Karlsruhe Institute of Technology

Analytic Initial State Radiation Calculation

ILC IDT-WG3: Physics and Detectors Open Meeting, 2021

Kay Schönwald | July 15, 2021

TTP KARLSRUHE

[based on: Ablinger, Blümlein, De Freitas, Schönwald (Nucl. Phys. B955 (2020))
and Blümlein, De Freitas, Schönwald (Phys. Lett. B816 (2021))]

Outline

Motivation

- Corrections due to initial state radiation (ISR) can be large, especially due to large logarithmic corrections

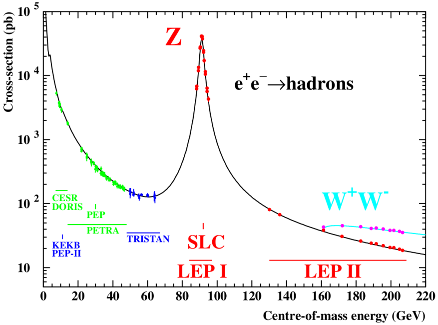
$$L = \ln(s/m_e^2) \approx 10.$$

- These corrections are important e.g.
  - for the prediction of the Z-boson peak
  - for $t\bar{t}$ production
  - associated Higgs production through $e^+ e^- \to Z^* H^0$

  at future $e^+ e^-$ colliders.

- We extend the known $O(\alpha^2)$ ISR corrections up to $O(\alpha^6 L^5)$, including the first three subleading logarithmic corrections at lower orders.

- We extend the ISR corrections for the forward-backward asymmetry at leading logarithmic order to $O(\alpha^6 L^6)$.



Previous Calculations

- 1988: First calculation to $O(\alpha^2)$ for the LEP analysis, through expansion of the phase space integrals.
  [Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))]
- 2012: New calculation up to $O(\alpha^2)$ using the method of massive operator matrix elements.
  [Blümlein, De Freitas, van Neerven (Nucl Phys. B855 (2012))]

$\Rightarrow$ Calculations do not agree at $O(\alpha^2 L^0)$!

- Errors in one of the calculations?
- Breakdown of factorization?

- We revisited the original calculation, doing the expansion in m_e at the latest stage
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The Method of Massive Operator Matrix Elements

The initial state radiation factorizes from the born cross section:

$$\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}s'} = \frac{\sigma^{(0)}(s')}{s} \sum_{l,k} \Gamma_{li}\left(z, \frac{\mu^2}{m_e^2}\right) \otimes \tilde{\sigma}_{lk}\left(z, \frac{s'}{\mu^2}\right) \otimes \Gamma_{kj}\left(z, \frac{\mu^2}{m_e^2}\right) + O\left(\frac{m_e^2}{s}\right) = \frac{\sigma^{(0)}(s')}{s} H_{ij}\left(z, \frac{s}{m_e^2}\right)$$

$$\left[f(z) \otimes g(z) = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 f(x_1) g(x_2) \delta(z - x_1 x_2),\ f(N) = \int_0^1 \mathrm{d}z\, z^{N-1} f(z)\right]$$

with $z = s'/s$, μ the factorization scale, into:

- massless (Drell-Yan) cross sections $\tilde{\sigma}_{ij}\left(z, \frac{s'}{\mu^2}\right)$
  [Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))]
  [Harlander, Kilgore (Phys. Rev. Lett. 88 (2002))]
  [Duhr, Dulat, Mistelberger (Phys. Rev. Lett. 125 (2020))]
- massive operator matrix elements $\Gamma_{ij}\left(z, \frac{\mu^2}{m_e^2}\right)$, which carry all mass dependence
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The Method of Massive Operator Matrix Elements

Massless cross sections and massive operator matrix elements obey renormalization group equations:

- massless cross sections $\tilde{\sigma}_{ij}$

$$\left[\left(\frac{\partial}{\partial\lambda} - \beta(a)\frac{\partial}{\partial a}\right)\delta_{kl}\delta_{jm} + \frac{1}{2}\gamma_{kl}(N)\delta_{jm} + \frac{1}{2}\gamma_{jm}(N)\delta_{kl}\right]\tilde{\sigma}_{lj}(N) = 0$$

- massive operator matrix elements Γ_{ij}

$$\left[\left(\frac{\partial}{\partial\Lambda} + \beta(a)\frac{\partial}{\partial a}\right)\delta_{jl} + \frac{1}{2}\gamma_{kl}(N)\right]\Gamma_{li}(N) = 0$$

  with $\lambda = \ln(s'/\mu^2)$, $\Lambda = \ln(\mu^2/m_e^2)$, the QED β-function $\beta(a)$ and $a = \alpha/4$

