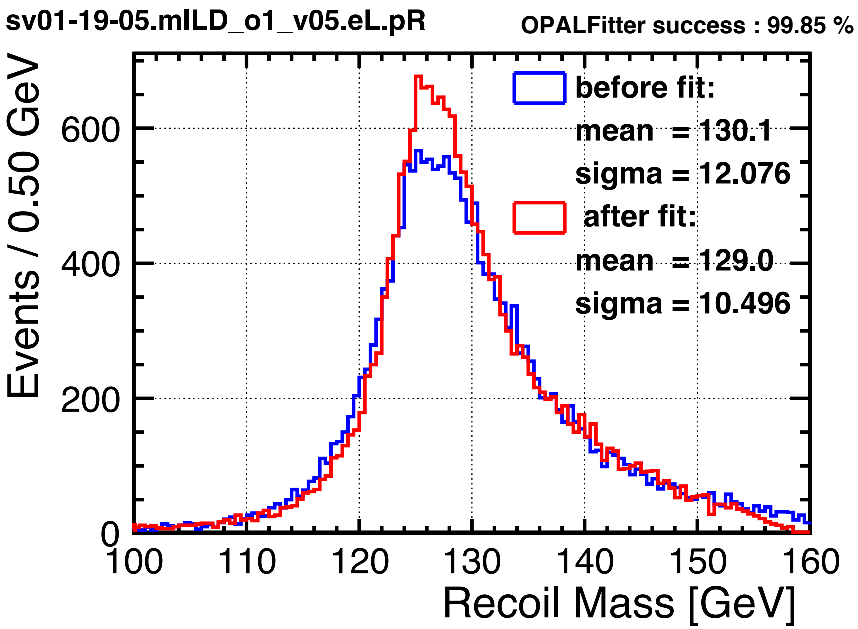
Result : Recoil mass

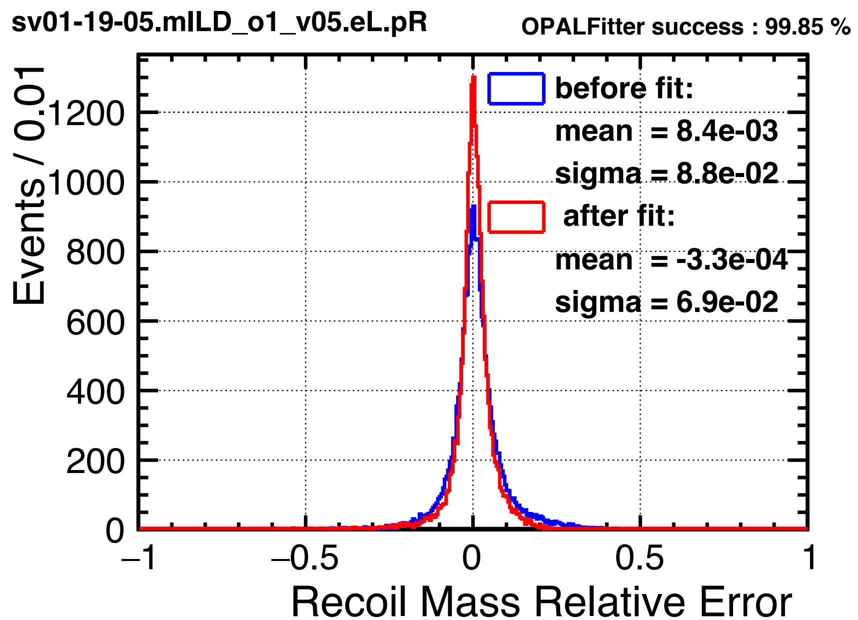
$$M_{rec} = \sqrt{(\sqrt{s} - E_Z)^2 - |\vec{p}_Z|^2}$$

