.

Forward Bhabha Scattering – Theoretical Problems

Tord Riemann, DESY, Zeuthen

FCAL Collaboration Meeting

Tel-Aviv University, 18–19 Sep 2005

A project in collaboration with

Michal Czakon Univ. Würzburg (and Katowice)

Janusz Gluza DESY (and Katowice)

See also: • NPB(PS) 135 (2004), hep-ph/0406203

• PRD 71 (2005), hep-ph/0412164

• http://www-zeuthen.desy.de/theory/research/bhabha/

- **What do we need?** $\rightarrow 10^{-4}$ **for** $d\sigma/d\cos\vartheta$ **at small** ϑ

- **Higher Order Corrections – Status**

- **Summary**

The Physics Needs

For more details see e.g.:

K. Mönig, "Bhabha scattering at the ILC"

talk at Mini-WS on Bhabha scattering, Univ. Karlsruhe, April 2005

/afs/ifh.de/user/m/moenig/public/www/bhabha_ilc.pdf

ILC – Need Bhabha cross-sections with 3–4 significant digits.

Why?

- **ILC**: $e^+e^- \to W^+W^-, f\bar{f}$ **with** $O(10^6)$ **events** $\to 10^{-3}$

- **GigaZ**: **relevant physics derived from** $Z \to$ **hadrons**, l^+l^-, **the latter with** $O(10^8)$ **events** $\to 10^{-4}$, **the systematic errors (luminosity!) influence this**

- **ILC**: $e^+e^- \to e^+e^-$, **a probe for New Physics with** $O(10^5)$ **events/year** $\to 10^{-3}$

Conclude: will need $\Delta\mathcal{L}/\mathcal{L} \approx 2 \times 10^{-4}$

The luminosity comes from very forward Bhabha scattering.

Some Kinematics

Need a cross-section prediction with **5 significant digits**.

Perturbative orders:

$$\left(\frac{\alpha}{\pi}\right) = 2 \times 10^{-3}$$

$$\left(\frac{\alpha}{\pi}\right)^2 = 0.6 \times 10^{-5}$$

Kinematics:

$$\sqrt{s} = 90 \ldots 1000 \text{ GeV}$$

$$\vartheta = 26 \ldots 82 \text{ mrad}$$

$$\cos\vartheta \sim 0.999\ 66 \ldots 0.996\ 64$$

$$T = \frac{s}{2}(1 - \beta^2 \cos\vartheta) > 1.36 \text{ GeV}|_{GigaZ},\ 42.2 \text{ GeV}|_{ILC500}$$

Conclude:

- t-**channel exchange of** γ **dominates everything else**

- $m_e^2/s\ <\ m_e^2/T\ \leq\ 10^{-5} \ldots 10^{-7}$

- **Calculate: 1-loop EWRC + 2-loop QED + corresp. bremsstrahlung**

.

The **1-loop electroweak** corrections (plus some leading higher order terms) are well-known, with rising technical precision, since about 1988/91.
Böhm, Denner, Hollik; Bardin, TR 1991 $\rightarrow$ **Fig. 2004 Lorca, TR**

2-loop Bhabha scattering: What to be done?

- **Calculate:**

$$\sigma \;=\; (2 \rightarrow 2) \;+\; (2 \rightarrow 3) \;+\; (2 \rightarrow 4)$$

$$\sigma \;=\; |\textbf{Born} \;+\; \textbf{1-loop} \;+\; \textbf{2-loop}|^2$$
$$+\; |(\textbf{Born} + \textbf{1-}\gamma) \;+\; (\textbf{1-loop} + \textbf{1-}\gamma)|^2$$
$$+\; |(\textbf{Born} + \textbf{2-}\gamma)|^2$$

- **Do not include:** $|\textbf{2-loop}|^2$
$$|(\textbf{1-loop} + \textbf{1-}\gamma)|^2$$

Status by end of 2004

Established: 10^{-3} MC programs for LEP, ILC

Introduction to NLLBHA by Trentadue and to BHLUMI by Jadach in:
Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, "Theoretical error of luminosity cross section at LEP",
hep-ph/0306083 [1]

- **BHLUMI v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997**

- **see also: Jadach, Melles, Ward, Yost: PLB 1996, thesis Melles 1996 [2]**

- **NLLBHA: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997,**
  CERN 96-01

- **SAMBHA: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211**

See e.g.: Table 1 of [1] and Figure 3.1 of [2] $\to$ Conclude:
The nonlogarithmic $O(\alpha^2)$ terms, originating from pure QED radiative 1-loop and from
2-loop diagrams are not completely covered.

They have to be calculated and integrated into the MC programs.

Beware:
$$m_e, m_\gamma, (d-4), E_\gamma$$

Results: Numerical comparison in all $f\bar{f}$

Bhabha $e^- e^+ \to e^- e^+ (\gamma)$ at LC: $\sqrt{s} = 500$ GeV, $E_{\max}(\gamma_{\text{soft}}) = \frac{\sqrt{s}}{10}$

$\cos\theta$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{Born}}$ (pb)	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\mathcal{O}(\alpha^3)=\text{Born+QED+weak+soft}}$	Group
-0.9999	0.21482 70434 05632 5	0.14889 12125 78083 7	$a\mathring{\text{I}}$TALC
-0.9999	0.21482 70434 05632 6	0.14889 12189 28404 0	*FeynArts*
-0.9	0.21699 88288 10920 5	0.19344 50785 26863 6	$a\mathring{\text{I}}$TALC
-0.9	0.21699 88288 10920 0	0.19344 50785 26862 2	*FeynArts*
-0.9	0.21699 88288 41513 1	0.19344 50785 62637 9	$m_e = 0$
$+0.0$	0.59814 23072 50330 3	0.54667 71794 69423 1	$a\mathring{\text{I}}$TALC
$+0.0$	0.59814 23072 50329 4	0.54667 71794 69421 8	*FeynArts*
$+0.0$	0.59814 23072 88584 4	0.54667 71794 99961 4	$m_e = 0$
$+0.9$	$0.18916\ 03223\ 32270\ 6 \cdot 10^3$	$0.17292\ 83490\ 66507\ 2 \cdot 10^3$	$a\mathring{\text{I}}$TALC
$+0.9$	$0.18916\ 03223\ 32270\ 6 \cdot 10^3$	$0.17292\ 83490\ 66508\ 0 \cdot 10^3$	*FeynArts*
$+0.9$	$0.18916\ 03223\ 31848\ 5 \cdot 10^3$	$0.17292\ 83490\ 61347\ 4 \cdot 10^3$	$m_e = 0$
$+0.9999$	$0.20842\ 90676\ 46142\ 9 \cdot 10^9$	$0.19140\ 17861\ 11341\ 6 \cdot 10^9$	$a\mathring{\text{I}}$TALC
$+0.9999$	$0.20842\ 90676\ 46436\ 4 \cdot 10^9$	$0.19140\ 17861\ 11979\ 0 \cdot 10^9$	*FeynArts*

