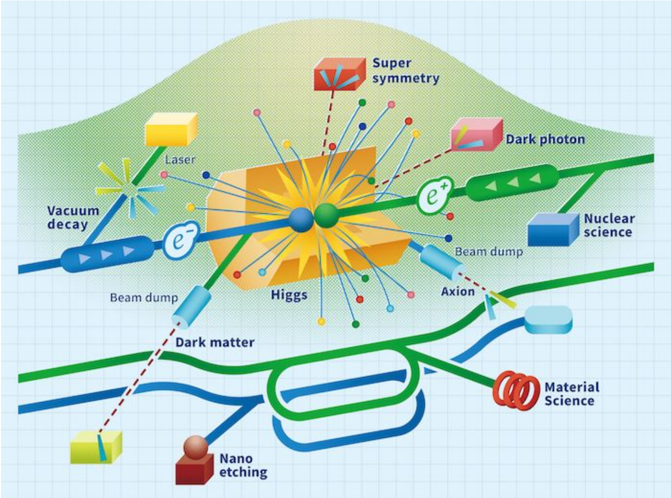
Electroweak Precision Observables at future e^+e^- colliders

- EWPOs, current vs. future.
- New accomplished three-loop calculations to EWPOs.
- Linking EWPOs to the Z line shape.

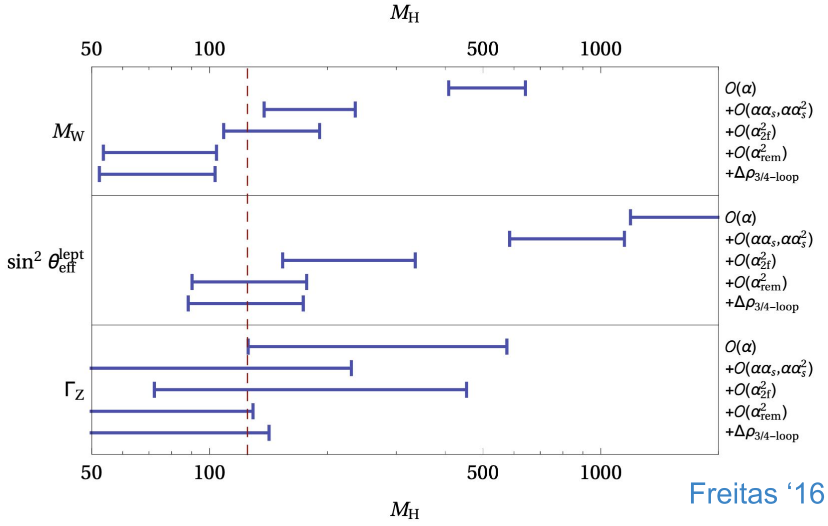


Lisong Chen, in collaboration with Prof. Ayres Freitas.

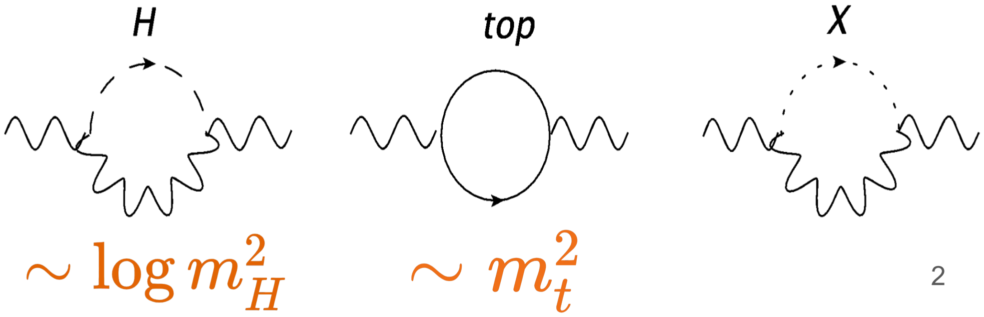
Precision Test of Electroweak Precision Observables (EWPOs)

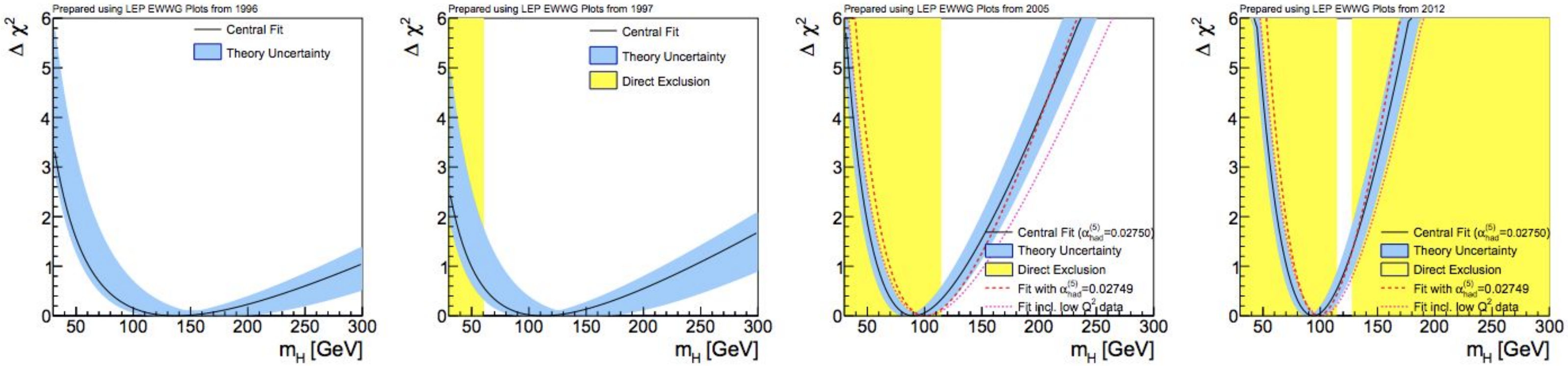
- The Standard Model can only be tested by considering higher-order corrections when confronting experimental high precision data.

- New physics unknown by experiments directly might be sensitive to quantum corrections.



Freitas '16

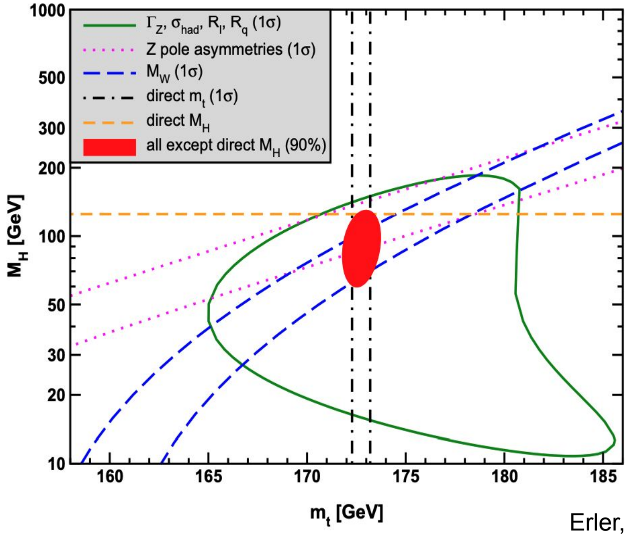




J.Erler '19

- A way of checking the inner consistency of the SM.

e.g. Constraints of $m_H - m_t$ by various set of EWPOs.



Erler, Schott'20

EWPOs

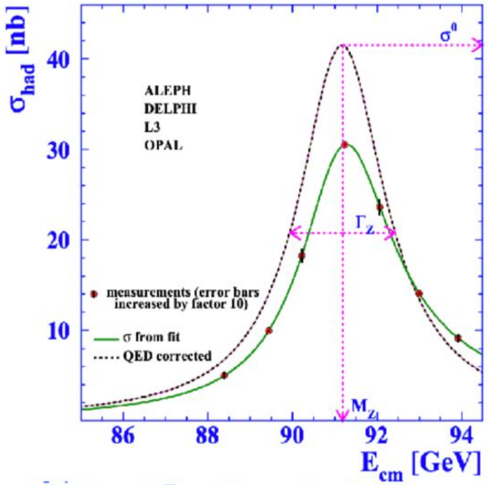
M_W Measured via W-boson pair production

Γ_Z By fitting the cross section of $e^+e^- \to f\bar{f}$ $\quad \Gamma_Z = \sum_f \Gamma_{f\bar{f}}$

$$\sigma_{hard} = \sigma_Z + \sigma_\gamma + \sigma_{\gamma Z} + \sigma_{box}$$

$$\sigma_Z = \sigma_{f\bar{f}}^{\text{peak}} \frac{s\Gamma_Z^2}{(s-m_Z^2)^2 + s^2\frac{\Gamma_Z^2}{m_Z^2}},$$

$$\text{where} \quad \sigma_{f\bar{f}}^{\text{peak}} = \frac{1}{\mathcal{R}_{QED}}\sigma_{f\bar{f}}^0, \quad \text{and} \quad \sigma_{f\bar{f}}^0 = \frac{12\pi}{m_Z^2}\frac{\Gamma_{ee}\Gamma_{f\bar{f}}}{\Gamma_Z^2}$$



$$\sigma[e^+e^- \to f\bar{f}] = \mathcal{R}_{QED}(s,s') \otimes \sigma_{hard}(s')$$

$\sin^2\theta^l_{eff}$ Extracted from the measured asymmetries, which are defined based on $e^+e^- \to f\bar{f}$ at Z resonance peak.

$$A^f_{\text{FB}} = \frac{\sigma_f(\theta < \frac{\pi}{2}) - \sigma_f(\theta > \frac{\pi}{2})}{\sigma_f(\theta < \frac{\pi}{2}) + \sigma_f(\theta > \frac{\pi}{2})} \equiv \frac{3}{4}\mathcal{A}_e\mathcal{A}_f,$$

$$A^f_{\text{LR}} = \frac{\sigma_f(P_e < 0) - \sigma_f(P_e > 0)}{\sigma_f(P_e < 0) + \sigma_f(P_e > 0)} \equiv \mathcal{A}_e|P_e|$$

$$\mathcal{A}_f = 2\frac{g_{V_f}/g_{A_f}}{1 + (g_{V_f}/g_{A_f})^2} = \frac{1 - 4|Q_f|\sin^2\theta^f_{\text{eff}}}{1 - 4|Q_f|\sin^2\theta^f_{\text{eff}} + 8(|Q_f|\sin^2\theta^f_{\text{eff}})^2} \quad (f = \ell, b, \ldots)$$

● How Far Have We Got?