- Here the usual anomalous dimensions, i.e. Mellin transforms of the splitting functions, contribute:

$$\gamma_{ij}(N) = -\int_0^1 \mathrm{d}z\, z^{N-1} P_{ij}(z)$$

The Method of Massive Operator Matrix Elements

$$\frac{\mathrm{d}\sigma_{e^+e^-}}{\mathrm{d}s'} = \frac{\sigma^{(0)}(s')}{s} H_{e^+e^-}(z, L) = \frac{\sigma^{(0)}(s')}{s} \sum_{i=0}^{\infty} \sum_{k=0}^{i} a^i L^k c_{i,k}$$

The radiators:

$$c_{1,1} = -\gamma_{ee}^{(0)},$$

$$c_{1,0} = \tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)},$$

$$c_{2,2} = \frac{1}{2}\gamma_{ee}^{(0)2} + \frac{\beta_0}{2}\gamma_{ee}^{(0)} + \frac{1}{4}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)},$$

$$\ldots$$

$$c_{3,1} = -\gamma_{ee}^{(2)} - 2\Gamma_{ee}^{(0)}\gamma_{ee}^{(1)} - \Gamma_{ee}^{(0)}\gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)} - \gamma_{e\gamma}^{(1)}\Gamma_{\gamma e}^{(0)} - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(1)} - \beta_1\tilde{\sigma}_{ee}^{(0)} - \gamma_{ee}^{(1)}\tilde{\sigma}_{ee}^{(0)}$$
$$- \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{ee}^{(0)} - 2\Gamma_{ee}^{(0)}\gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} - \gamma_{\gamma e}^{(1)}\tilde{\sigma}_{e\gamma}^{(0)} - \Gamma_{\gamma e}^{(0)}\gamma_{\gamma\gamma}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} - \gamma_{\gamma e}^{(0)}\tilde{\sigma}_{\gamma e}^{(1)} + \beta_0\Big[-2\Gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)}$$
$$- 2\tilde{\sigma}_{ee}^{(1)} - 2\Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)}\Big] - \gamma_{ee}^{(0)}\Big[\Gamma_{ee}^{(0)2} + 2\Gamma_{ee}^{(1)} + 2\Gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} + \tilde{\sigma}_{ee}^{(1)} + \Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)}\Big],$$

$$\ldots$$

For the first three logarithmic orders we need the following ingredients:

- splitting functions γ_{ij} up to three-loop order
  [E.G. Floratos, D.A. Ross, C.T. Sachrajda (Nucl. Phys. B129 (1977))]
  [A. Gonzalez-Arroyo, C. Lopez, F.J. Yndurain (Nucl. Phys. B153 (1979))]
  ...
  [S. Moch, J. Vermaseren, A. Vogt (Nucl.Phys.B 688/691 (2004))]
- massless (Drell-Yan) cross sections $\tilde{\sigma}_{ij}$ up to two-loop order
  [Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))]
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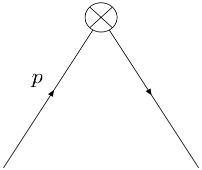
- massive operator matrix elements Γ_{ij} up to two-loop order[1]
  [Blümlein, De Freitas, van Neerven (Nucl. Phys. B855 (2012))]

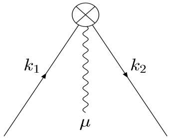
$\Rightarrow$ $\Gamma_{\gamma e}$ was only considered up to one-loop order

[1] In the case of massless external states massive operator matrix elements have been considered in the context of DIS. [Buza, Matiounine, Smith, Migneron, van Neerven (Nucl. Phys. B472 (1996)), Bierenbaum, Blümlein, Klein (Nucl. Phys. B820 (2009)), ...]

The Missing Operator Matrix Element $\Gamma_{\gamma e}$



$\not{\Delta}(\Delta.p)^{N-1}$



$-e\Delta_\mu \sum_{j=1}^{N-1}(\Delta.k_1)^{j-1}(\Delta.k_2)^{N-1-j}$

$\frac{1+(-1)^N}{2}(\Delta.p)^{N-2}\left[g_{\mu\nu}(\Delta.p)^2-(\Delta_\mu p_\nu+\Delta_\nu p_\mu)\Delta.p+p^2\Delta_\mu\Delta_\nu\right]$

$$\Gamma_{e^+e^+} = \Gamma_{e^-e^-} = \langle e|\, O_F^{\mathrm{NS,S}}\, |e\rangle\,,$$
$$\Gamma_{e^+\gamma} = \Gamma_{e^-\gamma} = \langle \gamma|\, O_F^{\mathrm{S}}\, |\gamma\rangle\,,$$
$$\Gamma_{\gamma e^+} = \Gamma_{\gamma e^-} = \langle e|\, O_V^{\mathrm{S}}\, |e\rangle\,,$$

$$O^{\mathrm{NS,S}}_{F;\mu_1,\ldots,\mu_N} = i^{N-1}\mathrm{S}\left[\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\right] - \text{traces},$$
$$O^{\mathrm{S}}_{V;\mu_1,\ldots,\mu_N} = 2i^{N-2}\mathrm{S}\left[F_{\mu_1\alpha}D_{\mu_2}\ldots D_{\mu_{N-1}}F^{\alpha}_{\mu_N}\right] - \text{traces}$$