Great independent agreement up to 14 digits! : **limit** in double precision

Previous agreement with *FeynArts*: **11 digits** hep–ph/0307132, SANC: 10 digits hep–ph/0207156

Thanks to T. Hahn, numbers supplied with *FeynArts + FormCalc + LoopTools*

Zinnowitz, 28/04/2004

A. Lorca —Automatization and width effects with $a\mathring{\text{I}}$TALC

Table 2:

The differential Bhabha cross section in nbarn as function of the scattering angle and the cms-energy.

$M_Z = 91.16$ GeV, $m_t = 150$ GeV, $M_H = 100$ GeV.

Upper rows: DZ, lower rows: H.

δ_m: *largest relative deviation in per mille.*

$\sqrt{s}$ (GeV)	60	89	91.16	93	200
θ					
15°	129.6	65.11	57.93	49.00	11.82
	129.6	65.11	57.93	49.00	11.82
45°	1.451	1.376	1.755	.4833	11.67
	1.451	1.377	1.756	.4837	11.68
60°	.4303	.6124	1.125	.2697	.03075
	.4305	.6129	1.126	.2699	.03077
75°	.1717	.3627	.8718	.2232	.01072
	.1718	.3630	.8720	.2233	.01072
90°	.08873	.2768	.7790	.2088	.004862
	.08876	.2769	.7787	.2087	.004855
105°	.05917	.2690	.8082	.2157	.002858
	.05918	.2690	.8074	.2157	.002853
120°	.04906	.3053	.9323	.2429	.002077
	.04906	.3051	.9309	.2426	.002074
135°	.04671	.3626	1.111	.2838	.001743
	.04672	.3624	1.109	.2833	.001742
165°	.04839	.4638	1.425	.3590	.001539
	.04839	.4635	1.422	.3584	.001540
δ_m	0.6	0.8	1.8	2.0	1.7

Bhabha scattering

Bardin,Hollik,T.R., Z.PhysikC49(1991)485

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Status 2005

Know the constant term $(m_e = 0)$
from 2-loop Bhabha scattering

A. Penin, Two-Loop Corrections to Bhabha Scattering, hep-ph/0501120 v.3, $\to$ PRL

Transform the massless 2-loop results of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by $D = 4 - 2\epsilon$ into the on-mass-shell renormalization with $m_e \to 0$ and IR regulation by $\lambda = m_\gamma \neq 0$

Use IR-properties of amplitudes (see Penin):

[A] Exponentiation of the IR logarithms (Sudakov 1956,...)

[B] Factorization of the collinear logarithms into expernal legs (Frenkel, Taylor 1976)

[C] Non-renormalization of the IR exponents (YFS 1961, ...)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani et al. 2005, Penin)

If all this is correct, the constant term in m_e is known for the MCs (but the radiative one-loops with 5-point functions).

$$m = 0$$

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.
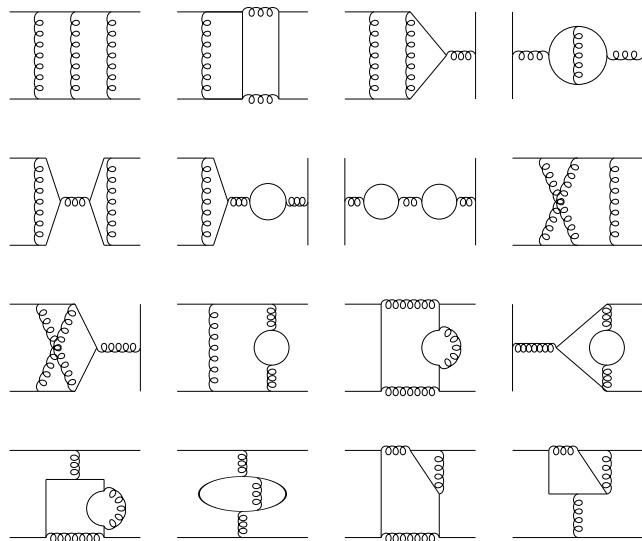
In this calculation all particles massless.

The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Two-loop integral inheritance chart