M_W

- ° mixed QCD/EW 2-loop corrections √. Djouai, Verzegnassi'87;Djouadi'88; Kniehl,Kühn, Stuar'99;Kniehl,Sirlin'93;Djouadi,Gambino'94
- ° complete EW 2-loop corrections √. Freitas, Hollik, Walter, Weiglein'00;Awramik,Czakon '02;Onishchenko,Vertin '02
- ° improvements by 3-loop and 4-loop $\Delta\rho$ √. Avdeev et al.'94; Chetyrkin, Kühn, Steinhauser '95; v.d.Bij et al. '05; Schröder, Steinhauser '06; Faisst et al. '03; Boughezal, Tausk, v.d.Bij '05

$$\Longrightarrow \qquad \Delta M_W \sim 4 \text{ MeV}$$

Γ_Z

- ° complete EW 1-loop and fermionic 2-loop √. Freitas'13'14
- ° mixed QCD/EW 2-loop corrections √. Djouai, Verzegnassi'87;Halzen Kniehl'91; Djouadi,Gambino'94; Chetyrkin, Kühn'96; Fleischer et al. '92
- ° improvements by 3-loop and 4-loop $\Delta\rho$ √. Avdee et al. 94'; v.d.Bij et al. '05;Schröder, Steinhauser'06;Faisst et al.'03;Boughezal, Tausk, v.d.Bij '05
- ° EW complete 2-loop corrections . √. $\mathcal{O}(\alpha_{bos}^2)$ Dubovyk, Freitas,Gluza, Riemann Usovitsch. '18

$$\Longrightarrow \qquad \Delta\Gamma_Z \sim 0.5 \text{ MeV}$$

$\sin^2\theta^l_{eff}$

- ° mixed QCD/EW 2-loop and 3-loop $\Delta\rho$ Corrections as for M_W
- ° EW complete 2-loop corrections √. Awramik, Czakon, Freitas, Weiglein '04; Hollik,Meier, Uccirati'05;Awramik,Czakon, Freitas '06

$$\Longrightarrow \qquad \sin^2\theta^l_{eff} \sim 4.5 \times 10^{-5}$$

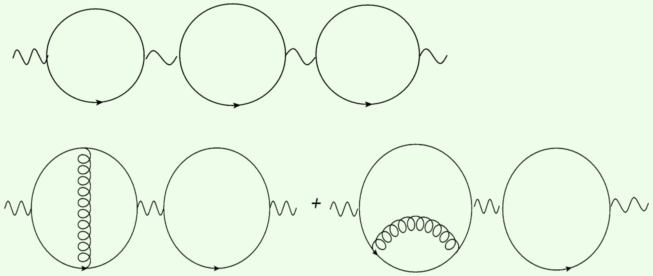
Experimental Uncertainties Given by Future Electron-Positron Colliders

	Global fits at LEP/SLD/LHC	Current intrinsic theo. error	CEPC	FCC-ee	ILC/GigaZ
M_W[MeV]	12	$4(\alpha^3, \alpha^2\alpha_s)$	1	$0.5 \sim 1$	2.5
Γ_Z[MeV]	2.3	$0.4(\alpha^3, \alpha^2\alpha_s, \alpha\alpha_s^2)$	0.5	0.1	1.0
$\sin^2\theta_{\text{eff}}^f\,[10^{-5}]$	16	$4.5(\alpha^3, \alpha^2\alpha_s)$	2.3	0.6	1

- Due to the lack of knowledge of theory error estimation, we need $|\Delta^{th}| \ll |\Delta^{obs}|$
- Current theoretical predictions are inadequate.
- The calculation of the next perturbative order $\mathcal{O}(\alpha^3, \alpha^2\alpha_s)$ for the EWPOs will be necessary!!

Why Leading Fermionic Corrections?

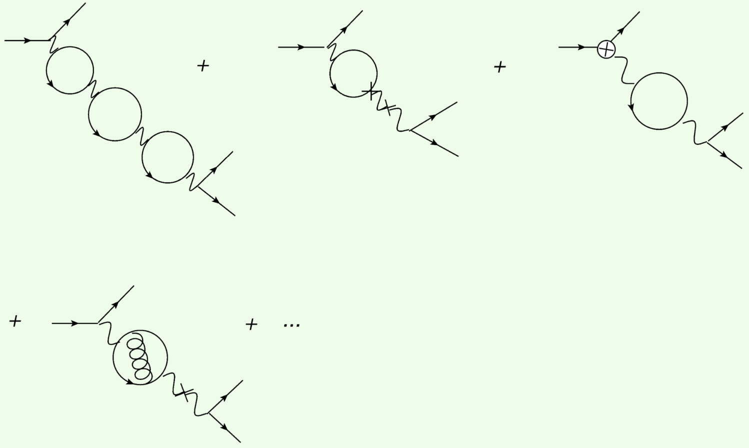
- Enhancement by power of Top Mass.
- Enhancement by power of flavor numbers N_f



Considerably the leading numerical contribution!

Computing EWPOs

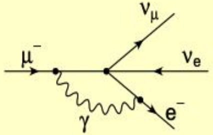
- G_μ is determined from measuring muon decay after subtracting QED corrections within 4-Fermi theory.
- Then move on to the SM, G_μ receives corrections depicted on the right hand side. One can then use such a relation to predict W-boson mass.



G_F from μ decay in Fermi Model

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} F\left(\frac{m_e^2}{m_\mu^2}\right)(1+\Delta q)$$

QED corrections (2-loop)



Ritbergen, Stuart '98
Pak, Czarnecki '08

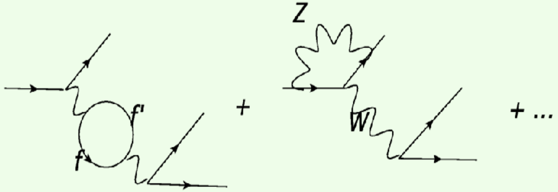
G_F decay in Standard Model

$$\frac{G_F^2}{\sqrt{2}} = \frac{e^2}{8 s_W^2 M_W^2}(1+\Delta r)$$

electroweak corrections

Freitas '18talk

$$\Delta r(M_W, M_Z, M_H, \dots) =$$



One gets an implicit relation between W-boson mass and G-Fermi:

$$\overline{M_W}^2 = \overline{M_Z}^2\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\alpha\pi}{\sqrt{2}G_\mu \overline{M_Z}^2}(1+\Delta r)}\right)$$

- We have seen parity-violating asymmetry can be determined by effective weak-mixing angle $\sin^2\theta^f_{eff}$. It relates to the ratio between dressed vector and axial-vector coupling.

- Using the decay rate equation in terms of dressed vector and axial-vector couplings. We can derive the total and partial width of Z-boson.

Using optical theorem

$$\Im\Sigma_Z = \frac{1}{3\overline{M}_Z}\sum_f \sum_{spins}\int d\Phi(|g_V^f|^2 + |g_A^f|^2)$$

Plugging what we have from OS condition in complex pole scheme.

$$\overline{\Gamma}_Z = \frac{N_c^f}{12\pi\overline{M}_Z} C_Z(\mathcal{R}_V^f|g_V^f|^2 + \mathcal{R}_A^f|g_A^f|^2)$$

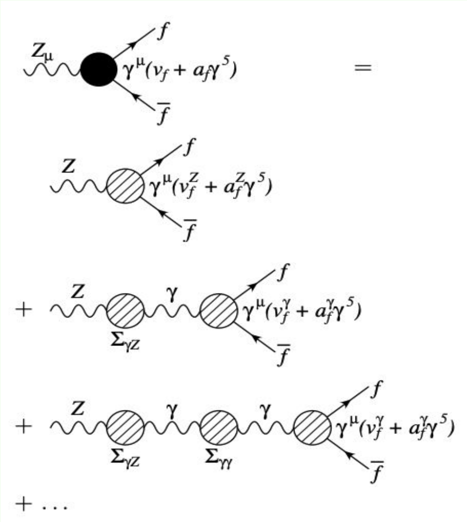
where C_Z features all self-energy contributions, and $\mathcal{R}^f_{V,A}$ feature final-state QCD and QED corrections. Here for closed fermionic loops we set them to 1.