- The technique has been used to derive deep-inelastic scattering (DIS) structure functions in the asymptotic limit $Q^2 \gg m^2$ up to $O(\alpha_s^3)$.
- In the context of DIS proven to work at α_s^2 in the
  - non-singlet process
    [Buza, Matiounine, Smith, van Neerven (Nucl.Phys. B485 (1997))
    Blümlein, Falcioni, De Freitas (Nucl.Phys. B910 (2016))]
  - pure-singlet process
    [Blümlein, De Freitas, Raab, Schönwald (Nucl.Phys. B945 (2019))]

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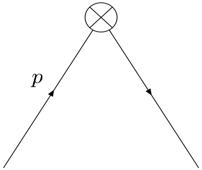

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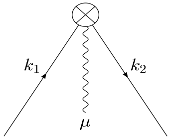

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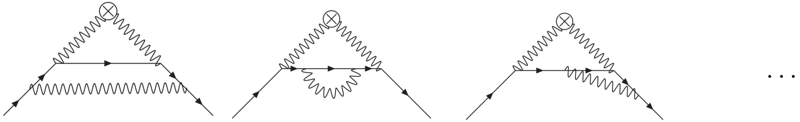

- We have to compute on-shell 2-point functions with local operator insertions ($\Delta^2 = 0$).
- The operator can be resummed into a propagator like term:

$$\sum_{N=0}^{\infty} t^N (\Delta.k)^N = \frac{1}{1 - t\Delta.k}.$$

- The calculation can now follow standard techniques:
  - Integration-By-Parts reduction to master integrals.
  - Calculation of the master integrals via differential equations in the resummation variable t.
  - We find the Mellin space expression by symbolically computing the N-th coefficient.
- For the calculations we make use of the packages `Sigma` [C. Schneider (Sem. Lothar. Combin.56 (2007))] and `HarmonicSums` [J. Ablinger et al. (arXiv:1011.1176)] .

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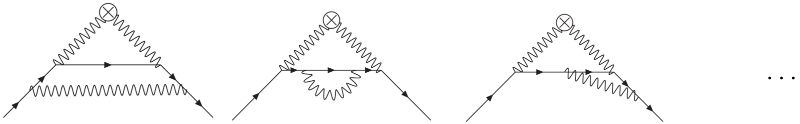

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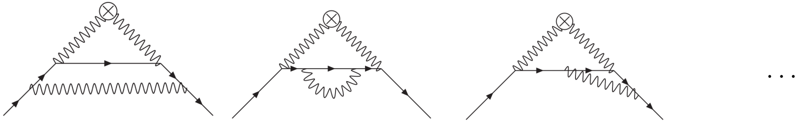

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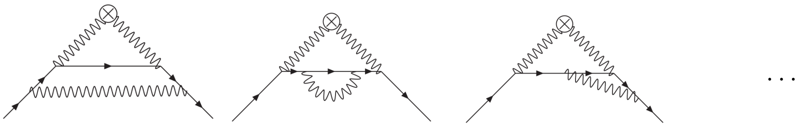

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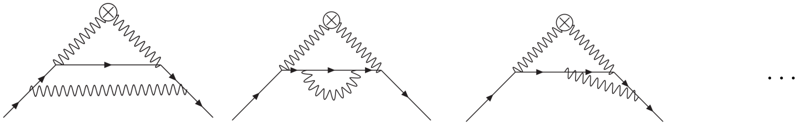

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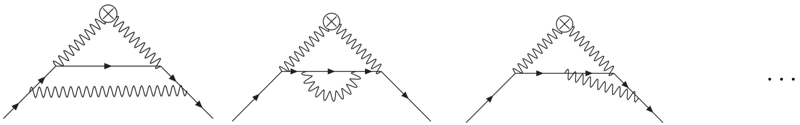

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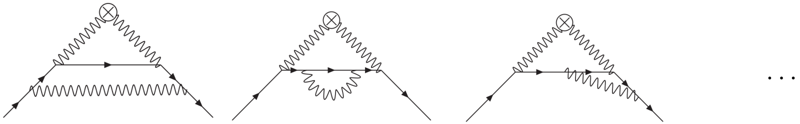