Xbox$_{1,2}$ Pbox$_{1,2}$ pentabox

Xtri   tri_sym   pbdy   Btie

Dbox   Bbox

Btri

Sset

.
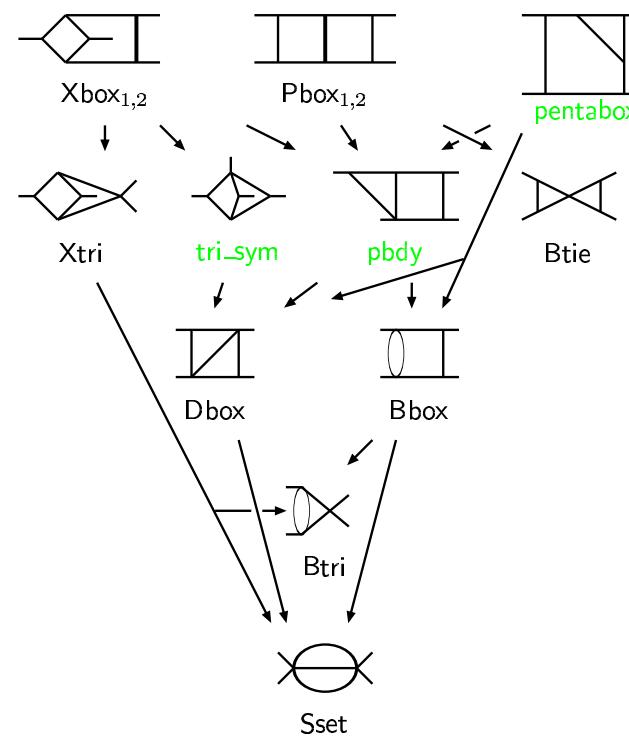
The massive 2-loop contributions

We are interested in a calculation of the virtual second order corrections to

$$\frac{d\sigma}{d\cos\vartheta}(e^+e^- \to e^+e^-)$$

We are using a scheme with

(1) $m_e \neq 0$ **(good with the MC's)**

(2) $m_\gamma = 0$ **(bad with the MC's; $\to$ Mastrolia, Remiddi 2003)**

(3) dim.reg. for UV and IR divergences

Also:

Argeri, Bonciani, Ferroglia, Mastrolia, Remiddi, v.d.Bij: all but many 2-boxes
Heinrich, Smirnov: Calculation of selected complicated Feynman integrals

.

There are few topologies only:

- 1-loop: **5**

- 2-loop self energies: **5 (3 for external legs)**

- 2-loop vertices: **5**

- 2-loop boxes: **6** $\rightarrow$ **next slide**

The many Feynman integrals may be reduced to 'few' **master integrals** by sophisticated methods (**Remiddi-Laporta algorithm, 1996/2000** $\rightarrow$ **IdSolver** (Czakon 2003)).

The massive diagrams (See also webpage)

Assume 3 leptonic flavors, do not look at loops in external legs.

Not too many QED diagrams:

- Born diagrams: **2**

- 1-loop diagrams: **14**

- 2-loop diagrams: **154** (with **68 double-boxes**) interfere with Born

The two-loop box diagrams for massive Bhabha scattering
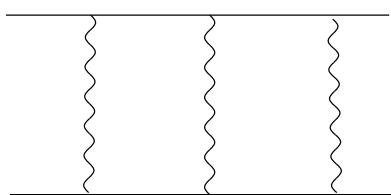
B1
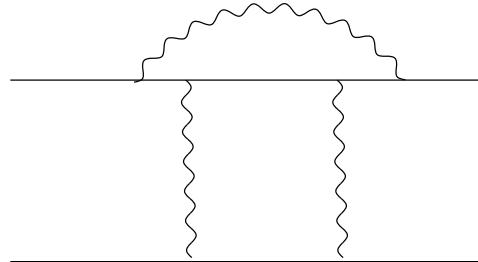
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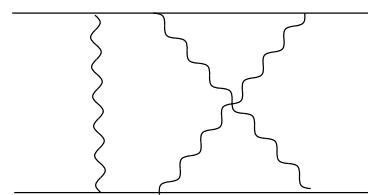
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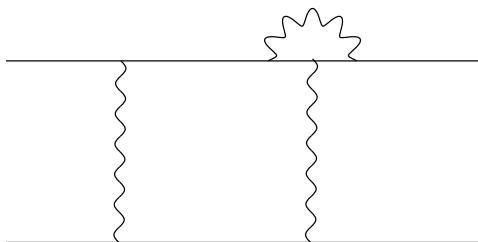
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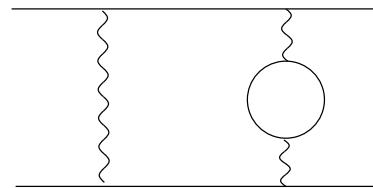
B5
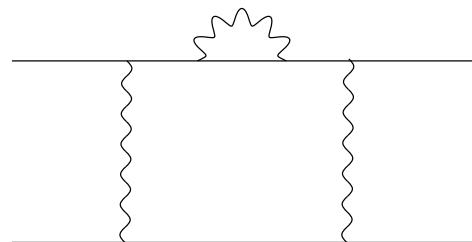
B6

- **B5**: Completely known (2004)

  Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij: hep-ph/0405275, hep-ph/0411321

  Czakon, Gluza, Riemann: *http://www-zeuthen.desy.de/.../MastersBhabha.m* (unpubl.)

- **B1–B3**: Few masters known (Smirnov, Heinrich 2002,2004)

- **B4, B6**: Not much known (Czakon et al. 2004)

The basic planar 2-box master of **B1**, **B7l4m**, was a breakthrough

The two-loop Feynman integrals

One has to solve many, very complicated Feynman integrals with $L = 2$ loops and $N \leq 7$ internal lines:

$$G(X) \;=\; \frac{1}{\left(i\pi^{d/2}\right)^2} \int \frac{d^D k_1 d^D k_2 \; X}{(q_1^2 - m_1^2)^{\nu_1} \dots (q_j^2 - m_j^2)^{\nu_j} \dots (q_N^2 - m_N^2)^{\nu_N}},$$

$$X \;=\; 1, (k_1 P), (k_1 k_2), (k_2 P), \cdots$$

where P is some external momentum: $p_1, ..p_4$

A completely numerical approach might be possible Passarino 2004.

For checks in the Euclidean region $(s < 0, t < 0)$ this has been proven to be a powerful tool Binoth, Heinrich 2000/03

We prefer to calculate the integrals analytically (where possible)
Derive a minimal set of so-called master integrals and algebraic expressions in terms of them for all the other Feynman integrals

.