$$\delta M_{rec} = \frac{M_{rec}^{result} - M_{rec}^{mc}}{M_{rec}^{mc}}$$



improve recoil mass resolution ~20%





Outline

Motivation

- BSM search using Higgs→invisible at ILC
- Kinematic fit

Evaluate jet energy resolution

ILD model: large/small

- check jet energy & cos θ dependence

also evaluate jet angle resolution → apply to kinematic fit

Kinematic fit

fit variables : $E_{j1}, \theta_{j1}, \phi_{j1}, E_{j2}, \theta_{j2}, \phi_{j2}$

constraint : $m_{jj} = m_Z = 91.2\ \mathrm{GeV}$

use MarlinKinfit - fitter engine : OPALFitter

apply jet energy/angle resolution

- check effect & accuracy of fit

Improve analysis performance

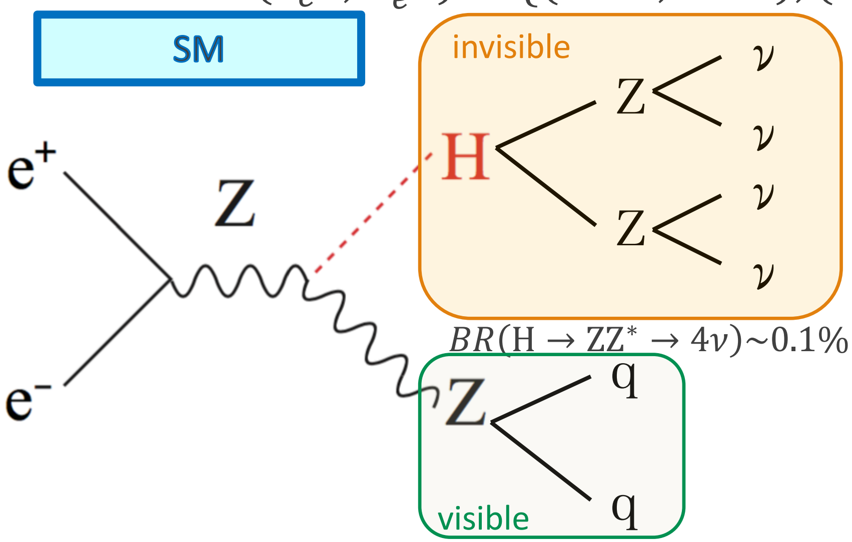
Motivation

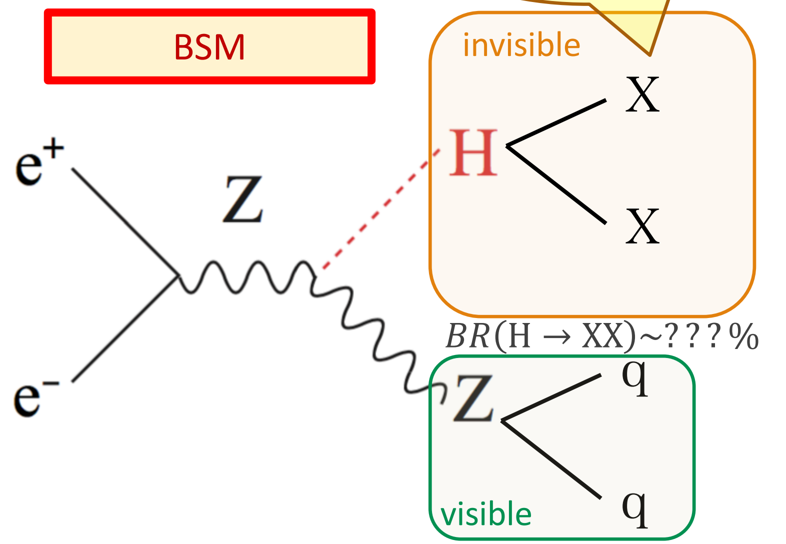
- In SM, Higgs decays invisibly through
  $H \to ZZ^* \to 4\nu$ ($BR(H \to inv.) \sim 0.1\%$)
- If $BR(H \to inv.)$ exceeds SM prediction ,
  it signifies new physics beyond SM (BSM)
- We estimate upper limit of $BR(H \to inv.)$ in SM
- Compare results in ILC between

$$(P_{e^-}, P_{e^+}) = \{(-0.8, +0.3), (+0.8, -0.3)\}$$

Previous study(A. Ishikawa)
(95% CL, 250fb^{-1})
left pol. : right pol.
0.95% : 0.69%

Dark Matter…
SUSY…





➢ A. Ishikawa (Tohoku Univ.),
"Search for Invisible Higgs Decays at the ILC" LCWS2014@Belgrade

Analysis Setup

- Simulation
  - ILCSoft: v01-19-05
  - Samples: DBD sample + Signal sample ($e^+e^- \to qqH, H \to ZZ^* \to 4\nu$)
  - Detector: ILD full simulation (ILD_o1_v05)
  - $\sqrt{s} = 250$ GeV, $\int Ldt = 250$ fb^{-1} , $(P_{e^-}, P_{e^+}) = (-0.8, +0.3), (+0.8, -0.3)$ "Left" "Right"

- Flow of analysis
  1. Particle flow reconstruction (PandoraPFA)
  2. Isolated lepton finder (veto)
  3. Durham jet finder (forced 2 jets)
  4. Kinematic fit with MarlinKinfit (OPALFitter)
  5. Event selection
     - Optimized assuming signal BR(H→invisible) = 10%
  6. Estimate upper limit of BR (95% CL)

Cut table $(P_{e^-}, P_{e^+}) = (-0.8, +0.3)$

$$significance = \frac{N_s}{\sqrt{N_s + N_b}}$$

cut condition	S/√S+B	signal	all bkg	ZZ	WW	$\nu\nu Z$	other bkg
No Cut	0.84	5255	3.93×10^7	214211	2748230	67952	3.63×10^7
$N_{lep} = 0$	1.00	5249	2.74×10^7	165399	1276030	67853	2.59×10^7
Pre-Cut	7.54	5026	439363	35027	69535	33852	300949
$N_{pfo} > 15 \& N_{charged} > 6$	9.66	4947	256873	34332	67457	33236	121848
$p_{Tjj} \in (20, 80)\mathrm{GeV}$	12.48	4688	136149	30207	56149	29166	20627
$M_{jj} \in (80, 100)\mathrm{GeV}$	13.50	3919	80266	23533	29210	23675	3848
$\|\cos\theta_{jj}\| < 0.9$	13.94	3768	69199	20457	24817	21246	2679

w/o kinematic fit

cut condition	S/√S+B	signal	all bkg	ZZ	WW	$\nu\nu Z$	other bkg
common part	13.94	3768	69199	20457	24817	21246	2679
$M_{recoil} \in (100, 160)\mathrm{GeV}$	13.95	3765	69002	20438	24748	21174	2642
$\mathrm{BDT} > -0.0718$	15.54	3388	44086	12604	14941	14676	1865