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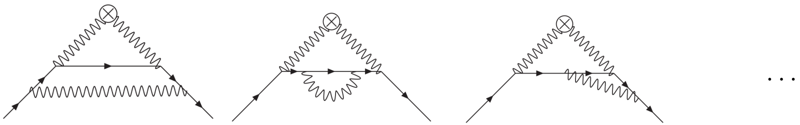

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$$
\begin{aligned}
\Gamma_{\gamma e}^{(1)}(N) = {} & \frac{P_8}{27(N-4)(N-3)(N-2)(N-1)N^4(N+1)^4} + \left(\frac{2P_7}{9(N-4)(N-3)(N-2)(N-1)N^3(N+1)^3} + \frac{2\left(N^2+N+2\right)}{(N-1)N(N+1)} S_2\right) S_1 \\
& + \frac{P_3}{3(N-2)(N-1)N(N+1)^2} S_1^2 + \frac{2\left(N^2+N+2\right)}{3(N-1)N(N+1)} S_1^3 + \frac{P_6}{3(N-2)(N-1)N^2(N+1)^2} S_2 + \frac{4\left(N^2+N+2\right)}{3(N-1)N(N+1)} S_3 \\
& + \frac{3 \cdot 2^{6+N}}{(N-2)(N+1)^2} S_{1,1}\left(\frac{1}{2}, 1\right) + \frac{2^{6-N} P_5}{3(N-3)(N-2)(N-1)^2 N^2} \Big(S_2(2) + S_1 S_1(2) - S_{1,1}(1,2) - S_{1,1}(2,1) \Big) \\
& - \frac{32\left(N^2+N+2\right)}{(N-1)N(N+1)} \left[S_1(2) S_{1,1}\left(\frac{1}{2}, 1\right) + S_{1,2}\left(\frac{1}{2}, 2\right) - S_{1,1,1}\left(\frac{1}{2}, 1, 2\right) - S_{1,1,1}\left(\frac{1}{2}, 2, 1\right) - \frac{\zeta_2}{2} S_1(2) \right] \\
& - \frac{48\left(N^2+N+2\right)}{(N-1)N(N+1)} S_{2,1} + \frac{4P_4}{(N-2)(N-1)N^2(N+1)^2} \zeta_2
\end{aligned}
$$

harmonic sums:

$$
S_{a,\vec{b}} = S_{a,\vec{b}}(N) = \sum_{i=1}^{N} \frac{\mathrm{sgn}(a)^i}{i^a} S_{\vec{b}}(i)
$$

generalized harmonic sums:

$$
S_{a,\vec{b}}(c,\vec{d}) = S_{a,\vec{b}}(c,\vec{d};N) = \sum_{i=1}^{N} \frac{(\mathrm{sgn}(a) \cdot c)^i}{i^a} S_{\vec{b}}(\vec{d};i)
$$

The Missing Operator Matrix Element $\Gamma_{\gamma e}$

Karlsruhe Institute of Technology

Analytic Mellin-inversion with `HarmonicSums`:

$$
\begin{aligned}
\Gamma^{(1)}_{\gamma e}(z) = {} & \frac{P_9}{135z^3} - \frac{320-335z+231z^2}{15z}\mathrm{H}_0 + \frac{12+23z}{6}\mathrm{H}_0^2 + \frac{2-z}{3}\mathrm{H}_0^3 + 32(2-z)\left(\frac{(2-z)^2}{3z^2} - \mathrm{H}_0\right)\left(\tilde{\mathrm{H}}_{-1}\tilde{\mathrm{H}}_0 - \tilde{\mathrm{H}}_{0,-1}\right) \\
& - 8(2-z)\mathrm{H}_{0,0,1} - \frac{96-190z+118z^2-41z^3}{3z^2}\mathrm{H}_1^2 - 32(2-z)\left(\tilde{\mathrm{H}}_{-1}\tilde{\mathrm{H}}_0 - \tilde{\mathrm{H}}_{0,-1}\right)\tilde{\mathrm{H}}_1 \\
& - \left(\frac{2\left(32-48z+36z^2-13z^3\right)}{3z^2} + 4(2-z)\mathrm{H}_0\right)\mathrm{H}_{0,1} - \left(\frac{2P_{10}}{45z^4} - \frac{2\left(32-48z+12z^2+7z^3\right)}{3z^2}\mathrm{H}_0\right)\mathrm{H}_1 \\
& + \frac{2\left(2-2z+z^2\right)}{z}\left(\frac{\mathrm{H}_1^3}{3} + 8\mathrm{H}_1\mathrm{H}_{0,1} + 16\tilde{\mathrm{H}}_0\tilde{\mathrm{H}}_{0,-1} - 32\tilde{\mathrm{H}}_{0,0,-1} - 16\mathrm{H}_{0,1,1} + 8\tilde{\mathrm{H}}_0\zeta_2\right) + \left(\frac{4\left(32-48z+24z^2-3z^3\right)}{3z^2}\right. \\
& \left. - 8(2-z)\left(\mathrm{H}_0 + 2\tilde{\mathrm{H}}_1\right)\right)\zeta_2 + \frac{8\left(12-10z+5z^2\right)}{z}\zeta_3
\end{aligned}
$$