So we need a **A table of master integrals**

We use **IdSolver** with the Laporta/Remiddi algorithm:

Derive with integration-by-parts (and Lorentz-invariance) identities a system of *algebraic* **equations for the Feynman integrals and solve the system.**

- 1-loop: **5** masters (all known)

- 2-loop self energies: **6** masters (all known)

- 2-loop vertices: **19** masters (all known)

- 2-loop boxes: **33** masters $\to$ (**O(5) published**, maybe more known)   **see table**

The **calculation of the master integrals** is mainly done with two methods:

- derive and solve (systems of) **differential equations** (with boundary conditions)

- derive and solve (up to 8-dimensional) **Mellin-Barnes integral representations** for single Feynman integrals

From Czakon et al., PRD 71 (2004): 4-point MIs entering basic two-loop box diagrams. An asterisk denotes one-loop MI. MIs with a dagger: know singular parts only

MI	B1	B2	B3	B4	B5	B6	ref.
B714m1	+	−	−	−	−	−	Smirnov:2001cm
B714m1N	+	−	−	−	−	−	Heinrich:2004iq
B714m2	−	+	−	−	−	−	Heinrich:2004iq†
B714m2[d1--d3]	−	+	−	−	−	−	
B714m3	−	−	+	−	−	−	Heinrich:2004iq†
B714m3[d1--d2]	−	−	+	−	−	−	
B613m1	+	−	+	−	−	−	
B613m1d	+	−	+	−	−	−	
B613m2	−	+	−	+	−	−	
B613m2d	−	+	−	+	−	−	
B613m3	−	−	+	−	−	−	
B613m3[d1--d5]	−	−	+	−	−	−	
B512m1	+	−	+	−	−	−	Czakon:2004tg
B512m2	−	+	−	+	−	+	Sec. IIIE1†
B512m2[d1--d2]	−	+	−	+	−	+	Sec. IIIE1†
B512m3	+	−	+	−	−	−	
B512m3[d1--d3]	+	−	+	−	−	−	Sec. IIIE1†
B513m	−	+	+	+	−	−	
B513m[d1--d3]	−	+	+	+	−	−	
B514m	−	+	+	+	+	−	Bonciani:2003cj
B514md	−	+	+	+	+	−	Sec. IIIE
B412m*	−	−	−	+	+	+	'tHooft:1972fi,Bonciani:2003cj
total = 33+1*	9	15	22	11+1*	2+1*	3+1*	

A simple example: A class of scalar self-energy integrals

$$\texttt{SE312m}(a,b,c,d) \quad = \quad -\frac{e^{2\epsilon\gamma_E}}{\pi^D} \int \frac{d^D k_1 d^D k_2 (k_1 k_2)^{-d}}{[(k_1 + k_2 - p)^2 - m^2]^b [k_1^2]^a [k_2^2 - m^2]^c}.$$

The two Master Integrals are:

$$\texttt{SE312m} = \texttt{SE312m}(1,1,1,0)$$

$$\texttt{SE312md} = \texttt{SE312m}(1,1,2,0)$$

In Bonciani et al. 2003 it is used instead as a master integral:

$$\texttt{SE312mN} = \texttt{SE312m}(1,1,1,-1)$$

By an algebraic relation, valid for $m^2 = 1$ and $p^2 = s$,

$$\texttt{SE312md} \quad = \quad \frac{-(1+s) + \epsilon(2+s)}{s-4} \, \texttt{SE312m} + \frac{2(1-\epsilon)}{s-4} \left(\texttt{T111m}^2 + 3 \, \texttt{SE312mN} \right),$$

one may derive then $\texttt{SE312md}$.

Because the integral $\texttt{SE312md}$ is one of our masters, we reproduce it here explicitly:

$$\texttt{SE312md}(x) \quad = \quad \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} - \left(\frac{1-\zeta_2}{2} + \frac{1+x}{1-x} H[0,x] + \frac{1+x^2}{(1-x)^2} H[0,0,x] \right) + \mathcal{O}(\epsilon).$$

At our webpage, there is a file with all the master integrals we have determined so far.
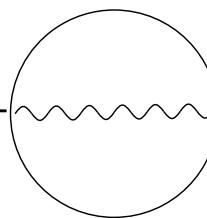
SE3l2m
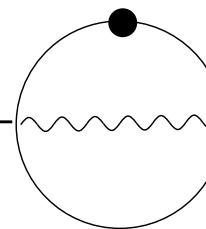
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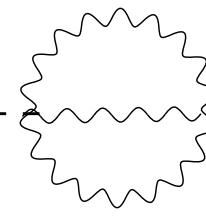
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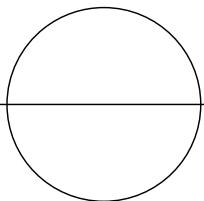
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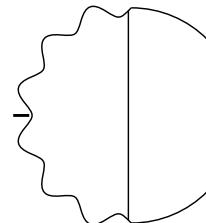
SE5l3m
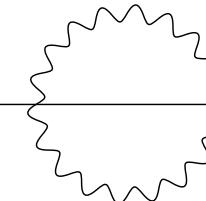
SE3l1m