w/ kinematic fit

cut condition	S/√S+B	signal	all bkg	ZZ	WW	$\nu\nu Z$	other bkg
common part	13.94	3768	69199	20457	24817	21246	2679
$M_{recoil} \in (100, 160)\mathrm{GeV}$	15.10	3766	58404	15873	21289	18665	2577
$\mathrm{BDT} > -0.0867$	16.26	3425	40893	11086	14030	13903	1874

$$significance = \frac{N_s}{\sqrt{N_s + N_b}}$$

Cut table $(P_{e^-}, P_{e^+}) = (+0.8, -0.3)$

cut condition	S/√S+B	signal	all bkg	ZZ	WW	$\nu\nu Z$	other bkg
No Cut	0.76	3549	2.16×10^7	116792	189591	23127	2.13×10^7
$N_{lep} = 0$	0.95	3545	1.39×10^7	89111	88065	23092	1.37×10^7
Pre-Cut	7.33	3391	210605	16373	4918	8970	180344
$N_{pfo} > 15 \& N_{charged} > 6$	10.03	3331	106837	16028	4773	8786	77250
$p_{Tjj} \in (20, 80)\mathrm{GeV}$	15.59	3144	37369	14018	4022	7793	11536
$M_{jj} \in (80, 100)\mathrm{GeV}$	17.17	2632	20815	10828	2087	6121	1779
$\lvert\cos\theta_{jj}\rvert < 0.9$	17.81	2535	17670	9387	1806	5373	1104

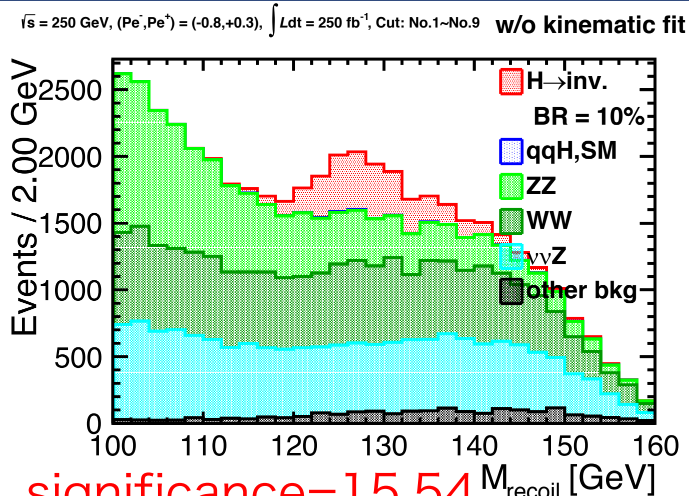
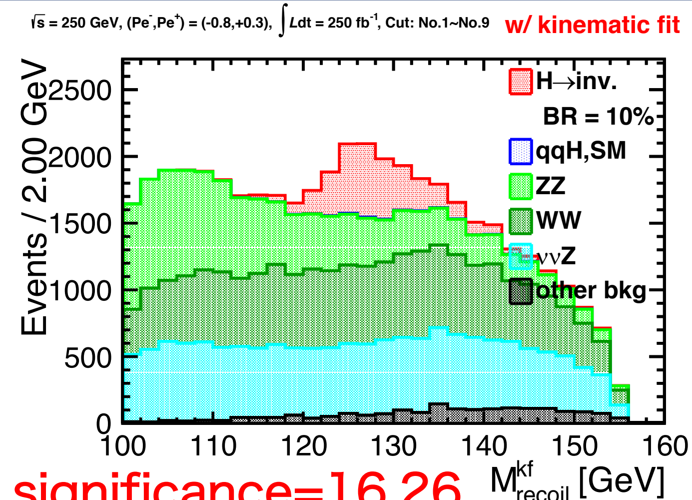
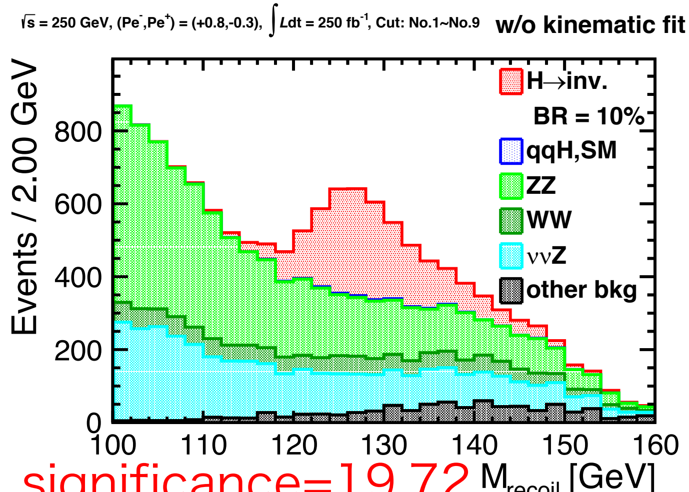
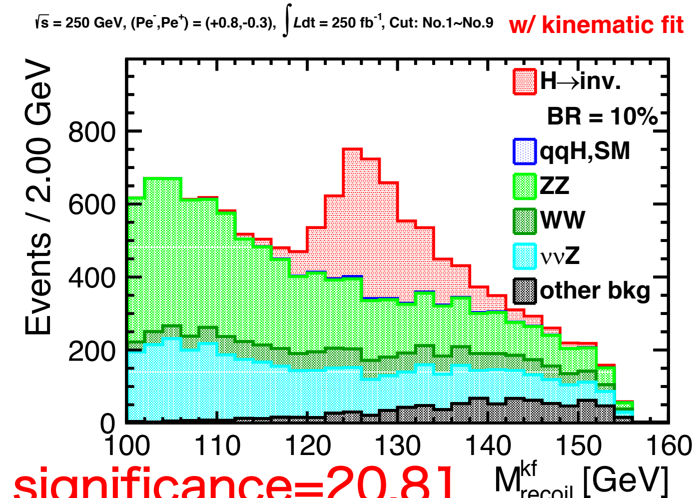
w/o kinematic fit

