Theory needs for future e^+e^- colliders

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- "Other" electroweak parameters ("input" parameters)
- Z pole & WW
- Electroweak precision at $\sqrt{s} = 250$ GeV
- Higgs physics

Need for theory input

- Comparison of EWPOs / HPOs with SM to **probe new physics**
  $\longrightarrow$ multi-loop corrections in full SM

- Extraction of EWPOs / HPOs (**pseudo-observables**) from **real observables**
  $\longrightarrow$ QED/QCD, MC tools $\longrightarrow$ talk by S. Jadach

- "Other" eletroweak parameters (**"input" parameters**)
  $\longrightarrow$ m_t, α_s, etc. extracted from other processes

Other electroweak precision parameters

Reviews: 1906.05379, 2012.11642

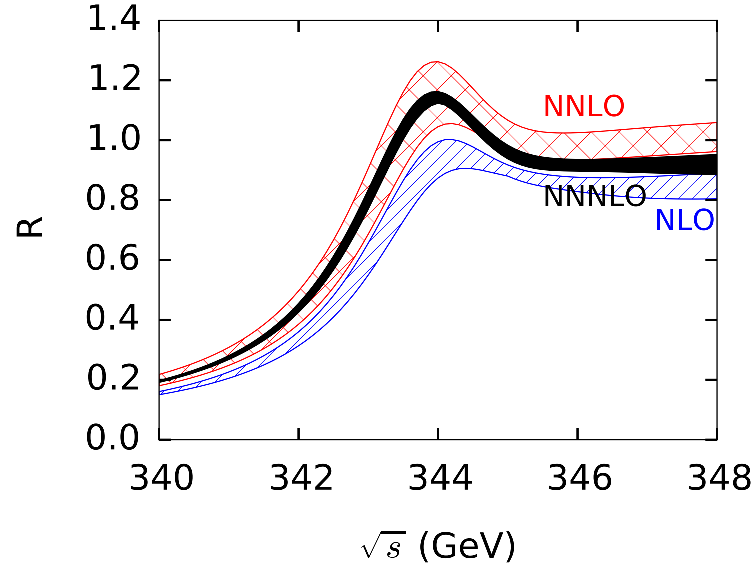
- M_Z, Γ_Z: From $\sigma(\sqrt{s})$ lineshape; δM_Z, $\delta\Gamma_Z \sim 0.1$ MeV at FCC-ee
  $\rightarrow$ Main theory uncertainties: QED ISR $\rightarrow$ talk by S. Jadach

- m_t: Current status $\delta m_t \sim 0.3$ GeV at LHC PDG '20
  $\rightarrow$ Additional theory uncertainties? Butenschoen et al. '16
  Ferrario Ravasio, Nason, Oleari '18

From $e^+e^- \rightarrow t\bar{t}$ at $\sqrt{s} \sim 350$ GeV

today:

$$\delta m_t^{\overline{\text{MS}}} = [\quad]_{\text{exp}} \oplus [50\text{ MeV}]_{\text{QCD}} \oplus [10\text{ MeV}]_{\text{mass def.}} \oplus [70\text{ MeV}]_{\alpha_s} > 100\text{ MeV}$$



Beneke et al. '15

Other electroweak precision parameters

Reviews: 1906.05379, 2012.11642

- $M_Z,\ \Gamma_Z$: From $\sigma(\sqrt{s})$ lineshape; $\delta M_Z,\ \delta\Gamma_Z \sim 0.1$ MeV at FCC-ee
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From $e^+e^- \to t\bar{t}$ at $\sqrt{s} \sim 350$ GeV

today:	**future:**
$\delta m_t^{\overline{\rm MS}} = [\quad]_{\rm exp}$	$[20\text{ MeV}]_{\rm exp}$
$\oplus\ [50\text{ MeV}]_{\rm QCD}$	$\oplus\ [30\text{ MeV}]_{\rm QCD}$ (h.o. resummation)
$\oplus\ [10\text{ MeV}]_{\text{mass def.}}$	$\oplus\ [10\text{ MeV}]_{\text{mass def.}}$
$\oplus\ [70\text{ MeV}]_{\alpha_s}$	$\oplus\ [15\text{ MeV}]_{\alpha_s}$ ($\delta\alpha_s \lesssim 0.0002$)
> 100 MeV	$\lesssim 50$ MeV

Other electroweak precision parameters

- $m_b,\ m_c$: From quarkonia spectra using Lattice QCD
  $\delta m_b^{\overline{\text{MS}}} \sim 30$ MeV, $\delta m_b^{\overline{\text{MS}}} \sim 25$ MeV — LHC HXSWG '16

  $\rightarrow$ estimated improvements $\delta m_b^{\overline{\text{MS}}} \sim 13$ MeV, $\delta m_b^{\overline{\text{MS}}} \sim 7$ MeV
  Lepage, Mackenzie, Peskin '14

- M_{H}: from kinematic constraint fits $HZ(\ell\ell)$, $H(b\bar{b})Z$
  $\rightarrow \delta M_{\text{H}} \sim 10...20$ MeV
  $\rightarrow$ theory errors subdominant