The six two-loop 2-point MIs. External solid (dashed) lines describe on (off) -shell momenta.
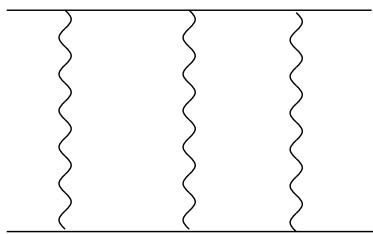
B7l4m1
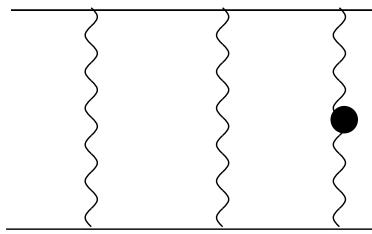
B7l4m1d
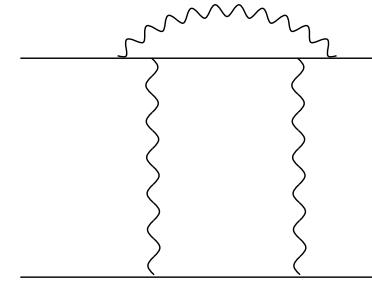
B7l4m2
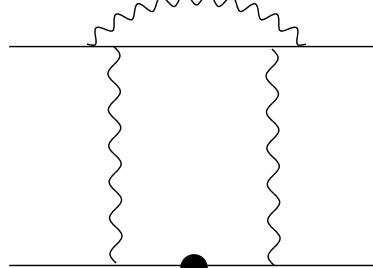
B7l4m2d1
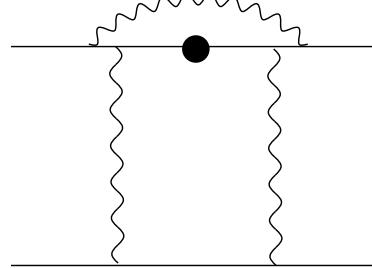
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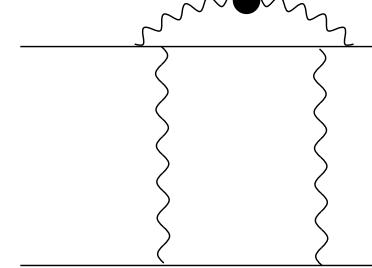
B7l4m2d3
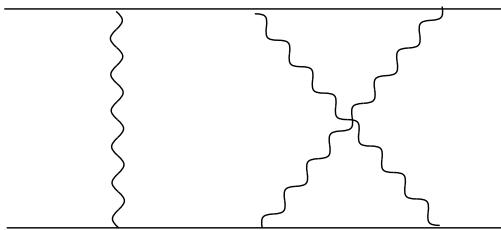
B7l4m3
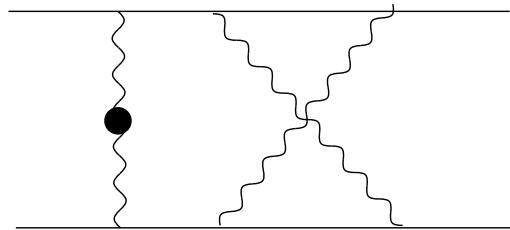
B7l4lm3d1
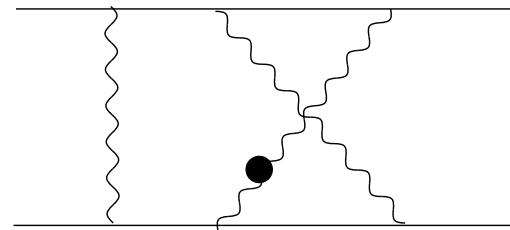
B7l4m3d2

The nine two-loop box MIs with seven internal lines.

The simplest diagram is the tadpole:

$$
\begin{aligned}
\texttt{T1l1m} \;&=\; \frac{e^{\epsilon \gamma_E}}{i \pi^{D/2}} \int \frac{d^D q}{q^2 - 1} \\[2mm]
&=\; \frac{1}{\epsilon} + 1 + \left(1 + \frac{\zeta_2}{2} \right) \epsilon + \left(1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3} \right) \epsilon^2 + \dots
\end{aligned}
$$
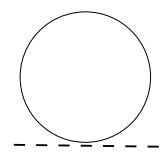
T1l1m
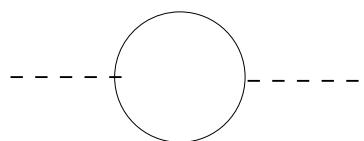
SE2l2m
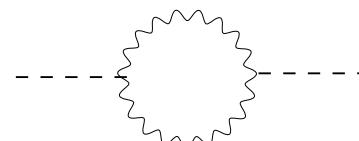
SE2l0m
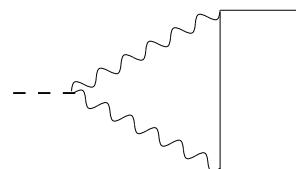
V3l1m
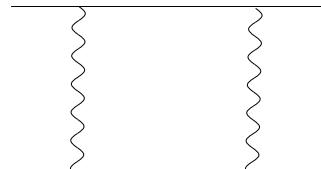
B4l2m

.

How to calculate 2-loop Bhabha masters?

- **Self-energies and vertices and (very few) 2-boxes:**
  use **differential equations** and **Harmonic Polylogarithms**, introduced by Remiddi,
  Vermaseren, plus . . .)

- **Some 7-line 2-boxes**
  use **Mellin-Barnes technique**, sum up **multiple series**, use numerical checks in
  Euclidean space (s, t negative)

- **For the unsolved 2-boxes:**
  Combination of both methods: present study

There are other methods not used here:
difference equations
pure numerical approaches
. . .

.

The 2-boxes with 5 lines

The completely known 2-boxes with 5 lines are B5l4m (Bonciani et al., Czakon et al. 2004), B5l2m1 (Czakon et al. 2004) :
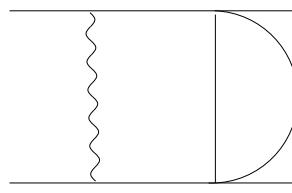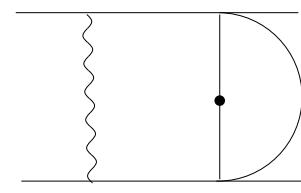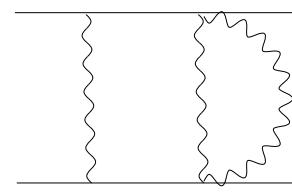
B5l4m1   B5l4m1d1   B5l2m1