$$\sin^2\theta^f_{eff} = \frac{1}{4|Q_f|}\left(1 - \Re\frac{g_V^f}{g_A^f}\right)_{s=\overline{M}_Z^2}$$

$$g_V^f = Z_e(v_f^Z - Q_f\sqrt{Z_{\gamma Z}}) - v_f^\gamma\frac{\hat{\Sigma}_{\gamma Z}}{s+\hat{\Sigma}_{\gamma\gamma}}$$

$$g_A^f = Z_e a_f^Z - a_f^\gamma\frac{\hat{\Sigma}_{\gamma Z}}{s+\hat{\Sigma}_{\gamma\gamma}}$$

Decompstion of the effective Zff vertex



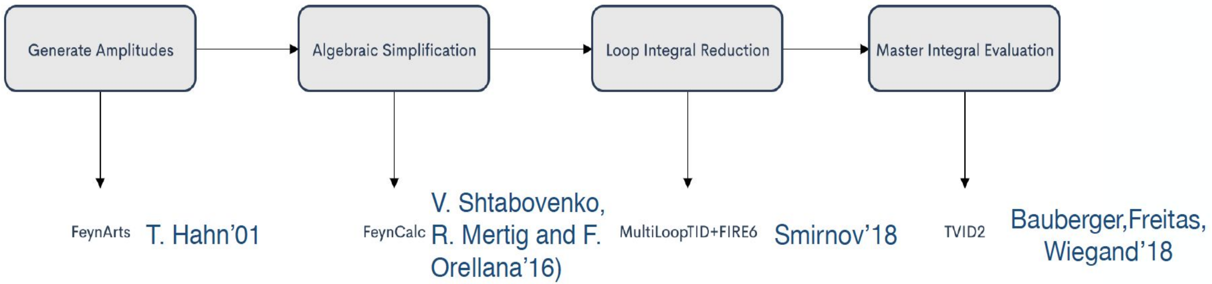
Technical Aspects

- ❑ In pure EW case, All loop integrals can be written as 1-loop scalar master integrals and their derivatives up to second order.
- ❑ Exact agreement at 2-loop was found comparing to previous work (hep-ph:004091;0202131;0407317;13102256) , except one missing term as the second term in the following:

$$\mathfrak{R}\Sigma'_{ZZ(2)}(s) - \frac{d}{ds}\left(\frac{\mathfrak{I}\Sigma^2_{\gamma Z(1)}(s)}{s}\right)$$

of which numerical impact shall be investigated.

- ❑ Unlike pure EW, mixed EW-QCD at 3-loop order features non-unique master integral (2-loop) basis. (difficult to cross-check symbolically)
- ❑ Integral reduction is non-trivial. (IBP and technique from G.Weiglein,R.Scharf et.al.hep-ph:9310358 were adopted in this work in parallel)
- ❑ The derivative of 2-loop master integral is needed.
- ❑ Both cases have been carried out in two independent implementations.



Numeric and Algebraic Cross-check

- Two schemes of renormalization (on shell and on shell+ $\overline{MS}$)
- Due to the ambiguity in the choice of master integrals, only the UV part can be checked algebraically. The finite parts are carried out numerically in TVID2.1.

- Some $\mathcal{O}(4-D)$ coefficients from scalar one-loops' Laurent series have been computed.

Numerical Inputs

- We turn-off the CKM mixing due to its negligible numerical impact.
- For $\overline{MS}$ scheme, we change out top mass into $m_t(\mu = m_t) = 163.229\ \mathrm{GeV}$
- Due to the internal relation between G_μ and W-boson mass, one can treat either one as induced from another. (Usually W-boson mass is predicted from G_μ due to high precision G given by measurement.)

$$
\begin{aligned}
\left.\begin{aligned} M_\mathrm{Z} &= 91.1876\ \mathrm{GeV} \\ \Gamma_\mathrm{Z} &= 2.4952\ \mathrm{GeV} \end{aligned}\right\} &\Rightarrow \overline{M}_\mathrm{Z} = 91.1535\ \mathrm{GeV} \\
\left.\begin{aligned} M_\mathrm{W} &= 80.358\ \mathrm{GeV} \\ \Gamma_\mathrm{W} &= 2.089\ \mathrm{GeV} \end{aligned}\right\} &\Rightarrow \overline{M}_\mathrm{W} = 80.331\ \mathrm{GeV} \\
M_t &= 173.0\ \mathrm{GeV} \\
M_{f\neq t} &= 0 \\
\alpha_s &= 0.1179 \\
\alpha &= 1/137.035999084 \\
\Delta\alpha &= 0.05900 \\
G_\mu &= 1.1663787 \times 10^{-5}\ \mathrm{GeV}^{-2}
\end{aligned}
$$

Numerical Results

❖ On Shell Scheme

❑ On-shell in pure EW case

❑ On-shell in mixed EW-QCD case

❑ the parametric shift of G_μ can goes into W-boson mass.

❑ Similarly, one gets effective weak mixing angle and Z widthith leading W-boson mass shift

	$\Delta\overline{M}_{\mathrm{W}}$ (MeV)	$\Delta\sin^2\theta_{\mathrm{eff}}$	$\Delta'\sin^2\theta_{\mathrm{eff}}$	$\Delta\overline{\Gamma}_{\mathrm{tot}}$ [MeV]	$\Delta'\overline{\Gamma}_{\mathrm{tot}}$ [MeV]
$\mathcal{O}(\alpha^3)$	-0.389	1.34×10^{-5}	2.09×10^{-5}	0.331	0.255
$\mathcal{O}(\alpha^2\alpha_s)$	1.703	1.31×10^{-5}	-1.98×10^{-5}	-0.103	0.229
Sum	1.314	2.65×10^{-5}	0.11×10^{-5}	0.228	0.484

❖ Comparing between two schemes

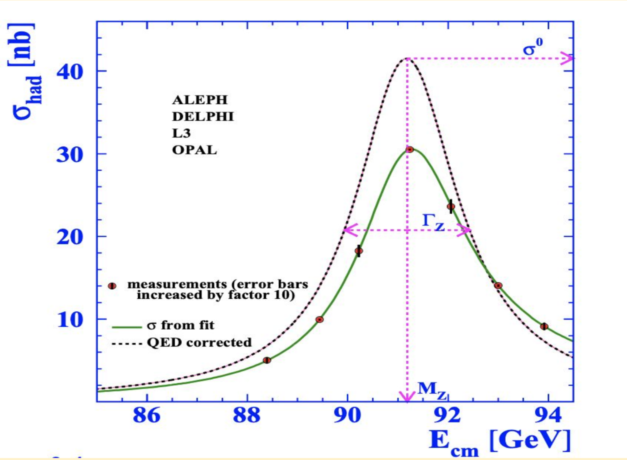
	on-shell M_t		$\overline{\mathrm{MS}}$ m_t	
	$\mathcal{O}(\alpha^2)$	$\mathcal{O}(\alpha^2\alpha_s)$	$\mathcal{O}(\alpha^2)$	$\mathcal{O}(\alpha^2\alpha_s)$
Δr $[10^{-4}]$	7.85	−1.09	7.56	−0.50
$\Delta \sin^2\theta^f_{\mathrm{eff}}$ $[10^{-5}]$	30.98	1.31	31.18	0.75
$\Delta\overline{\Gamma}_\ell$ [MeV]	0.2412	−0.0157	0.2284	−0.0003
$\Delta\overline{\Gamma}_\nu$ [MeV]	0.4145	−0.0002	0.4152	0.0009
$\Delta\overline{\Gamma}_{\mathrm{d}}$ [MeV]	0.6666	−0.0049	0.6780	−0.0018
$\Delta\overline{\Gamma}_{\mathrm{u}}$ [MeV]	0.4964	−0.0203	0.4911	−0.0029
$\Delta\overline{\Gamma}_{\mathrm{tot}}$ [MeV]	4.951	−0.103	4.947	−0.0093

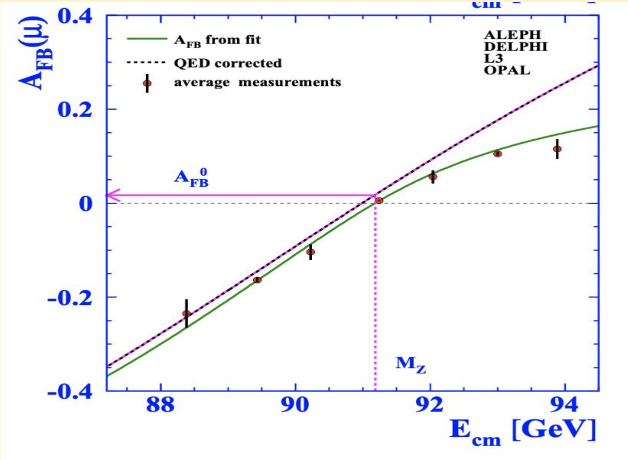
❑ $\overline{MS}$ Top mass must be used at previous order $\mathcal{O}(\alpha^2)$ when using $\overline{MS}$ renormalization scheme for top mass.

❑ A better convergence behavior from $\overline{MS}$ is observed. Also the numerical size of corrections at given order gets reduced comparing to on shell scheme.

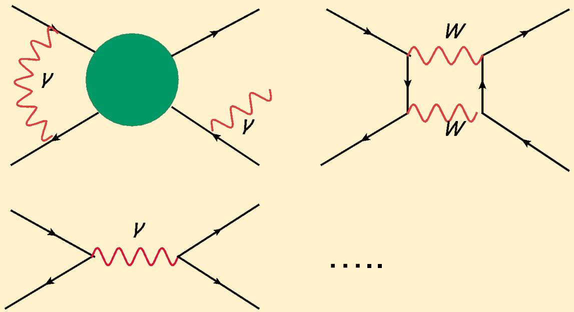
How to connect precision observables with measurements?