cut condition	S/√S+B	signal	all bkg	ZZ	WW	$\nu\nu Z$	other bkg
common part	17.81	2535	17670	9387	1806	5373	1104
$M_{recoil} \in (100, 160)\mathrm{GeV}$	17.83	2532	17598	9376	1800	5366	1056
$\mathrm{BDT} > -0.0840$	19.72	2298	11231	5800	1170	3463	798

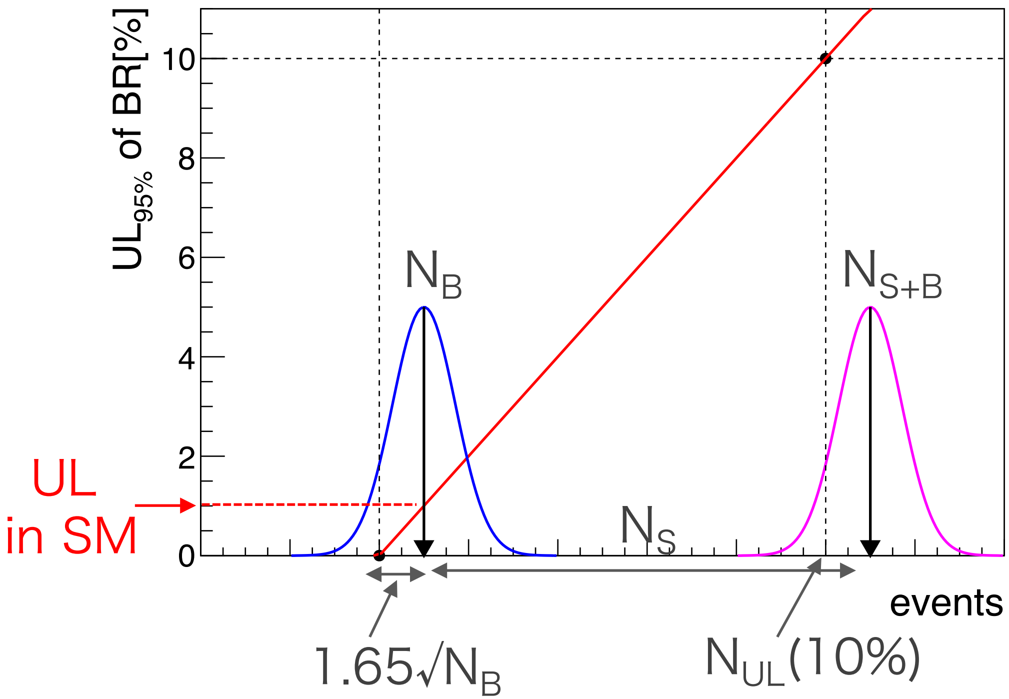
w/ kinematic fit

cut condition	S/√S+B	signal	all bkg	ZZ	WW	$\nu\nu Z$	other bkg
common part	17.81	2535	17670	9387	1806	5373	1104
$M_{recoil} \in (100, 160)\mathrm{GeV}$	19.61	2533	14104	7196	1549	4274	1085
$\mathrm{BDT} > -0.1162$	20.81	2395	10806	5431	1187	3318	870

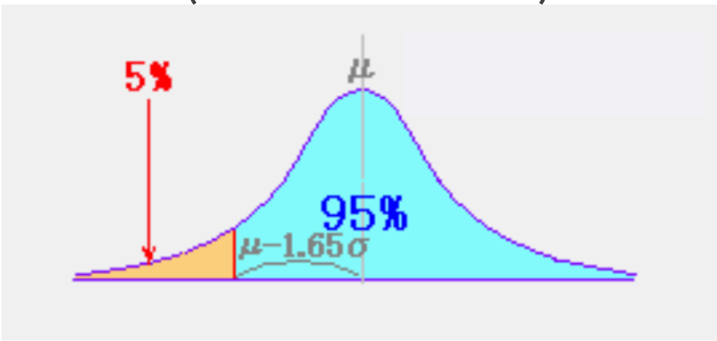
Result : Recoil mass distribution

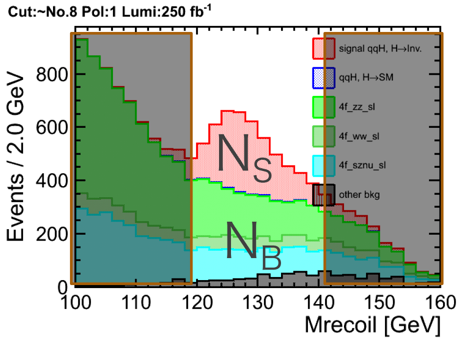
	w/o kinematic fit	w/ kinematic fit
Left polarization	 significance=15.54	 significance=16.26
Right polarization	 significance=19.72	 significance=20.81