The divergent parts of the B5l2m2 and B5l2m3 type are known (Czakon et al. 2004):
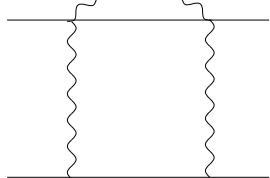
B5l2m3   B5l2m3d1
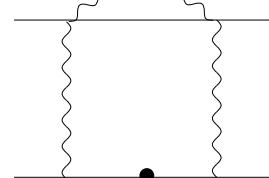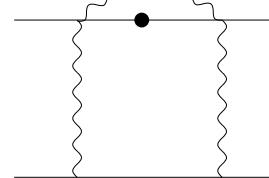
B5l2m2   B5l2m2d1   B5l2m2d2   B5l2m3d2   B5l2m3d3
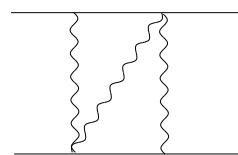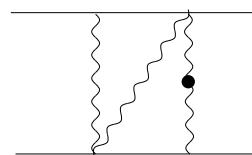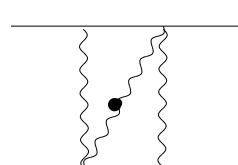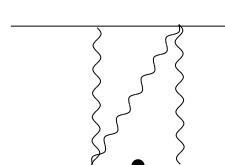
B5l3m: The divergences in $D - 4 = -2\epsilon$

B7l4m1   ×2   B7l4m2   ×4   B7l4m3   ×3
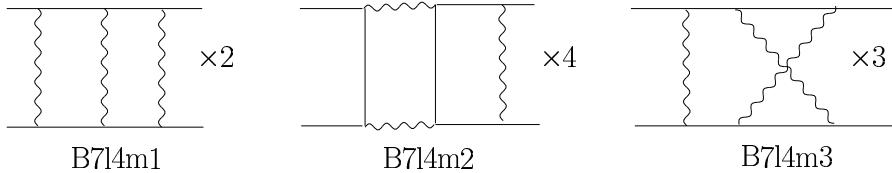
The B5l3m boxes, contribute to B2 (2nd planar 2-box) (shrink two lines...)

B5l3m   B5l3md1   B5l3md2
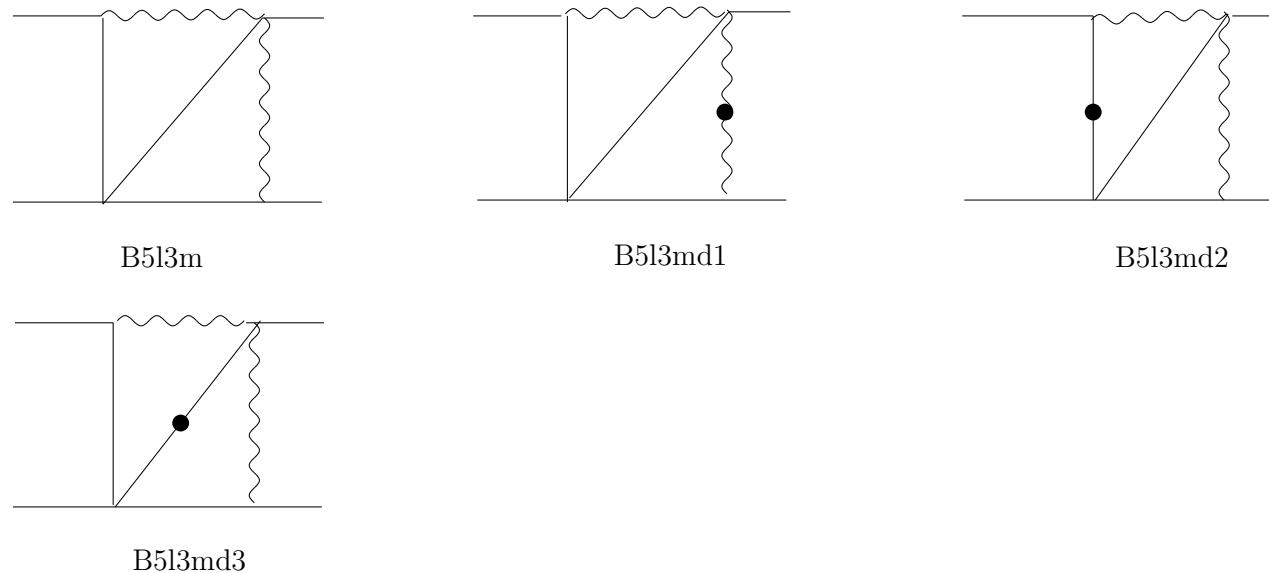
B5l3md3

The B5l3md2 topology appears twice as a master
but the B5l3md1 does not!

The B5l3m topology: Gross features

$$MB5l3m[x, y] = Sum[B5l3m[k, x, y] * ep^k, k, 0, 1]; \tag{1}$$

$$MB5l3md1[x, y] = Sum[B5l3md1[k, x, y] * ep^k, k, -2, 1]; \tag{2}$$

$$MB5l3md2[x, y] = Sum[B5l3md2[k, x, y] * ep^k, k, -2, 1]; \tag{3}$$

$$MB5l3md2a[x, y] = Sum[B5l3md2a[k, x, y] * ep^k, k, -2, 1]; \tag{4}$$

$$MB5l3md3[x, y] = Sum[B5l3md3[k, x, y] * ep^k, k, -1, 1]; \tag{5}$$

Note:

- B5l3m – the basic master is finite

- B5l3md2 – use 4-dim. MB-Representation

- B5l3md2' – the same, but $(s \leftrightarrow t)$

- B5l3md1, B5l3md3 – system of 2 coupled differential eqns
  Only BLB5l3md1 has $1/\epsilon^2$ (so decouples), and last step is the two $1/\epsilon$ coefficients of B5l3md1 and B5l3md3.
  The first one is found by algebraic manipulations (see Czakon et al. LCWS Paris 2004), the second then fulfils a diff.eqn