- EWPOs are “pseudo-observables”.
- Most of them connect to the Z boson lineshape and asymmetries. ---need theory input to extract. (Fixed-order+MC tools)





LEP EWWG ‘05



shall be removed in determining EWPOs.

- In LEP/SLC era

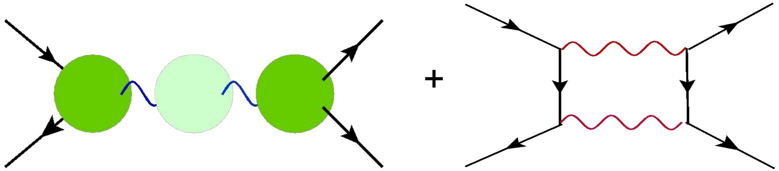
- ZFITTER(D. Bardin et al) ,TOPAZ0(G.Passarino et al), and BHM/WOH(W.Hollik et al, not public) are developed for computing relevant observables.

- In future electron-positron colliders' era

- Formally gauge invariant setup .
- Extendability.

➔ Motivates this project! (GRIFFIN: Gauge-Resonance-In-Four-Fermion-INteraction)

Gauge invariance can easily be violated.



$$\mathcal{A}(s) = \frac{Z_i^{(0)} Z_f^{(0)} + Z_i^{(1)} Z_f^{(0)} + Z_i^{(0)} Z_f^{(1)}}{s - M_Z^2 - \Sigma_Z^{(1)}(s)} + B^{(1)}(s)$$

Dyson-resumming truncated 1-PI self-energy.

An *exact* S-matrix description gives

$$\mathcal{A} = \frac{R}{s - s_0} + B(s)$$

where the pole is strictly defined:

$$s_0 - M_0^2 - \Sigma(s_0) = 0$$

Renormalization can only affect the renormalized mass but not the location of the pole.

- EWPOs, like Z-bosn mass, are defined gauge-invariant.

- Need a gauge invariant theoretical description up to any given accuracy to compare with the measured Z-resonance lineshape, where all EWPOs are extracted from (R.G. Stuart 91).

- Gives a model-independent profile of four-fermion interaction with gauge resonance.

$$s - M_0^2 - \Sigma(s) = \frac{1}{F(s)}(s - s_0)$$

The pole has to be located at the lower half plane of complex s, defined as

$$s_0 = M_Z^2 - iM_Z\Gamma_Z$$

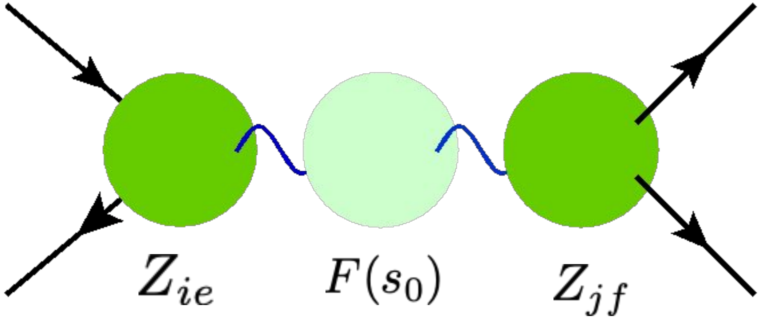
Performing Laurent expansion around the pole:

$$\mathcal{A}(s) = \frac{\overbrace{Z_i(s_0)F(s_0)Z_f(s_0)}^{R}}{s - s_0} + S + (s - s_0)S' + B(s)$$

Resonant Part

$$s_0 = M_Z^2 - iM_Z\Gamma_Z$$

$$\mathcal{A}_{ij} = \frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0)S'_{ij}$$



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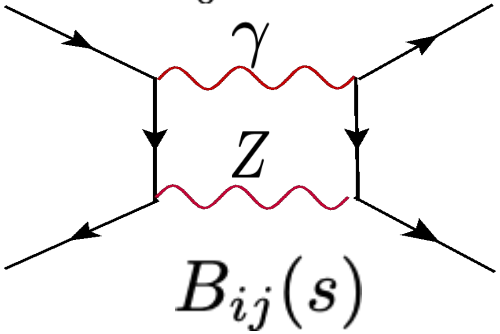
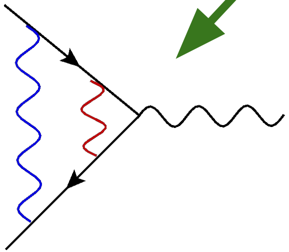
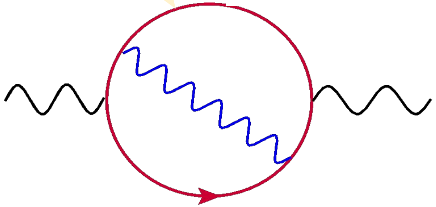


$$R_{ij} = Z_{ie}Z_{jf}F(s_0)$$

Freitas, Hollik, Walter, Weiglein'00;Amramik, Czakon '02;Onishchenko, Vertin '02; Dubovyk, Freitas, Gluza, Riemann Usovitsch '18; Freitas '14; 13Awramik, Czakon, Freitas, Weiglein '04;Hollik, Meier, Uccirati'05; Awramik,Czakon, Feritas '06...

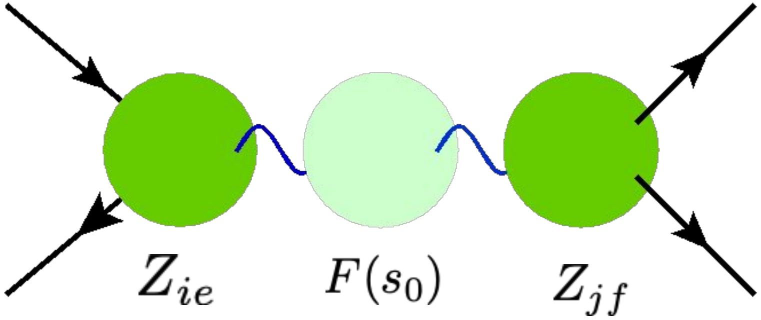
For NNLO





... already exists!

$$s_0 = M_Z^2 - iM_Z\Gamma_Z \qquad \mathcal{A}_{ij} = \frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0)S'_{ij}$$



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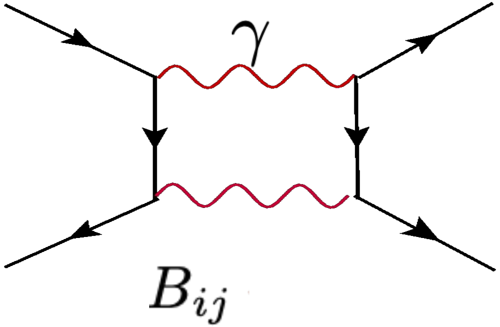
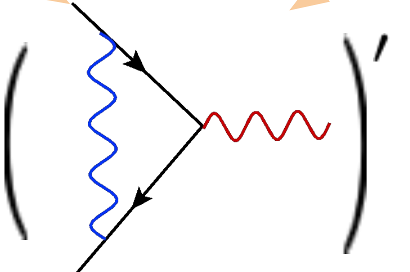


$$S_{ij} = Z'_{ie}Z_{jf}F(s_0) + Z_{ie}Z'_{jf}F(s_0) + Z_{ie}Z_{jf}F'(s_0) + B_{ij} \quad ..$$

For NNLO :



$+ \quad \ldots$

the derivative of triangle-loop needs to be carried out.

Non-Resonant and QED background.

- ❑ QED contributions are fully taken care by MC tools (e.g. CEEX S. Jadach, B.F.L.Ward,Z.Was).

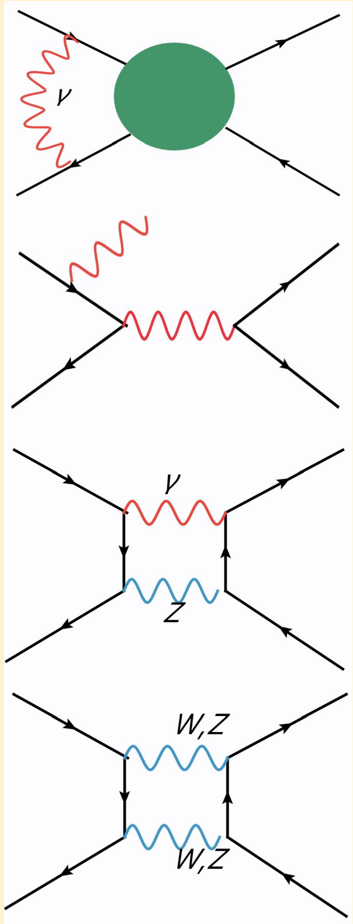
  Most sizable QED effect comes from ISR.

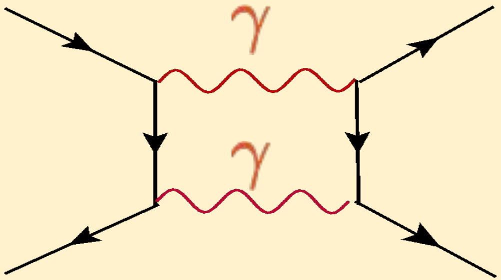
  Forms a gauge-invariant subset together with photonic boxes.