How to set Upper Limit



definition of 95% C.L.
(one-sided test)





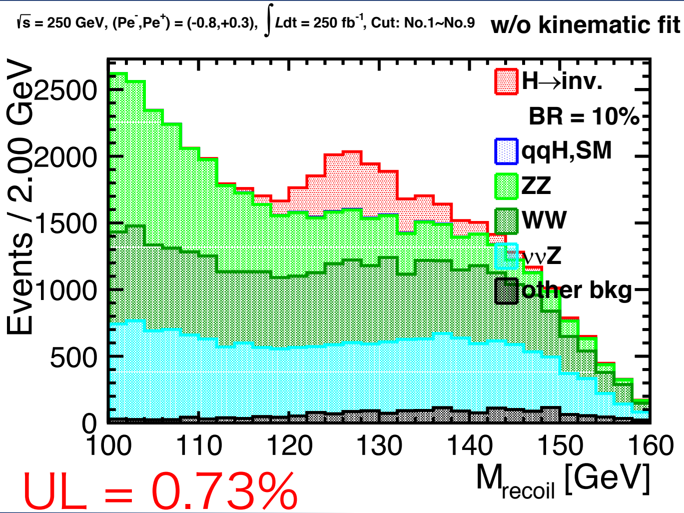
Mrecoil∈[120,140] GeV

$$UL(\%) \equiv \frac{10(\%)}{N_S(10\%)} \times 1.65\sqrt{N_B}$$

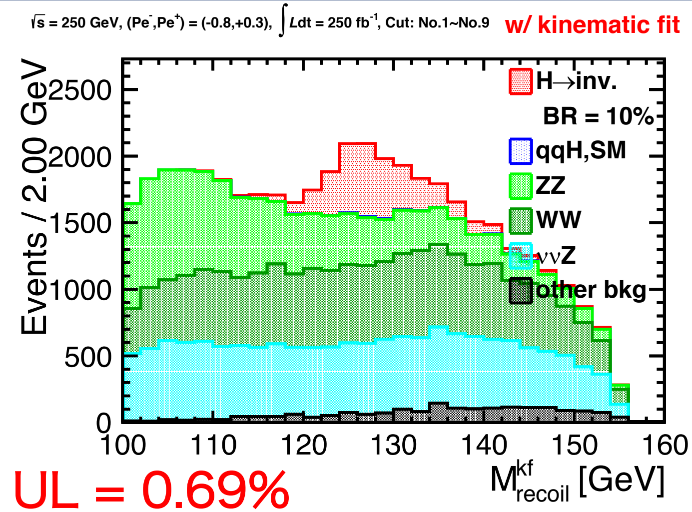
Result : Upper limit of BR (95% CL)