harmonic polylogarithms of argument z and $1-z$ ($\tilde{\mathrm{H}}(z) = \mathrm{H}(1-z)$):

$$
H_{a,\vec{b}} = H_{a,\vec{b}}(z) = \int_0^1 \mathrm{d}\tau f_a(\tau) H_{\vec{b}}(\tau), \quad \text{with} \quad f_0(\tau) = \frac{1}{\tau},\ f_1(\tau) = \frac{1}{1-\tau},\ f_{-1}(\tau) = \frac{1}{1+\tau}
$$

The Radiators

$$\frac{\mathrm{d}\sigma_{e^+e^-}}{\mathrm{d}s'} = \frac{\sigma^{(0)}(s')}{s} H_{e^+e^-}(z,L) = \frac{\sigma^{(0)}(s')}{s} \sum_{i=0}^{\infty} \sum_{k=0}^{i} a^i L^k c_{i,k}$$

- The radiators do not depend on the factorization scale, i.e. no collinear singularities for massive electrons.
- The analytic structures directly translate from the different ingredients.
- Radiators are distributions in z-space:

$$c_{i,j}(z) = c_{i,j}^{\delta}\delta(1-z) + c_{i,j}^{+} + c_{i,j}^{\mathrm{reg}}$$

$$c_{3,3}^{\delta} = \frac{572}{9} - \frac{704}{3}\zeta_2 + \frac{512}{3}\zeta_3,$$

$$c_{3,3}^{+} = \left(\frac{5744}{27} - 256\zeta_2\right)\mathcal{D}_0 + \frac{1408}{3}\mathcal{D}_1 + 256\mathcal{D}_2,$$

$$\mathcal{D}_k = \left(\frac{\ln^k(1-z)}{1-z}\right)_+,$$

$$c_{3,3}^{\mathrm{reg}} = \left\{\frac{16\mathrm{H}_0 P_{104}}{9(z-1)} - \frac{4P_{131}}{27z} + \frac{8\left(3-19z^2\right)\mathrm{H}_0^2}{3(z-1)} + \left[\frac{16P_{105}}{9z} - \frac{128\left(1+z^2\right)\mathrm{H}_0}{z-1}\right]\mathrm{H}_1 - 128(1+z)\mathrm{H}_1^2 - \frac{352}{3}(1+z)\mathrm{H}_{0,1} + \frac{736}{3}(1+z)\zeta_2\right\}$$

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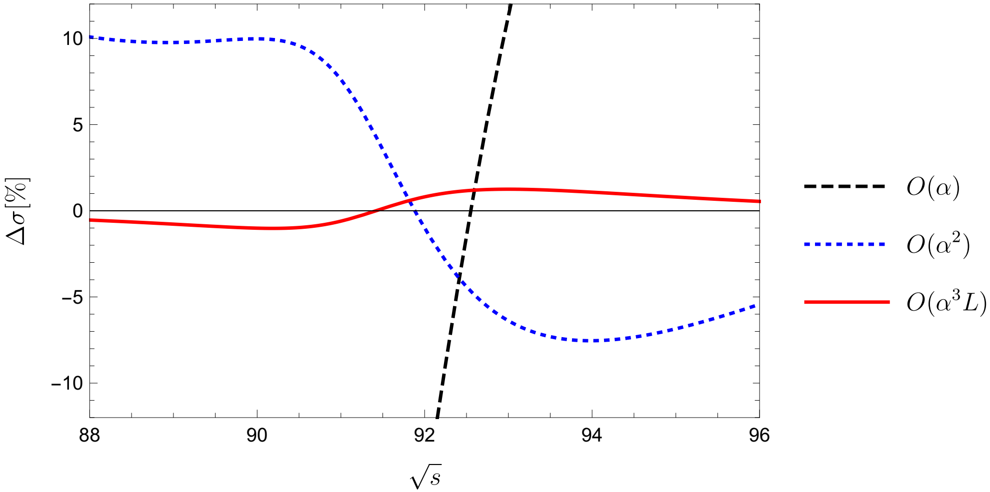
Numerical Results

- $\Delta\sigma$ is the change in the total cross section from one order to the other for $z_0 = 4m_\tau^2$