- ❑ photon-Z boxes needs special care since they also contribute to resonant part.

$$B_{\gamma Z} = \frac{B_{ij}^{R} + B_{\gamma Z}^{RL}\log\left(1-\frac{s}{s_0}\right)}{s-s_0} + B_{\gamma Z}^{S} + \ldots$$

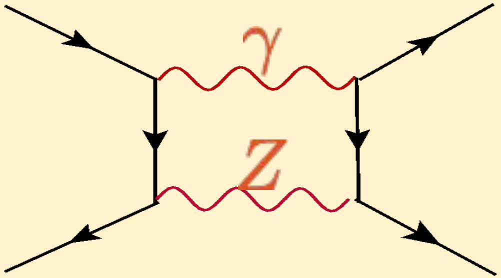
- ❑ Other finite EW boxes,as pure background, are computed.





- IR divergence are handled in MC (e.g. CEEX S. Jadach, B.F.L.Ward,Z.Was).

$$B_{\gamma\gamma} \rightarrow B_{\gamma\gamma} - F_{IR}(t, u, m_\gamma)$$

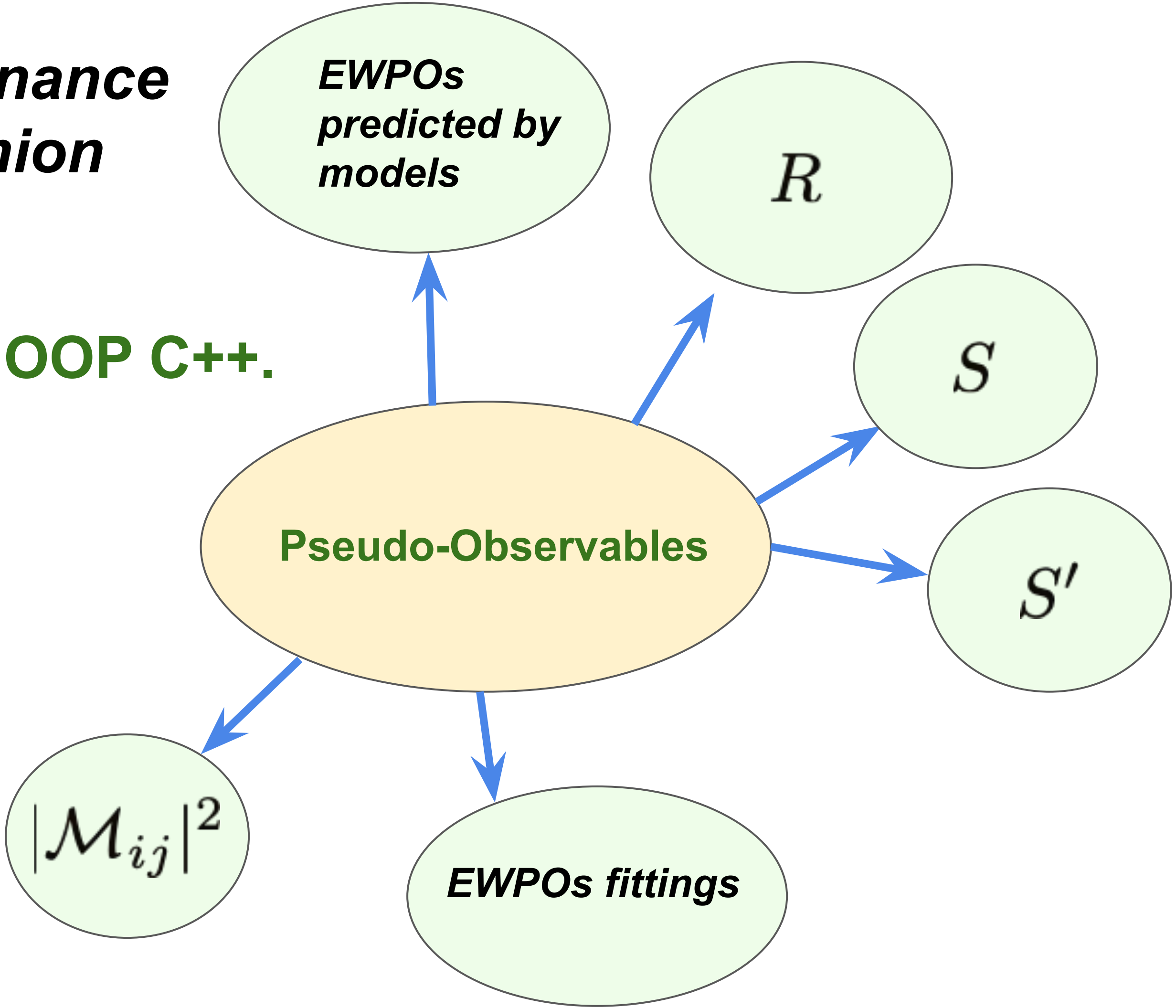


- $B_{\gamma Z}$ features a singular pole $s - s_0$, cancelled by considering ISR*FSR interference, which is also taken care by MC(S.Jadach, B.F.L.Ward,Z.Was, 00')

$$B_{\gamma Z} \rightarrow B_{\gamma Z} - F_{IR}^{\gamma Z} - F_{IFI}(s, t, u)$$

Gauge Resonance In Four-Fermion INteraction

Realization in OOP C++.



- ❑ Vertex form factors and their derivatives, Boxes' amplitudes on V-A basis are done. IR subtraction of photon-exchange boxes has been taken care of.
- ❑ Implementation on C++ is on-going.

- ❑ Direct use for the SM NNLO precision test at the CEPC, FCC-ee, ILC/GigaZ,... And can extend to orders beyond NNLO consistently.

- ❑ Becomes handy in study of EW constraint of BSM, SMEFT. i.e. contributions given by BSM models etc. can also be implemented in our software framework.

- ❑ Transplant to other 4-fermion interaction processes where gauge resonance study is also important. (e.g. Drell-Yan at the HL-LHC)

Summary and Outlook

- EWPOs measurements at future electron-positron colliders require higher order corrections beyond 2-loop level.

- We present the results for contributions with maximal closed fermionic loops at order $\mathcal{O}(\alpha^3, \alpha^2\alpha_s)$.

- The new results are smaller than one would naïvely expect (some accidental cancellations between diagrams) --- Other missing three-loop contributions are needed.

- Linking EWPOs to measurements requires model-independent and gauge-invariant description of key processes (ee->Z->ff, in this case).

- A new package is developing for this purpose. Stay tuned!

THANK YOU.

Backup Slides

Renormalization

❖ Two schemes are considered.

❑ On-Shell(OS) with complex pole mass $\mathcal{O}(\alpha^3, \alpha^2\alpha_s)$
❑ OS+ $\overline{MS}$ for top mass $\mathcal{O}(\alpha^2\alpha_s)$

➔ Complex pole mass is a must for gauge-invariance.

➔ OS top mass closely connects to experiments, while suffers from renormalon issue and non-perturbative QCD. $\overline{MS}$ top mass is preferable from theory point of view.

➔ Top masses calculated from two schemes related by a finite transformation.

➔ No asymptotic massive gauge boson, hence field renormalization of Z,W can be neglected.

complex-pole

$$s_0 \equiv \overline{M^2} - i\overline{M\Gamma}$$

The inverse dressed propagator (W/Z/H)

$$D(p^2) = p^2 - \overline{M^2} - \delta Z(p^2 - \overline{M^2}) + \Sigma(p^2) - \delta\overline{M^2}$$

yield mass counter term and widths

$$\delta\overline{M^2} = \frac{Re\Sigma(\overline{M^2} - i\overline{M\Gamma})}{Z} \qquad \overline{\Gamma} = \frac{Im\Sigma(\overline{M^2} - i\overline{M\Gamma})}{Z\overline{M}}$$

mass ratio between two schemes

$$\frac{M^{OS}}{M^{\overline{MS}}} = 1 + \alpha_s C_F \frac{3\log {M^{OS}}^2/\mu^2 - 4}{4\pi} + \mathcal{O}(\alpha_s^2)$$

Ward-Identity yields $\quad Z_e = \left(\sqrt{Z_{\gamma\gamma}} + \frac{\sin\theta_W}{\cos\theta_W}\sqrt{Z_{Z\gamma}}\right)^{-1}$

Weak-Mixing Angle $\quad s_W + \delta s_W = \sqrt{1 - \frac{\overline{M_W^2} + \delta\overline{M_W^2}}{\overline{M_Z^2} + \delta\overline{M_Z^2}}}$

- ❖ the self-energy function $\Sigma(p^2)$ composed by 1-PI at desired order. Only transverse part contributes, longitudinal part cancels against unphysical amplitude. (Slavnov-Taylor Identity)

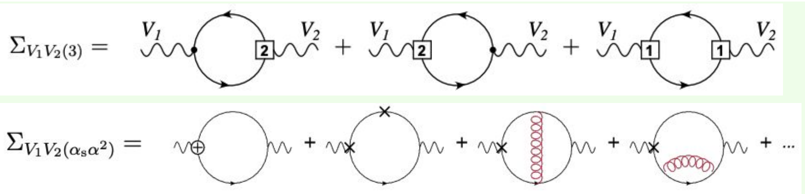
- ❖ $\gamma - Z$ mixing is included. Mixing counterterms are defined at poles