w/o kinematic fit | w/ kinematic fit

Left polarization

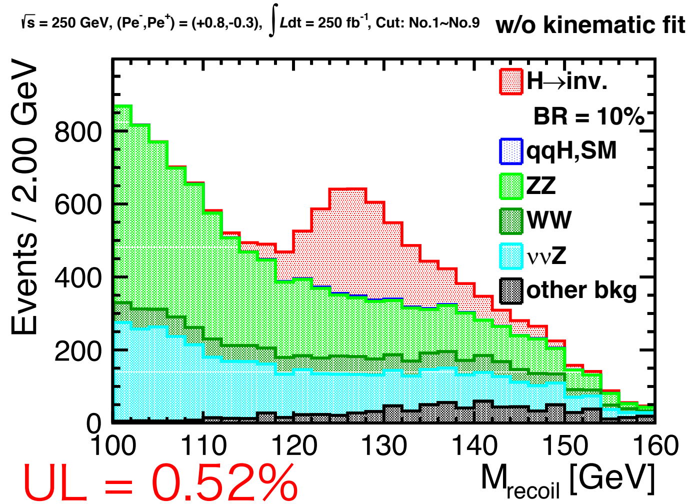


UL = 0.73%

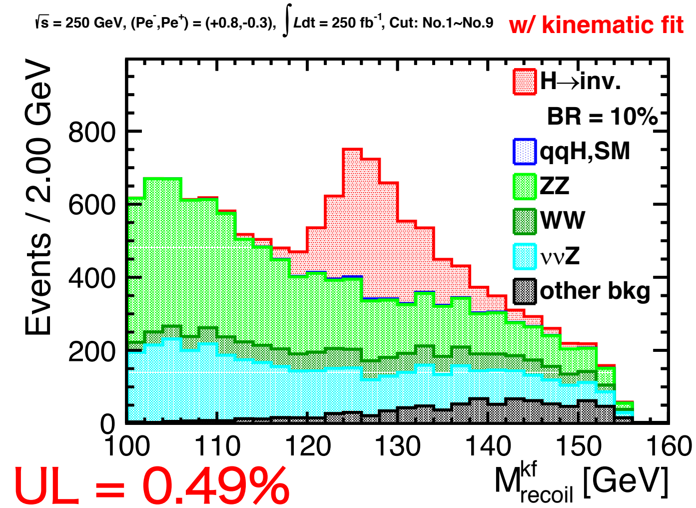


UL = 0.69%

Right polarization



UL = 0.52%



UL = 0.49%

Summary

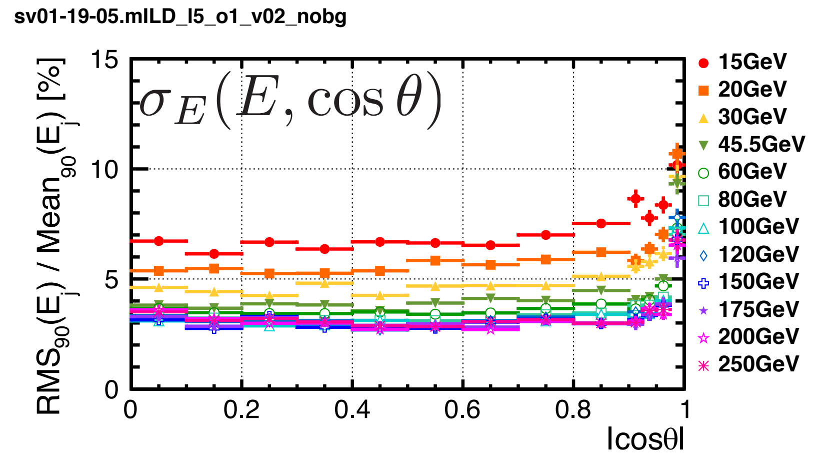
Evaluate jet energy resolution

ILD model : ILD_l(s)5_v02

- jet energy & cos θ dependence

evaluate jet angle resolution also

→ apply to kinematic fit



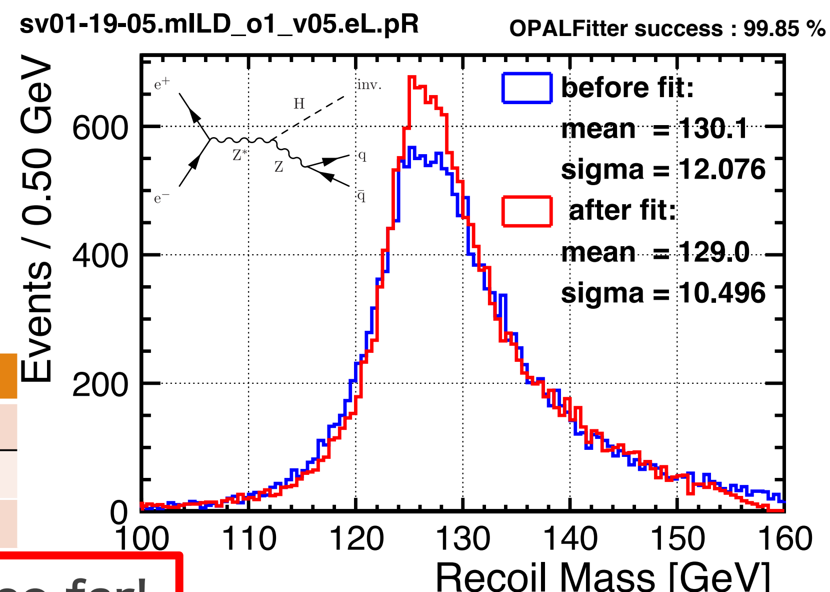
Kinematic fit

fit variables : $E_{j1}, \theta_{j1}, \phi_{j1}, E_{j2}, \theta_{j2}, \phi_{j2}$

constraint : $m_{jj} = m_Z = 91.2\ \mathrm{GeV}$

MarlinKinfit : OPALFitter

apply jet resolution



Higgs→invisible

Estimated upper limit of BR(H→inv.)

Improvement by kinematic fit

UL of BR [%] (95%CL)	Left polarization	Right polarization
Previous study	0.95	0.69
w/o kinematic fit	0.73	0.52
w/ kinematic fit	0.69	0.49

Achieve best sensitivity in the results so far!

Plans

Evaluate jet energy resolution

- More detailed evaluation of the end cap part : more statistics
- Add c-jet & b-jet information
- Use jet clustering : $E_{j1} \neq E_{j2}$
- Add jet mass dependence

kinematic fit

- Improve fit accuracy : check underestimation of JER
- Implement soft constraint : Γ_Z
- Apply to other processes