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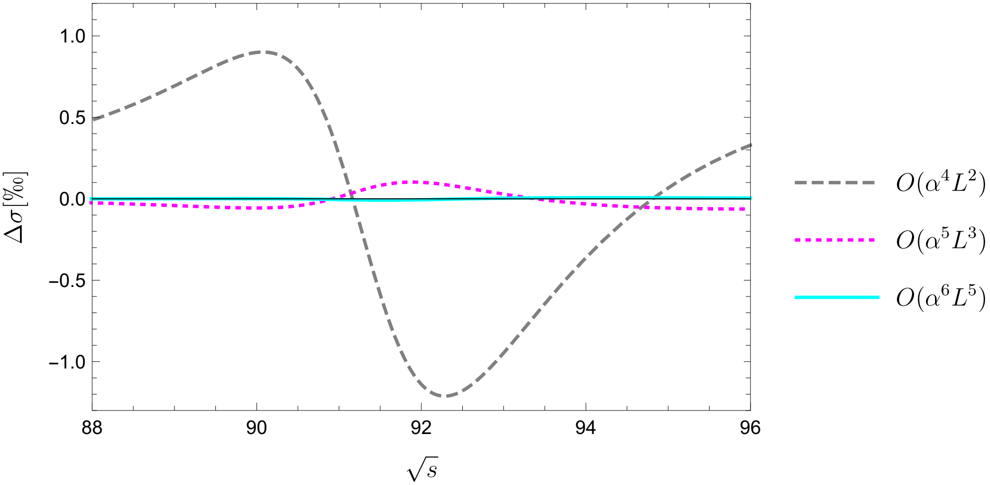

Application to the Forward-Backward Asymmetry A_{FB}

- The forward-backward asymmetry is defined by:

$$A_{FB}(s) = \frac{\sigma_F(s) - \sigma_B(s)}{\sigma_F(s) + \sigma_B(s)},$$

with

$$\sigma_F(s) = 2\pi \int_0^1 \mathrm{d}\cos(\theta) \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}, \qquad \sigma_B(s) = 2\pi \int_{-1}^0 \mathrm{d}\cos(\theta) \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega},$$

and θ the angle between the incoming e^- and outgoing μ^-.

- The technique of radiators can also be used for A_{FB}: [Böhm et al. (LEP Physics Workshop 1989, p.203–234)]

$$A_{FB}(s) = \frac{1}{\sigma_F(s) + \sigma_B(s)} \int_{z_0}^1 \mathrm{d}z \frac{4z}{(1+z)^2} H_{FB}(z) \sigma_{FB}^{(0)}(zs)$$

- Due to the angle dependence the radiators are not the same as in the total cross-section.

Application to the Forward-Backward Asymmetry A_{FB}

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- At leading logarithmic (LL) accuracy the radiators are given by:

$$H_{FB}^{LL} = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \frac{(1+z)^2}{(x_1+x_2)^2} \Gamma_{ee}^{LL}(x_1) \Gamma_{ee}^{LL}(x_2) \delta(z - x_1 x_2).$$

- Due to the additional angle dependence the integral does not factorize with the Mellin-transform.
- At subleading logarithmic accuracy the integral will likely become more involved due to additional angle dependence of the cross-sections.
- The integrals can be solved analytically in Mellin and momentum fraction space.

Application to the Forward-Backward Asymmetry A_{FB}

- A direct integration in terms of iterated integrals at argument z is complicated by singularities at $z \to 0$ and $z \to 1$.
- A direct computation of the Mellin transform

$$\mathcal{M}[\tilde{H}^{LL}_{FB}(z)](N) = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, x_1^N x_2^N \left(\frac{(1+x_1x_2)^2}{(x_1+x_2)^2} - 1 \right) \Gamma^{LL}_{ee}(x_1) \Gamma^{LL}_{ee}(x_2)$$

  is complicated by the symbolic powers of N.
- We chose to calculate the generating function:

$$\mathcal{G}[\tilde{H}^{LL}_{FB}(z)](t) = \sum_{N=0}^{\infty} t^N \mathcal{M}[H^{LL}_{FB}(z)](N) = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, \frac{1}{1 - t\, x_1 x_2} \left(\frac{(1+x_1x_2)^2}{(x_1+x_2)^2} - 1 \right) \Gamma^{LL}_{ee}(x_1) \Gamma^{LL}_{ee}(x_2)$$

- The integrations as well as the extraction of the Mellin space result and the inverse Mellin transform to momentum fraction space can be performed with `HarmonicSums` together with `Sigma`.