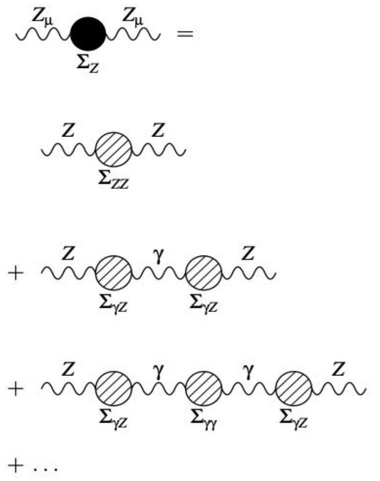
$$\hat{\Sigma}_{\gamma Z}(0) = 0 \qquad \Re\hat{\Sigma}_{\gamma Z}(\overline{M_Z}^2 - i\overline{M_Z}\,\overline{\Gamma_Z}) = 0$$

- ❖ Different Breit-Wigner forms
- ❖ In Experiment / In Theory

$$\sigma \sim \frac{1}{(s-M^2)^2+s^2\Gamma^2/M^2} \qquad \sigma \sim \frac{1}{(s-\overline{M}^2)^2+\overline{\Gamma}^2\,\overline{M}^2}$$

$$\overline{M} = M/\sqrt{1+\Gamma^2/M^2} \qquad \overline{\Gamma} = \Gamma/\sqrt{1+\Gamma^2/M^2}$$





$$\Sigma_Z(p^2) = \Sigma_{ZZ}(p^2) - \frac{[\hat{\Sigma}_{\gamma Z}(p^2)]^2}{p^2 + \hat{\Sigma}_{\gamma\gamma}(p^2)},$$

$$\hat{\Sigma}_{\gamma Z}(p^2) = \Sigma_{\gamma Z}(p^2) + \tfrac{1}{2}\delta Z^{Z\gamma}(p^2 - \overline{M}_Z^2 - \delta\overline{M}_Z^2) + \tfrac{1}{2}\delta Z^{\gamma Z}p^2,$$

$$\hat{\Sigma}_{\gamma\gamma}(p^2) = \Sigma_{\gamma\gamma}(p^2) + \tfrac{1}{4}(\delta Z^{Z\gamma})^2(p^2 - \overline{M}_Z^2 - \delta\overline{M}_Z^2).$$

- Charge renormalization needs a special care. We need α around $q^2 \sim M_Z^2$, while it's defined at Thomson limit ($q^2 \sim 0$).
- light-quark masses are inherently ill-defined in EW Lagrangian due to non-perturbative feature at the given mass scale $q^2 \sim m_{u,d,s}^2$.
- Alternative methods needs to apply to carry out the contribution given by light quarks. Dispersion relation is the one frequently use. Other possible ways: Lattice QCD or Bhabha scattering.

Charge Counterterms

Pure EW

$$\delta Z_e^{(3)} = \frac{5}{2}\delta Z_e^{(1)}$$

Mixed EW-QCD

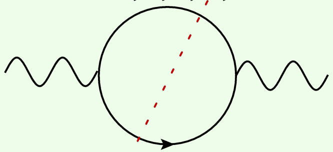
$$\delta Z_e^{(3)} = 3\delta Z_{e(\alpha)}\delta Z_{e(\alpha_s\alpha)}$$

$\delta Z_e = -\frac{1}{2}\delta Z_{\gamma\gamma} = \frac{1}{2}\Sigma'_{\gamma\gamma}(0)$ at one-loop level

$$\Sigma'_{\gamma\gamma}(0) \equiv \Pi(0) = \sum_f \frac{\alpha N_c Q_f^2}{3\pi}\left(\frac{2}{4-D} - \gamma_E - \log\frac{m_f^2}{4\pi\mu^2}\right)$$

$$\hat{\Pi}(s = M_Z^2) = \Pi(0) - \Re\Pi(M_Z^2) = \underbrace{\Pi^{lf}(0) - \Pi^{lf}(M_Z^2)}_{\Delta\alpha = \Delta\alpha_{lep} + \Delta_{had}} + \hat{\Pi}^{top}(M_Z^2)$$

Dispersion relation reads:



$$\Delta_{had} = -\frac{\alpha}{3\pi}s\int_{4m_\pi^2}^{\infty} ds' \frac{R_{\gamma\gamma}(s')}{s'(s'-s-i\epsilon)}\Big|_{s=M_Z^2}$$

$$R_{\gamma\gamma}(s') = \frac{\sigma(e^+e^- \to \gamma^* \to hadrons)}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

- non-perturbative quantity, apply to ALL order.
- Good precision ~ 0.0001

- Mass counterterms: By assuming $\Gamma_{W,Z}/M_{W,Z} \sim \mathcal{O}(\alpha)$, the imaginary part contributes to counterterms.

- Pure EW corrections at 3-loop order

$$\begin{aligned}
\delta\overline{M}^2_{Z(3)} &= \mathrm{Re}\,\Sigma_{ZZ(3)}(\overline{M}^2_Z) + [\mathrm{Im}\,\Sigma_{ZZ(2)}(\overline{M}^2_Z)]\,[\mathrm{Im}\,\Sigma'_{ZZ(1)}(\overline{M}^2_Z)] \\
&+ [\mathrm{Im}\,\Sigma_{ZZ(1)}(\overline{M}^2_Z)]\Big\{\mathrm{Im}\,\Sigma'_{ZZ(2)}(\overline{M}^2_Z) - [\mathrm{Im}\,\Sigma'_{ZZ(1)}(\overline{M}^2_Z)]\,[\mathrm{Re}\,\Sigma'_{ZZ(1)}(\overline{M}^2_Z)] \\
&\qquad - \tfrac{1}{2}[\mathrm{Im}\,\Sigma_{ZZ(1)}(\overline{M}^2_Z)]\,[\mathrm{Re}\,\Sigma''_{ZZ(1)}(\overline{M}^2_Z)] \\
&\qquad - \frac{\mathrm{Im}\,\Sigma_{\gamma Z(1)}(\overline{M}^2_Z)}{\overline{M}^2_Z}[2\,\mathrm{Re}\,\Sigma'_{\gamma Z(1)}(\overline{M}^2_Z) + \delta Z^{\gamma Z}_{(1)} + \delta Z^{Z\gamma}_{(1)}]\Big\} \\
&+ \frac{\mathrm{Im}\,\Sigma_{\gamma Z(1)}(\overline{M}^2_Z)}{\overline{M}^2_Z}\Big\{2\,\mathrm{Im}\,\Sigma_{\gamma Z(2)}(\overline{M}^2_Z) - \frac{\mathrm{Im}\,\Sigma_{\gamma Z(1)}(\overline{M}^2_Z)}{\overline{M}^2_Z}[\mathrm{Im}\,\Sigma_{\gamma\gamma(1)}(\overline{M}^2_Z)]\Big\} \\
&+ \tfrac{1}{2}\overline{M}^2_Z\,\delta Z^{\gamma Z}_{(1)}\,\delta Z^{\gamma Z}_{(2)}\,.
\end{aligned}$$

Mixed EW-QCD corrections

$$\begin{aligned}
\delta\overline{M}^2_{Z(\alpha_s\alpha^2)} &= \mathrm{Re}\,\Sigma_{ZZ(\alpha_s\alpha^2)}(\overline{M}^2_Z) + [\mathrm{Im}\,\Sigma_{ZZ(\alpha_s\alpha)}(\overline{M}^2_Z)]\,[\mathrm{Im}\,\Sigma'_{ZZ(\alpha)}(\overline{M}^2_Z)] \\
&+ [\mathrm{Im}\,\Sigma_{ZZ(\alpha)}(\overline{M}^2_Z)]\,[\mathrm{Im}\,\Sigma'_{ZZ(\alpha_s\alpha)}(\overline{M}^2_Z)] \\
&+ \frac{2}{\overline{M}^2_Z}[\mathrm{Im}\,\Sigma_{\gamma Z(\alpha_s\alpha)}(\overline{M}^2_Z)]\,[\mathrm{Im}\,\Sigma_{\gamma Z(\alpha)}(\overline{M}^2_Z)] + \frac{1}{2}\overline{M}^2_Z\,\delta Z^{\gamma Z}_{(\alpha)}\,\delta Z^{\gamma Z}_{(\alpha_s\alpha)}\,.
\end{aligned}$$