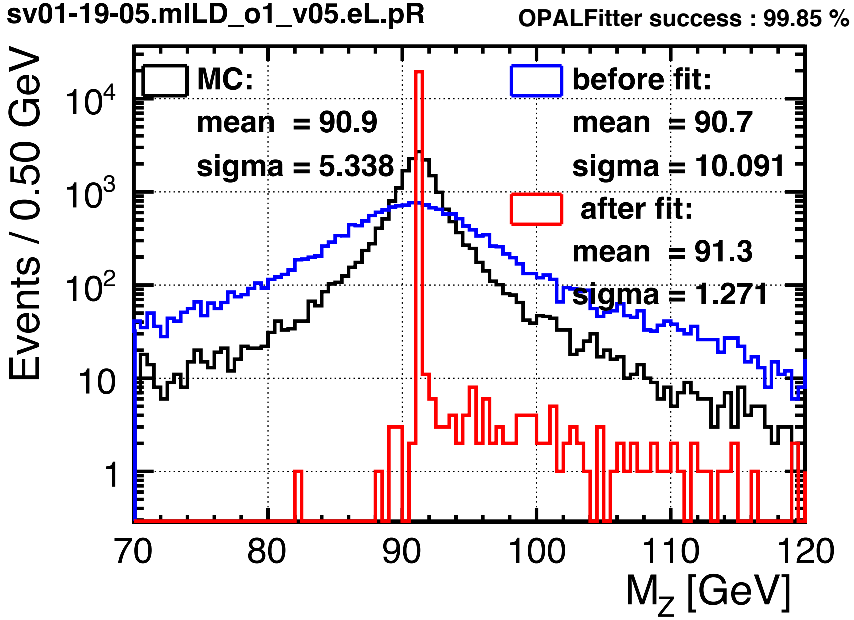
Higgs→invisible

- Use variables after fit for event selection
- Optimize recoil mass range used for estimation
- Set upper limit using profile likelihood ratio

backup

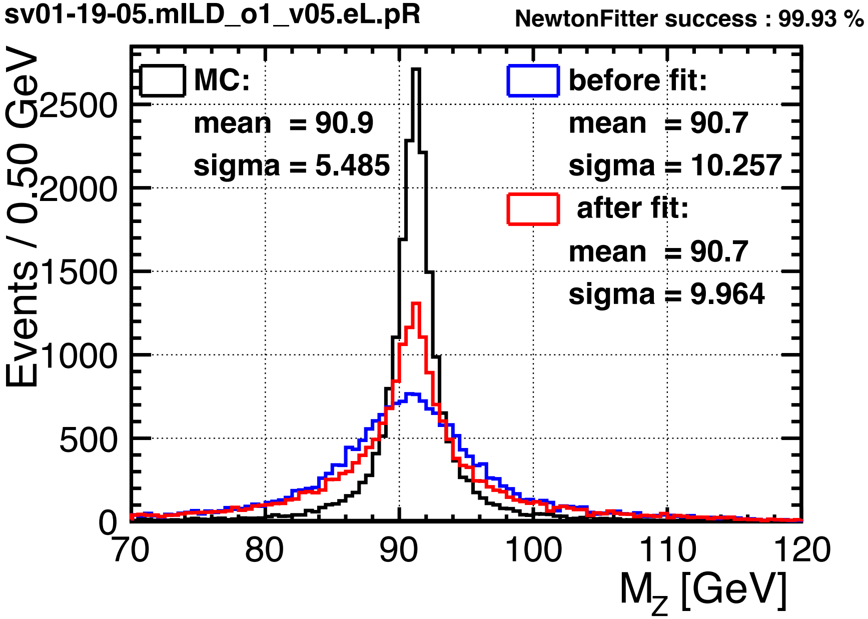
Preliminary : Soft Constraint Test

OPALFitter



Hard constraint

NewtonFitter



Soft constraint

Γ_Z=2.5 GeV

with SoftBWMassConstraint in MarlinKinfit

The Mathematics of the NewtonFitter

N parameters $a_i,\ i = 1 \ldots N$ Measured values $\vec{y}$, covarianve matrix V

K constraint funtions $\vec{f}\,(\vec{a})$

The total χ^2: $\chi^2_T\,(\vec{a},\vec{\lambda}) = \chi^2\,(\vec{a},\vec{y}) + \vec{\lambda}^T \cdot \vec{f}\,(\vec{a}).$

Seek stationary point, where all derivatives vanish:

$$\begin{aligned} \nabla_a \chi^2_T &= \nabla_a \chi^2 + \vec{\lambda}^T \cdot \nabla_a \vec{f}\,(\vec{a}) = \vec{0}, && (N \text{ equations}) \\ \nabla_\lambda \chi^2_T &= \vec{f}\,(\vec{a}) = \vec{0}, && (K \text{ equations}) \end{aligned}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial \chi^2_T}{\partial \vec{a}} \\ \frac{\partial \chi^2_T}{\partial \vec{\lambda}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \chi^2}{\partial a_i} + \sum_k \lambda_k \cdot \frac{\partial f_k}{\partial a_i} \\ f_k \end{pmatrix}$$

Newton-Raphson iterative method to solve *y*(*x*)=0:

$$x^{\nu+1} = x^\nu - \frac{y\,(x^\nu)}{y'\,(x^\nu)} \quad \Rightarrow \text{solve} \quad y'\,(x^\nu) \cdot (x^\nu - x^{\nu+1}) = y\,(x^\nu)$$

Here: Solve this system of equations in each step:

$$\left(\begin{array}{ccc|ccc} \frac{\partial^2 \chi^2}{\partial a_1 \partial a_1} + \lambda_k \cdot \frac{\partial^2 f_k}{\partial a_1 \partial a_1} & \ldots & \frac{\partial^2 \chi^2}{\partial a_1 \partial a_N} + \lambda_k \cdot \frac{\partial^2 f_k}{\partial a_1 \partial a_N} & \frac{\partial f_1}{\partial a_1} & \ldots & \frac{\partial f_K}{\partial a_1} \\ \ldots & & \ldots & \ldots & & \ldots \\ \frac{\partial \chi^2}{\partial a_N \partial a_1} + \lambda_k \cdot \frac{\partial^2 f_k}{\partial a_N \partial a_1} & \ldots & \frac{\partial^2 \chi^2}{\partial a_N \partial a_N} + \lambda_k \cdot \frac{\partial^2 f_k}{\partial a_N \partial a_N} & \frac{\partial f_1}{\partial a_N} & \ldots & \frac{\partial f_K}{\partial a_N} \\ \hline \frac{\partial f_1}{\partial a_1} & \ldots & \frac{\partial f_1}{\partial a_N} & 0 & \ldots & 0 \\ \ldots & & \ldots & \ldots & & \ldots \\ \frac{\partial f_K}{\partial a_1} & \ldots & \frac{\partial f_K}{\partial a_N} & 0 & \ldots & 0 \end{array}\right) \cdot \left(\begin{array}{c} a_1^\nu - a_1^{\nu+1} \\ \ldots \\ a_N^\nu - a_N^{\nu+1} \\ \hline \lambda_1^\nu - \lambda_1^{\nu+1} \\ \ldots \\ \lambda_K^\nu - \lambda_K^{\nu+1} \end{array}\right) = \left(\begin{array}{c} \frac{\partial \chi^2}{\partial a_1} + \lambda_k^\nu \cdot \frac{\partial f_k}{\partial a_1} \\ \ldots \\ \frac{\partial \chi^2}{\partial a_N} + \lambda_k^\nu \cdot \frac{\partial f_k}{\partial a_N} \\ \hline f_1 \\ \ldots \\ f_K \end{array}\right)$$

List of envelope parameters for ILD_l5_v02					
detector	inner radius	outer radius	half length min z, max z	additional parameters	
VXD	15.0	101.0	177.6	VXD_cone_min_z VXD_cone_max_z VXD_inner_radius_1	80.0 150.0 25.1
FTD	37.0	309.0	2350.0	FTD_outer_radius_1 FTD_outer_radius_2 FTD_min_z_0 FTD_min_z_1 FTD_min_z_2 FTD_cone_min_z FTD_cone_radius	152.8 299.7 177.7 368.2 644.2 230.0 192.0
SIT	152.9	324.6	644.1	SIT_outer_radius_1 SIT_half_length_1	299.8 368.1
TPC	329.0	1769.8	2350.0		
SET	1769.9	1804.3	2350.0		
Ecal	1804.8	2028.0	2350.0	Ecal_Hcal_symmetry Ecal_symmetry	8 8
EcalEndcap	400.0	2095.84	2411.8, 2635.0	EcalEndcap_symmetry	8
EcalEndcapRing	250.0	390.0	2411.8, 2635.0		
Hcal	2058.0	3395.5	2350.0	Hcal_inner_symmetry	8
HcalEndcap	350.0	3225.5	2650.0, 3937.0		
HcalEndcapRing	2145.84	2980.0	2411.8, 2635.0	HcalEndcapRing_symmetry	8
Coil	3425.0	4175.0	3872.0		
Yoke	4475.0	7776.0	4047.0	Yoke_symmetry	12
YokeEndcap	300.0	7776.0	4072.0, 7373.0	YokeEndcap_symmetry	12
YokeEndcapPlug	300.0	3395.54	3937.2, 4072.0	YokeEndcapPlug_symmetry	12
BeamCal	17.8	140.0	3115.0, 3315.0	BeamCal_thickness	200.0
LHCal	130.0	315.0	2680.0, 3160.0	LHCal_thickness	480.0
LumiCal	80.0	202.1	2411.8, 2540.5	LumiCal_thickness	128.7

List of envelope parameters for ILD_s5_v02					
detector	inner radius	outer radius	half length min z, max z	additional parameters	
VXD	15.0	101.0	177.6	VXD_cone_min_z VXD_cone_max_z VXD_inner_radius_1	80.0 150.0 25.1
FTD	37.0	309.0	2350.0	FTD_outer_radius_1 FTD_outer_radius_2 FTD_min_z_0 FTD_min_z_1 FTD_min_z_2 FTD_cone_min_z FTD_cone_radius	152.8 299.7 177.7 368.2 644.2 230.0 192.0
SIT	152.9	324.61	644.1	SIT_outer_radius_1 SIT_half_length_1	299.8 368.1
TPC	329.0	1426.8	2350.0		
SET	1426.9	1461.3	2350.0		
Ecal	1461.8	1685.0	2350.0	Ecal_Hcal_symmetry Ecal_symmetry	8 8
EcalEndcap	400.0	1717.92	2411.8, 2635.0	EcalEndcap_symmetry	8
EcalEndcapRing	250.0	390.0	2411.8, 2635.0		
Hcal	1715.0	3045.83	2350.0	Hcal_inner_symmetry	8
HcalEndcap	350.0	2875.83	2650.0, 3937.0		
HcalEndcapRing	1767.92	2656.92	2411.8, 2635.0	HcalEndcapRing_symmetry	8
Coil	3075.33	3825.33	3872.0		
Yoke	4125.33	7426.33	4047.0	Yoke_symmetry	12
YokeEndcap	300.0	7426.33	4072.0, 7373.0	YokeEndcap_symmetry	12
YokeEndcapPlug	300.0	3045.83	3937.2, 4072.0	YokeEndcapPlug_symmetry	12
BeamCal	17.8	140.0	3115.0, 3315.024	BeamCal_thickness	220.03
LHCal	130.0	315.0	2680.0, 3160.0	LHCal_thickness	480.0
LumiCal	80.0	202.09	2411.8, 2540.5	LumiCal_thickness	128.7