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Application to A_{FB} – Results

- In Mellin space we additionally encounter cyclotomic harmonic sums.
- In momentum fraction space we encounter cyclotomic harmonic polylogarithms, i.e. we have to introduce the additional letters:

$$f_{\{4,0\}}(\tau) = \frac{1}{1+\tau^2}, \qquad f_{\{4,1\}}(\tau) = \frac{\tau}{1+\tau^2}.$$

For example: $\left(S_{\vec{w}} \equiv S_{\vec{w}}(N)\right)$

$$H_{FB}^{(2),LL}(N) = \frac{8(3N^2+3N-1)P_1}{(N-1)N^2(N+1)^2(N+2)(2N-1)(2N+3)} - \frac{32(4N^2+4N-1)(-1)^N}{(2N-1)(2N+1)(2N+3)}[S_{-1}+\ln(2)],$$

$$H_{FB}^{(3),LL}(N) = -(-1)^N\frac{256(4N^2+4N-1)}{(2N-1)(2N+1)(2N+3)}\left[S_{-1,1} - \frac{1}{2}\ln^2(2) + \sum_{i=1}^{N}\frac{\ln(2)+S_{-1}(i)}{1+2i}\right] + \ldots .$$

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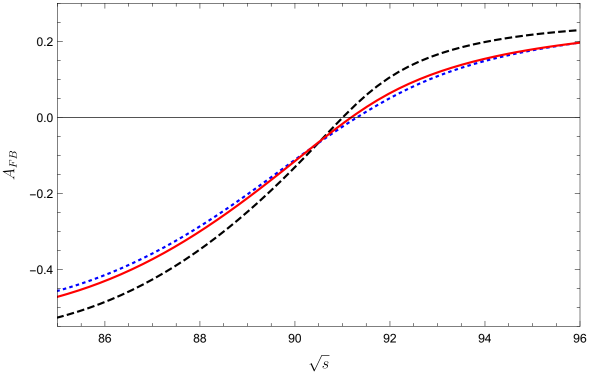
$$H_{FB}^{(2),LL}(z) = \frac{2(1-z)(1+z)^2}{z} + 2\pi\frac{(1-z)^2}{\sqrt{z}} - 8(1+z)H_0 - 8(1-z)^2\frac{H_{\{4,0\}}}{\sqrt{z}},$$

$$H_{FB}^{(3),LL}(z) = \frac{64(1-z)^2}{\sqrt{z}}\left[H_{1,\{4,0\}} - H_{-1,\{4,0\}} - H_{\{4,0\},\{4,1\}} + \frac{1}{2}H_{0,\{4,0\}}\right] + \ldots$$

Application to A_{FB} – Numerical Results

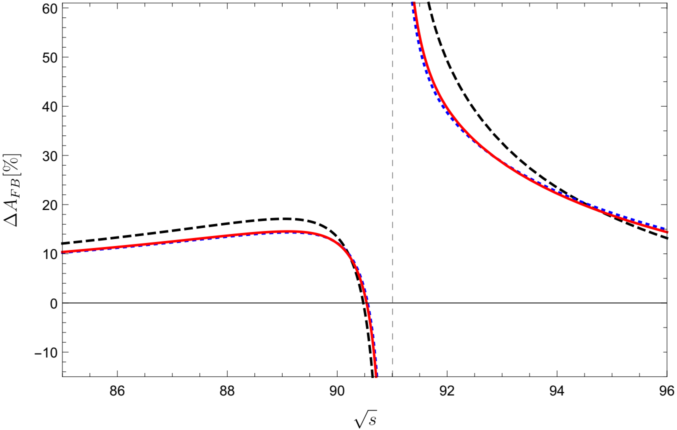


A_{FB} evaluated at $s_- = (87.9\,\mathrm{GeV})^2$, M_Z^2 and $s_+ = (94.3\,\mathrm{GeV})^2$ for the cut $z > 4m_\tau^2/s$.

	$A_{FB}(s_-)$	$A_{FB}(M_Z^2)$	$A_{FB}(s_+)$
$\mathcal{O}(\alpha^0)$	−0.3564803	0.0225199	0.2052045
$+\mathcal{O}(\alpha L^1)$	−0.2945381	−0.0094232	0.1579347
$+\mathcal{O}(\alpha L^0)$	−0.2994478	−0.0079610	0.1611962
$+\mathcal{O}(\alpha^2 L^2)$	−0.3088363	0.0014514	0.1616887
$+\mathcal{O}(\alpha^3 L^3)$	−0.3080578	0.0000198	0.1627252
$+\mathcal{O}(\alpha^4 L^4)$	−0.3080976	0.0001587	0.1625835
$+\mathcal{O}(\alpha^5 L^5)$	−0.3080960	0.0001495	0.1625911
$+\mathcal{O}(\alpha^6 L^6)$	−0.3080960	0.0001499	0.1625911

A_{FB} and its initial state QED corrections as a function of $\sqrt{s}$. Black (dashed) the Born approximation, blue (dotted) the $O(\alpha)$ improved approximation, red (full) also including the leading-log improvement up to $O(\alpha^6)$ for $s'/s \geq 4m_\tau^2/s$.