Differential equations

$$\frac{\partial B5l3md3[-1]}{\partial x} = \frac{1+x^2}{x(1-x^2)}B5l3md3[-1] - \frac{yH[0,y]}{(1-x^2)(1-y^2)} \tag{6}$$

with $s = -(1-x)^2/x$, $t = -(1-y)^2/y$

Solution:

$$B5l3md3[-1] = -\frac{xy}{(-1+x^2)(-1+y^2)}H[0,x]H[0,y] \tag{7}$$

with

$$H[0,x] = \ln(x) \tag{8}$$

The coefficients in the equation are of the form

$$\frac{A_1}{x-B_1} + \frac{A_2}{x-B_2} + .. \tag{9}$$

One may derive (systems of) differential equations for the masters, with inhomogeneity composed of simpler masters (Kotikov, Laporta, Remiddi)

$$\frac{\partial M_n}{\partial x} = A(x,y)\ M_n + I(x,y) \tag{10}$$

$$I(x,y) = \sum_{k=0,n-1} c_k\ M_k \tag{11}$$

Expand in ϵ $(D = 4 - 2\epsilon)$:

$$M_n = \sum_{i=-2,i_m} M_{n,i}\epsilon^i \qquad \text{etc.} \tag{12}$$

General solution for homogeneous eqn. $(M_h' = A\ M_h)$:

$$M_h'/M_h = A \tag{13}$$

$$\int (M_h'/M_h) = \ln M_h = \int A \tag{14}$$

$$= \int \sum \frac{a_i}{x - x_i} \sim \ln(x - x_i) \tag{15}$$

so:

$$M_h \sim \text{Polynomials} \tag{16}$$

Then the inhomogeneous solution is:

$$M(x,y) = M_h(x,y)\left(\text{const(y)} + \int \frac{I(x',y)}{M_h(x',y)}\right) \tag{17}$$

Result:

nested integrals over 'simple' iterated integrands

The method leads to the HPLs $H(\{a\}, x)$ and GPLs $G(\{a(y)\}, x)$

Harmonic Polylogarithms $H(x)$

$$
\begin{aligned}
H[-1, 1, x] &= \int_0^x \frac{dx''}{(1 + x'')} \int_0^{x''} \frac{dx'}{(1 - x')} \qquad (18) \\
&= Li_2\left(\frac{1 + x}{2}\right) + \ldots \qquad (19)
\end{aligned}
$$

Generalized Harmonic Polylogarithms $G(x, y)$ ...

but it works only if the polynomial structure is simple enough for a solution with this class of functions

Method is absolutely 'super' if it works.

But:

one needs complete chains of masters of lower complexity, and there are systems of up to 6 (!) potentially coupled 1st order equations

.

Mellin-Barnes representations

Boos, Davydychev 1991, Smirnov 1999, Tausk 1999, Smirnov book 2004

$$\frac{1}{(A+B)^\nu} \;=\; \frac{B^{-\nu}}{(1-(-A/B))^{-\nu}} \;=\; \frac{B^{-\nu}}{2\pi i \Gamma(\nu)} \int\limits_{-i\infty}^{i\infty} d\sigma\, A^\sigma\; B^{-\sigma}\; \Gamma(-\sigma)\Gamma(\nu+\sigma) \quad (20)$$

Is special case of a well-known Mellin-Barnes integral for hypergeometric functions

$$\frac{1}{(1-z)^\nu} \;=\; {}_2F_1(\nu,b,b',z)|_{b=b'} \tag{21}$$

$$= \; \frac{1}{2\pi i \Gamma(\nu)} \frac{\Gamma(b')}{\Gamma(b)} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \Gamma(\nu+\sigma)\Gamma(-\sigma)\frac{\Gamma(b+\sigma)}{\Gamma(b'+\sigma)} \tag{22}$$

with $-z = A/B$**.**

How can this be made useful here?

Introduce Feynman parameters

The momentum integrals of a Feynman diagram may be performed with Feynman parameters, one for each line.

In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for B7l4m2, the planar 2nd type 2-box:

allow for propagators with indices, $1/(k_1^2 - m_1^2)^{a_1}$ etc.

$$K_{\text{1-loop Box,off}} = \frac{(-1)^{a_{4567}}\Gamma(a_{4567} - d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int_0^\infty \Pi_{j=4}^7 dx_j\, x_j^{a_j-1} \frac{\delta(1 - x_4 - x_5 - x_6 - x_7)}{F^{a_{4567}-d/2}} \quad (23)$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function F is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$F = [-t]x_4 x_7 + [-s]x_5 x_6 + m^2(x_5 + x_6)^2 \quad (24)$$

$$+(m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7 x_5 \quad (25)$$

We want to apply now:

$$\int_0^1 \prod_i^4 dx_i\, x_i^{\alpha_i-1}\, \delta\left(1 - x_1 - x_2 - x_3 - x_4\right) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)}{\Gamma\left(\alpha_1 + \alpha_2 + \alpha_3 + +\alpha_4\right)} \quad (26)$$

with coefficients α_i dependent on a_i and on F

For this, we have to apply several MB-integrals here.

And repeat the procedure for the 2nd subloop.

For the 2nd planar 2-box, B7l4m2, one gets (Smirnov book 4.73):

$$B_{\text{pl},2} \;=\; \frac{\text{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[\frac{m^2}{-s}\right]^{z_5+z_6} \left[\frac{-t}{-s}\right]^{z_1} \prod_{j=1}^{6}[dz_j\Gamma(-z_j)] \frac{\prod_{k=7}^{18}\Gamma_k(\{z_i\})}{\prod_{l=19}^{24}\Gamma_l(\{z_i\})} \tag{27}$$

with $a = a_1 + \ldots + a_7$ and

$$z_i \;=\; \text{const} + i\,\Im m(z_i) \tag{28}$$

$$d \;=\; 4 - 2\epsilon \tag{29}$$

$$\text{const} \;=\; \frac{(i\pi^{d/2})^2(-1)^a(-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d-a_{4567})} \tag{30}$$

The integrand includes e.g.:

$$\Gamma_2 \;=\; \Gamma(-z_2) \tag{31}$$

$$\Gamma_4 \;=\; \Gamma(-z_4) \tag{32}$$

$$\Gamma_7 \;=\; \Gamma(a_4 + z_2 + z_4) \tag{33}$$

$$\Gamma_8 \;=\; \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4) \tag{34}$$

$$\ldots \tag{35}$$

We now derive from B7l4m2 the MB-integral for B5l3m by setting $a_1 = 0$ (trivial, gives B6l3m2) and $a_4 = 0$.