- Total width of Z-boson at 3-loop order (Pure EW)

$$\begin{aligned}
\overline{\Gamma}_Z = \frac{1}{\overline{M}_Z}\Big\{&\mathrm{Im}\,\Sigma_{Z(1)} + \mathrm{Im}\,\Sigma_{Z(2)} - (\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Re}\,\Sigma'_{Z(1)}) \\
&+ \mathrm{Im}\,\Sigma_{Z(3)} - (\mathrm{Im}\,\Sigma_{Z(2)})(\mathrm{Re}\,\Sigma'_{Z(1)}) \\
&\quad + (\mathrm{Im}\,\Sigma_{Z(1)})\big[(\mathrm{Re}\,\Sigma'_{Z(1)})^2 - \mathrm{Re}\,\Sigma'_{Z(2)} - \tfrac{1}{4}(\delta Z^{\gamma Z}_{(1)})^2 - \tfrac{1}{2}(\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(1)})\big] \\
&+ \mathrm{Im}\,\Sigma_{Z(4)} - (\mathrm{Im}\,\Sigma_{Z(3)})(\mathrm{Re}\,\Sigma'_{Z(1)}) \\
&\quad + (\mathrm{Im}\,\Sigma_{Z(2)})\big[(\mathrm{Re}\,\Sigma'_{Z(1)})^2 - \mathrm{Re}\,\Sigma'_{Z(2)} - \tfrac{1}{4}(\delta Z^{\gamma Z}_{(1)})^2 - (\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(1)})\big] \\
&\quad + (\mathrm{Im}\,\Sigma_{Z(1)})\big[-(\mathrm{Re}\,\Sigma'_{Z(1)})^3 + 2(\mathrm{Re}\,\Sigma'_{Z(2)})(\mathrm{Re}\,\Sigma'_{Z(1)}) - \mathrm{Re}\,\Sigma'_{Z(3)} \\
&\qquad - \tfrac{1}{2}\delta Z^{\gamma Z}_{(1)}\,\delta Z^{\gamma Z}_{(2)} + \tfrac{1}{2}(\mathrm{Re}\,\Sigma'_{Z(1)})(\delta Z^{\gamma Z}_{(1)})^2 - \tfrac{1}{2}(\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(2)}) \\
&\qquad + \tfrac{3}{2}(\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Re}\,\Sigma'_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(1)}) + \tfrac{1}{6}(\mathrm{Im}\,\Sigma_{Z(1)})^2(\mathrm{Re}\,\Sigma'''_{Z(1)})\big]\Big\}_{s=\overline{M}^2_Z}\,.
\end{aligned}$$

- Also one will obtain unstable particles' total widths by imposing on shell condition. (as a consequence of optical theorem)

Numerical Evaluation of Master Integrals

- ❑ Difficult to evaluated master integrals beyond two-loop analytically in general.
- ❑ Various mass scales in denominators.

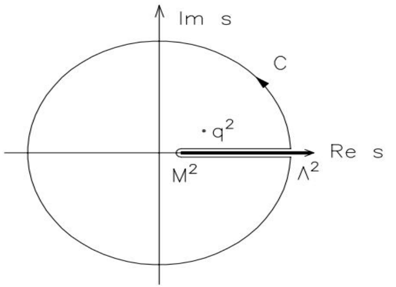
Dispersion Relation

- ❑ Amplitdues (form factors) can be reconstructed from the knowledge of their absorptive part along the branch cut.

- ❑ Excellent stability and convergence for two-loop self-energy (S.Bauberger et al. 95).

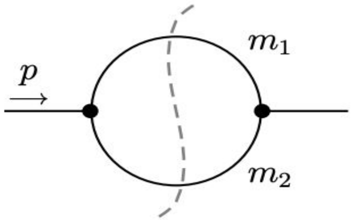
- ❑ Case-by-case treatment on removal of UV and IR divergences.

- ❑ Three-loop Vacuum Integral from Dispersion relation



$$F(q^2) = \frac{1}{\pi}\int_{M^2}^{\infty} ds \frac{\Im F(s)}{s - q^2 - i\epsilon}$$

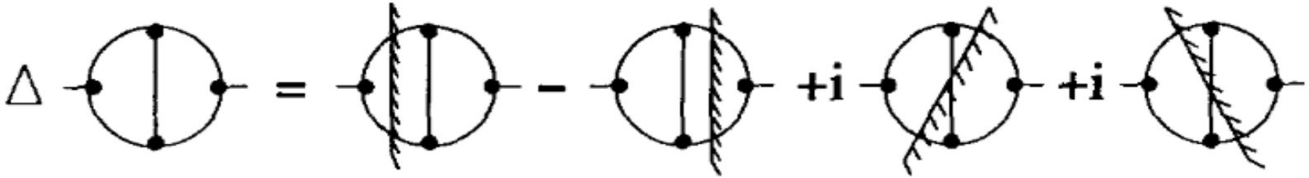
One-loop case



$$B_0(p^2, m_1^2, m_2^2) = \int_{(m_1+m_2)^2}^{\infty} \mathrm{d}s \frac{\Delta B_0(s, m_1^2, m_2^2)}{s - p^2 - i\varepsilon},$$

$$\Delta B_0(s, m_1^2, m_2^2) = \frac{\Gamma(D/2-1)}{\Gamma(D-2)} \frac{\lambda^{(D-3)/2}(s, m_1^2, m_2^2)}{s^{D/2-1}}$$

Two-loop case (topology T5a):

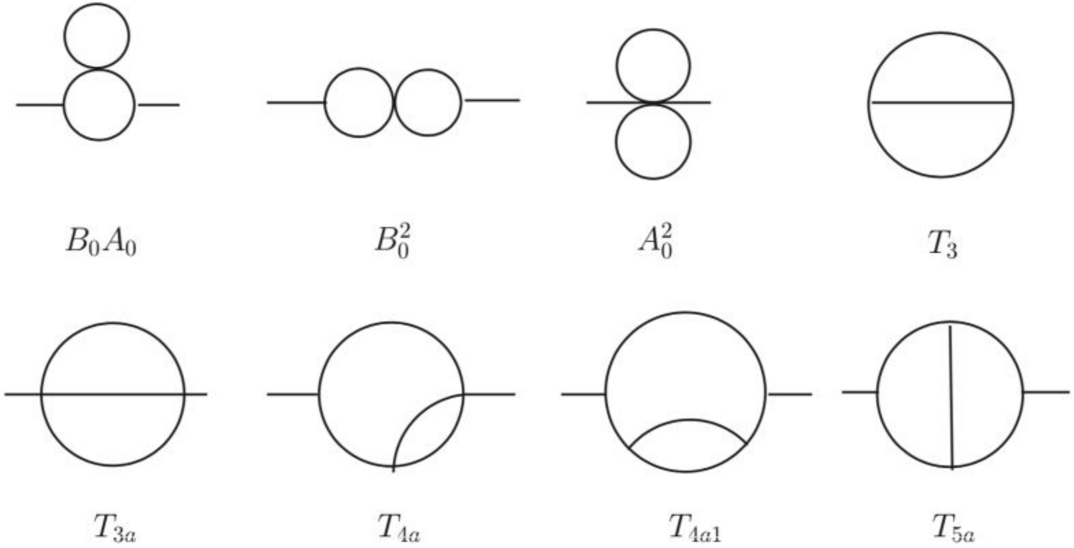


$$T_{5a}(p^2;m_i^2) = \frac{1}{2\pi i}\int_{s_0}^{\infty} ds \frac{\Delta T_{5a}(s;m_i^2)}{s-p^2-i\epsilon}$$

$$\Delta T_{5a}(s;m_i^2) = \Delta T_{5a1} + \Delta T_{5a2} + \Delta T_{5a3} + \Delta T_{5a4}$$

- ❑ **All discontinuities can be obtained by Cutkosky’s rule.**
- ❑ **Each discontinuity corresponds one choice of cut.**
- ❑ **Each cut features their own lower bound of integral associated with the threshold of the cut.**
- ❑ **Eventually these 2,3-pt disc can be written as 1-loop functions(discs).**

One of the two-loop master integral basis we use in calculations.



The derivative of Two-Loop Master Integral

Define a general two-loop scalar integral as

$$I(\nu_1,\nu_2,...,m_1,m_2,..,;p^2)$$
$$\equiv \int \frac{d^Dq_1\, d^Dq_2}{(q_1^2-m_1^2)^{\nu_1}((q_1+p)^2-m_2^2)^{\nu_2}((q_2-q_1)^2-m_3^2)^{\nu_3}(q_2^2-m_4^2)((q_2+p)^2-m_5^2)^{\nu_5}}$$

For $p^2 = 0$

$$\frac{\partial}{\partial p^2}I(...;p^2=0) = \frac{1}{2D}\frac{\partial^2}{\partial p_\mu \partial p^\mu}I(...;p^2)\Big|_{p^2=0}$$
$$= \frac{2}{D}\Big[\Big(1+\nu_2+\nu_5-\frac{D}{2}\Big)(\nu_2 I(\nu_2+1)+\nu_5 I(\nu_5+1))$$
$$+m_2^2\nu_2(\nu_2+1)I(\nu_2+2)+m_5^2\nu_5(\nu_5+1)I(\nu_5+2)$$
$$+\nu_2\nu_5((m_2^2-m_3^2+m_5^2)I(\nu_2+1,\nu_5+1)-I(\nu_2+1,\nu_3-1,\nu_5+1))\Big]_{p^2=0}$$

For $p^2 \neq 0$

$$\frac{\partial}{\partial p^2}I(...;p^2\neq 0) = -\frac{1}{2p^2}p^\mu\frac{\partial}{\partial p^\mu}I(...;p^2)$$
$$= -\frac{1}{2p^2}\big[(\nu_2+\nu_5)I-\nu_2 I(\nu_1-1,\nu_2+1)-\nu_5 I(\nu_4-1,\nu_5+1)$$
$$+\nu_2(m_2^2-m_1^2+p^2)I(\nu_2+1)+\nu_5(m_5^2-m_4^2+p^2)I(\nu_5+1)\big]$$

New $I(..;p^2)$ can be further reduced down to a linear combination of the chosen master integrals. Such a process can be carried out by using IBP technique.

❖ On-shell + $\overline{MS}$ in mixed EW-QCD case.