Application to A_{FB} – Numerical Results



$$\Delta A_{FB} = 1 - \frac{A_{FB}^{(l)}}{A_{FB}^{(0)}},$$

where (l) denotes the order of ISR-corrections considered

ΔA_{FB} in % as a function of $\sqrt{s}$. Black (dashed) the $O(\alpha)$ improved approximation, blue (dotted) the $O(\alpha^2 L^2)$ improved approximation, red (full) also including the leading-log improvement up to $O(\alpha^6)$ for $s'/s \geq 4m_\tau^2/s$.

Conclusions and Outlook

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Conclusions:

- We calculated the ISR corrections to the process $e^+e^- \to \gamma^*/Z^*$ up to $O(\alpha^6 L^5)$ including the first three logarithmic terms at lower orders.
- We calculated the leading logarithmic ISR corrections to the forward-backward asymmetry up to $O(\alpha^6 L^6)$.
- The corrections can become important at future e^+e^- machines running at high luminosities at or close to the Z-pole.
- The radiators can be used for other processes like $e^+e^- \to t\bar{t}$ and $e^+e^- \to ZH$.

Outlook:

- The massless Drell-Yan cross sections are known up to $O(\alpha^3)$
  $\Rightarrow$ An extension to the first four logarithmic orders is possible, but needs the calculation the operator matrix elements up to $O(\alpha^3)$ and the 4-loop splitting functions.
- The technique can be extended to subleading logarithmic corrections of A_{FB}.
- The method can be extended to QCD to study e.g. the heavy-quark initiated Drell-Yan process.

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Backup

The Method of Massive Operator Matrix Elements

- Usually it is more convenient to work in Mellin space:

$$M[f(z)](N) = \int_0^1 \mathrm{d}z z^{N-1} f(z)$$

- Here the convolution $\otimes$

$$f(z) \otimes g(z) = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 f(z_1) g(z_2) \delta(z - z_1 z_2)$$

factorizes:

$$M[f(z) \otimes g(z)](N) = M[f(z)](N) \cdot M[g(z)](N)$$

Numerical Results

	L^6	L^5	L^4	L^3	L^2	L	L^0
$O(\alpha)$						$\surd$	$\surd$
$O(\alpha^2)$					$\surd$	$\surd$	$\surd$
$O(\alpha^3)$				$\surd$	$\surd$	$\surd$	-
$O(\alpha^4)$			$\surd$	$\surd$	$\surd$	-	-
$O(\alpha^5)$		$\surd$	$\surd$	$\surd$	-	-	-
$O(\alpha^6)$	$\surd$	$\surd$	-	-	-	-	-

	Fixed width		s dep. width	
	Peak (MeV)	Width (MeV)	Peak (MeV)	Width (MeV)
$O(\alpha)$ correction	185.638	539.408	181.098	524.978
$O(\alpha^2 L^2)$:	– 96.894	–177.147	– 95.342	–176.235
$O(\alpha^2 L)$:	6.982	22.695	6.841	21.896
$O(\alpha^2)$:	0.176	– 2.218	0.174	– 2.001
$O(\alpha^3 L^3)$:	23.265	38.560	22.968	38.081
$O(\alpha^3 L^2)$:	– 1.507	– 1.888	– 1.491	– 1.881
$O(\alpha^3 L)$:	– 0.152	0.105	– 0.151	– 0.084
$O(\alpha^4 L^4)$:	– 1.857	0.206	– 1.858	0.146
$O(\alpha^4 L^3)$:	0.131	– 0.071	0.132	– 0.065
$O(\alpha^4 L^2)$:	0.048	– 0.001	0.048	0.001
$O(\alpha^5 L^5)$:	0.142	– 0.218	0.144	– 0.212
$O(\alpha^5 L^4)$:	– 0.000	0.020	– 0.001	0.020
$O(\alpha^5 L^3)$:	– 0.008	0.009	– 0.008	0.008
$O(\alpha^6 L^6)$:	– 0.007	0.027	– 0.007	0.027
$O(\alpha^6 L^5)$:	– 0.001	0.000	– 0.001	0.000

Table 1: Shifts in the Z-mass and the width due to the different contributions to the ISR QED radiative corrections for a fixed width of $\Gamma_Z = 2.4952$ GeV and s-dependent width using $M_Z = 91.1876$ GeV and $s_0 = 4m_\tau^2$.

Application to A_{FB} – Numerical Results

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