The latter do with care because of

$$\frac{1}{\Gamma(a_4)} \to \frac{1}{\Gamma(0)} = 0 \tag{36}$$

See by inspection that we will get factor $\Gamma(a_4)$ if $z_2, z_4 \to 0$.

$\to$ Start with the z_2, z_4 integrations by
taking the residues for closing the integration contours to the right:

$$
\begin{aligned}
I_{2,4} &= \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4 + z_2 + z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i) & (37) \\
&= \frac{1}{(2\pi i)} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,..} \frac{-(-1)^n}{n!} \frac{\Gamma(a_4 + z_2 + n)}{\Gamma(a_4)} R(z_i) & (38) \\
&= \sum_{n,m=0,1,..} \frac{(-1)^{n+m}}{n!m!} \frac{\Gamma(a_4 + n + m)}{\Gamma(a_4)} R(z_i) \to_{a=0} 1 & (39)
\end{aligned}
$$

So, setting $a_1 = a_4 = 0$ and eliminating $\int dz_2 dz_4$ with setting $z_2 = z_4 = 0$
we got a 4-fold Mellin-Barnes integral for B5l3m
with $24 - 3 = 21$ z_i-dependent Γ-functions which may yield residua within four-fold sums.

As mentioned:

This formula has to be calculated now explicitely for the case

$$B5l3md2 = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0 \qquad (40)$$

(B5l3md2 is a dotted master, with index $a_2 = 2$, all others are one)

Next tasks:

- Find a region of definiteness of the n-fold MB-integral

$$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10! \ (41)$$

- Then go to the physical region where $\epsilon << 1$ by distorting the integration path step by step (adding each crossed residuum – per residue this means one integral less!!!)

- Take integrals by sums over residua, i.e. introduce infinite sums

- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.

Here this means:

$$
\begin{aligned}
B5l3md2 \quad &\rightarrow \quad MB(\text{4-dim,fin}) + MB_3(\text{3-dim,fin}) && (42) \\
&+ \quad MB_{36}(\text{2-dim}, \epsilon^{-1}, fin) + MB_{365}(\text{1-dim}, \epsilon^{-2}, \epsilon^{-1,fin}) && (43) \\
&+ \quad MB_5(\text{3-dim,fin}) && (44)
\end{aligned}
$$

After these preparations e.g.:

$$
\begin{aligned}
MB_{365}(\text{1-dim}, \epsilon^{-2}) \quad &\sim \quad \frac{1}{\epsilon^2} \int dz_6 \frac{(-s)^{(z_6-1)}\Gamma(-z_6)^3\Gamma(1+z_6)}{8\Gamma(-2z_6)} && (45) \\
&\sim \quad \frac{1}{\epsilon^2} \sum_{n=0,\infty} -\frac{(-1)^n(-s)^n\Gamma(1+n)^3)}{8n!\Gamma(-2(-1-n))} && (46) \\
&= \quad -\frac{1}{\epsilon^2} \frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4-s}\sqrt{s}} && (47) \\
&= \quad \frac{1}{\epsilon^2} \frac{x}{4(1-x^2)}H[0,x] && (48)
\end{aligned}
$$

Here were residua at $z_6 = -n-1, n = 0, 1, ..$ taken

The divergent parts of the masters B5l3m are:

```
B5l3m[-2,x_,y_] = B5l3m[-1,x_,y_] = 0;


B5l3md1[-2,x_,y_] = ((-1 + x)^2*y*(-1 + y^2 + 2*y*H[0, y])))/
                       (8*x*(-1 + y)*(1 + y)^3);
B5l3md1[-1,x_,y_] = ((y*(6*(-1 + x - x^2 + x^3)*H[0, x]*
   (-1 + y^2 + 2*y*H[0, y]) - 6*(1 + x)*(-2 - 2*x^2 + 2*y^2 + 2*x^2*y^2 +
   y*z2 - 2*x*y*z2 + x^2*y*z2 + 2*(-2*x - y + 2*x*y - x^2*y - 2*x*y^2 +
   (-1 + x)^2*y*H[-1, -y] + 3*(-1 + x)^2*y*H[-1, y])*H[0, y] -
   6*(-1 + x)^2*y*H[0, -1, y] - 4*y*H[0, 0, y] + 8*x*y*H[0, 0, y] -
   4*x^2*y*H[0, 0, y] + 2*y*H[0, 1, y] - 4*x*y*H[0, 1, y] +
   2*x^2*y*H[0, 1, y])))/(24*x*(1 + x)*(-1 + y)*(1 + y)^3));


B5l3md2[-2,x_,y_] = -x/(1 - x^2)/4 H[0, x];
B5l3md2[-1,x_,y_] = ((x*(2*(1 + y^2)*H[0, x]*H[0, y] -
     (-1 + y^2)*(z2 + 6*H[-1, 0, x] - 4*H[0, 0, x] -
       2*H[1, 0, x]))))/(4*(-1 + x^2)*(-1 + y^2)));


B5l3md2a[-a,x_,y_] = B5l3md2[-a,y,x],    a=-2,-1;


B5l3md3[-2,x_,y_] = 0;
B5l3md3[-1,x_,y_] = -((x*y*H[0, x]*H[0, y])/((-1 + x^2)*(-1 + y^2)));
```

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Summary

A calculation of the constant 2-loop term for Bhabha scattering is derived from massless calculations Penin, Bonciani et al.

In parallel:

- A complete list of MASSIVE masters was derived (2004)

- Huge files with algebraic relations for all the reducible Feynman integrals needed for the interferences of boxes with Born (not complete, but fully understood)

- Essential progress for the massive 2-box master integral determination. Underway: Determination of all 2-box masters in a systematic approach use Generalized Harmonic Polylogarithms Remiddi, Vermaseren plus potentially ...)

- An unsolved problem is the systematic summation of the massive multiple sums after the MB-integral evaluation

It is also possible to do the massive 2-loop calculation with present computers.

Improve the existing MC-codes with that.

Care about the radiative 1-loops (with 5-point functions).