$\Delta r_{(\alpha^2\alpha_s)}$ $[10^{-4}]$	$\Delta M_{W(\alpha^2\alpha_s)}$ [MeV]
-0.50	0.78

X	$\Delta X_{(\alpha^2\alpha_s)}$	$\Delta' X_{(\alpha^2\alpha_s)}$
$\sin^2\theta_{\rm eff}$ $[10^{-5}]$	0.75	-0.76
Γ_ℓ [MeV]	-0.0003	0.0047
Γ_ν [MeV]	0.0009	0.0086
Γ_d [MeV]	-0.0018	0.0223
Γ_u [MeV]	-0.0029	0.0183
$\Gamma_{\rm tot}$ [MeV]	-0.0093	0.143

	on-shell M_t		$\overline{\rm MS}$ m_t	
	$\mathcal{O}(\alpha^2)$	$\mathcal{O}(\alpha^2\alpha_s)$	$\mathcal{O}(\alpha^2)$	$\mathcal{O}(\alpha^2\alpha_s)$
Δr $[10^{-4}]$	7.85	-1.09	7.56	-0.50
$\Delta \sin^2\theta^f_{\rm eff}$ $[10^{-5}]$	30.98	1.31	31.18	0.75
$\Delta\overline{\Gamma}_\ell$ [MeV]	0.2412	-0.0157	0.2284	-0.0003
$\Delta\overline{\Gamma}_\nu$ [MeV]	0.4145	-0.0002	0.4152	0.0009
$\Delta\overline{\Gamma}_d$ [MeV]	0.6666	-0.0049	0.6780	-0.0018
$\Delta\overline{\Gamma}_u$ [MeV]	0.4964	-0.0203	0.4911	-0.0029
$\Delta\overline{\Gamma}_{\rm tot}$ [MeV]	4.951	-0.103	4.947	-0.0093

❖ Comparing between two schemes

❑ $\overline{MS}$ Top mass must be used at previous order $\mathcal{O}(\alpha^2)$ when using $\overline{MS}$ renormalization scheme for top mass.

❑ A better convergence behavior from $\overline{MS}$ is observed. Also the numerical size of corrections at given order gets reduced comparing to on shell scheme.

❑ Numerical numbers given by two schemes at each order are partially compensate each other.

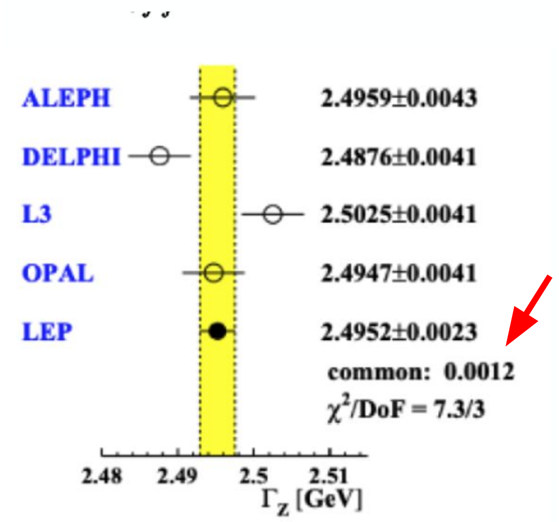
❑ Partial Widths of Z-boson decay in Pure EW case $\mathcal{O}(\alpha^3)$

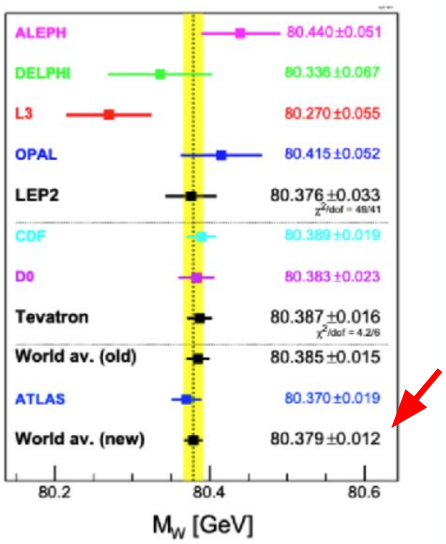
X	l^+l^-	$\nu\bar{\nu}$	$U\bar{U}$	$D\bar{D}$	Total
$\Delta\Gamma_X$ MeV	0.019	0.026	0.041	0.035	0.331
$\Delta'\Gamma_X$ MeV	0.017	0.022	0.029	0.024	0.255

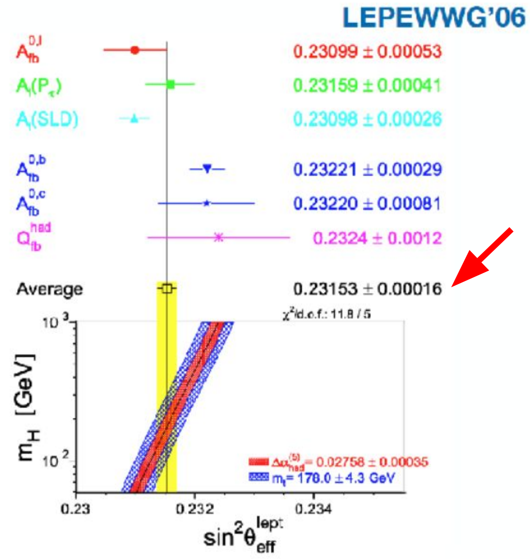
❑ partial Widths of Z-boson decay in Mixed EW-QCD case $\mathcal{O}(\alpha^2\alpha_s)$

X	l^+l^-	$\nu\bar{\nu}$	$U\bar{U}$	$D\bar{D}$	Total
$\Delta\Gamma_X$ MeV	-0.0157	-2.0E-4	-0.0049	-0.0203	-0.103
$\Delta'\Gamma_X$ MeV	-0.0049	0.0166	0.0475	0.0260	0.2296

- Current Status of Experimental Measurements







- ❑ The SM shows good consistency by comparing measured EWPOs and theory predictions.
- ❑ The theoretical uncertainties is under well-control comparing to the known measurements. But...

Theoretical uncertainty due to missing higher order

- Collect all common prefactors, such as couplings, Lie algebra number, particle multiplicities and mass ratios.

- Vary renormalization scale ($\overline{MS}$ only!), this is frequently used in QCD.

- Compare results from two different schemes.

- Extrapolate to higher order by assuming geometric series behavior of perturbation theory.

prefactor method yields

$$\mathcal{O}(\alpha_{\rm bos}) \sim \Gamma_{\rm Z}\alpha^2 \approx 0.13 \text{ MeV},$$
$$\mathcal{O}(\alpha^3) - \mathcal{O}(\alpha_{\rm t}^3) \sim \Gamma_{\rm Z}\alpha\alpha_{\rm t}^2 \approx 0.12 \text{ MeV},$$
$$\mathcal{O}(\alpha^2\alpha_{\rm s}) - \mathcal{O}(\alpha_{\rm t}^2\alpha_{\rm s}) \sim \Gamma_{\rm Z}\frac{\alpha\alpha_{\rm t}n_q}{\pi}\alpha_{\rm s}(m_{\rm t}) \approx 0.23 \text{ MeV},$$
$$\mathcal{O}(\alpha\alpha_{\rm s}^2) - \mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^2) \sim \Gamma_{\rm Z}\frac{\alpha n_q}{\pi}\alpha_{\rm s}^2(m_{\rm t}) \approx 0.35 \text{ MeV},$$
$$\mathcal{O}(\alpha\alpha_{\rm s}^3) - \mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^3) \sim \Gamma_{\rm Z}\frac{\alpha n_q}{\pi}\alpha_{\rm s}^3(m_{\rm t}) \approx 0.04 \text{ MeV}.$$

geometric series extrapolation yields

$$\mathcal{O}(\alpha_{\rm bos}) \sim [\mathcal{O}(\alpha_{\rm bos})]^2 \approx 0.10 \text{ MeV},$$
$$\mathcal{O}(\alpha^3) - \mathcal{O}(\alpha_{\rm t}^3) \sim \frac{\mathcal{O}(\alpha^2)}{\mathcal{O}(\alpha)}[\mathcal{O}(\alpha^2) - \mathcal{O}(\alpha_{\rm t}^2)] \approx 0.26 \text{ MeV},$$
$$\mathcal{O}(\alpha^2\alpha_{\rm s}) - \mathcal{O}(\alpha_{\rm t}^2\alpha_{\rm s}) \sim \frac{\mathcal{O}(\alpha\alpha_{\rm s})}{\mathcal{O}(\alpha)}[\mathcal{O}(\alpha^2) - \mathcal{O}(\alpha_{\rm t}^2)] \approx 0.30 \text{ MeV},$$
$$\mathcal{O}(\alpha\alpha_{\rm s}^2) - \mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^2) \sim \frac{\mathcal{O}(\alpha\alpha_{\rm s})}{\mathcal{O}(\alpha)}[\mathcal{O}(\alpha\alpha_{\rm s}) - \mathcal{O}(\alpha_{\rm t}\alpha_{\rm s})] \approx 0.23 \text{ MeV},$$
$$\mathcal{O}(\alpha\alpha_{\rm s}^3) - \mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^3) \sim \frac{\mathcal{O}(\alpha\alpha_{\rm s}^2)}{\mathcal{O}(\alpha)}[\mathcal{O}(\alpha\alpha_{\rm s}) - \mathcal{O}(\alpha_{\rm t}\alpha_{\rm s})] \approx 0.035 \text{ MeV}.$$

$$\delta_{th}\Gamma_Z \sim 0.5 \text{ MeV}$$