Strong coupling

- α_s: d'Enterria, Skands, et al. '15
  - Most precise determination using Lattice QCD:

    $\alpha_s = 0.1184 \pm 0.0006$ HPQCD '10

    $\alpha_s = 0.1185 \pm 0.0008$ ALPHA '17

    $\alpha_s = 0.1179 \pm 0.0015$ Takaura et al. '18

    $\alpha_s = 0.1172 \pm 0.0011$ Zafeiropoulos et al. '19

    $\rightarrow$ Difficulty in evaluating systematics

  - e^+e^- event shapes and DIS: $\alpha_s \sim 0.114$

    Alekhin, Blümlein, Moch '12; Abbate et al. '11; Gehrmann et al. '13

    $\rightarrow$ Subject to sizeable non-pertubative power corrections

    $\rightarrow$ Systematic uncertainties in power corrections?

  - Hadronic τ decays: $\alpha_s = 0.119 \pm 0.002$ PDG '18

    $\rightarrow$ Non-perturbative uncertainties in OPE and from duality violation

    Pich '14; Boito et al. '15,18

- α_s: d'Enterria, Skands, et al. '15
  - Electroweak precision ($R_\ell = \Gamma_Z^{had}/\Gamma_Z^\ell$):

    $\alpha_s = 0.120 \pm 0.003$ PDG '18

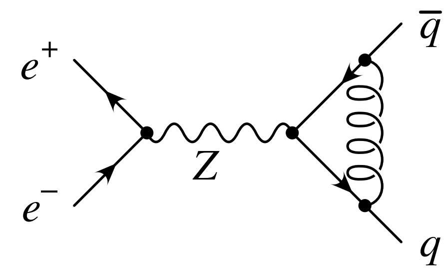
    $\rightarrow$ No (negligible) non-perturbative QCD effects

    FCC: $\delta R_\ell \sim 0.001$

    $\Rightarrow \quad \delta\alpha_s < 0.0001$

    Theory input: N^{3}LO EW corr. + leading N^{4}LO to keep $\delta_{th} R_\ell \lesssim \delta_{exp} R_\ell$

    Caviat: R_ℓ could be affected by new physics

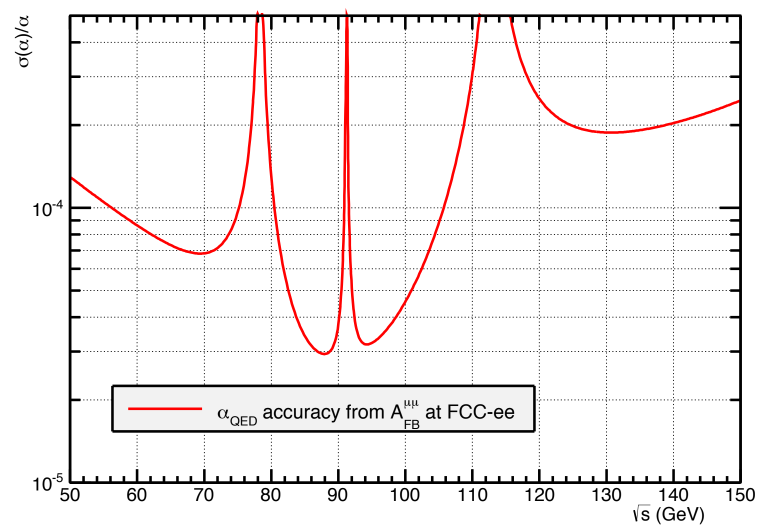


Shift of finestructure constant

- $\Delta\alpha_{\text{had}}$: Could be limiting factor
  - a) From $e^+e^- \to$ had. using dispersion relation
    Current: $\delta(\Delta\alpha_{\text{had}}) \sim 10^{-4}$
    Improvement to $\delta(\Delta\alpha_{\text{had}}) \sim 5 \times 10^{-5}$ likely
  - b) Direct determination at FCC-ee from $e^+e^- \to \mu^+\mu^-$ off the Z peak
    (i.e. $A_{\text{FB}}^{\mu\mu}$ at $\sqrt{s} \sim 88$ GeV and $\sqrt{s} \sim 95$ GeV)
    $\to \delta_{\text{th}}(\Delta\alpha_{\text{had}}) \sim 3 \times 10^{-5}$

Janot '15

Requires high-precision theory prediction for $e^+e^- \to \mu^+\mu^-$ including 2/3-loop corrections for γ-exchange and box contributions



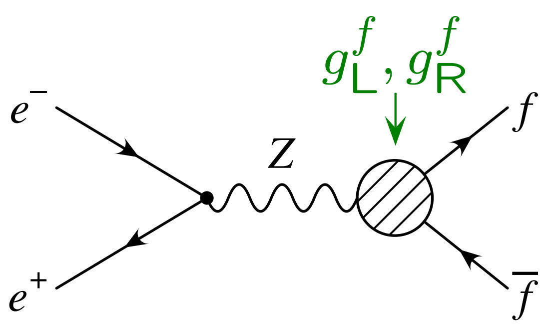
Z pole and WW

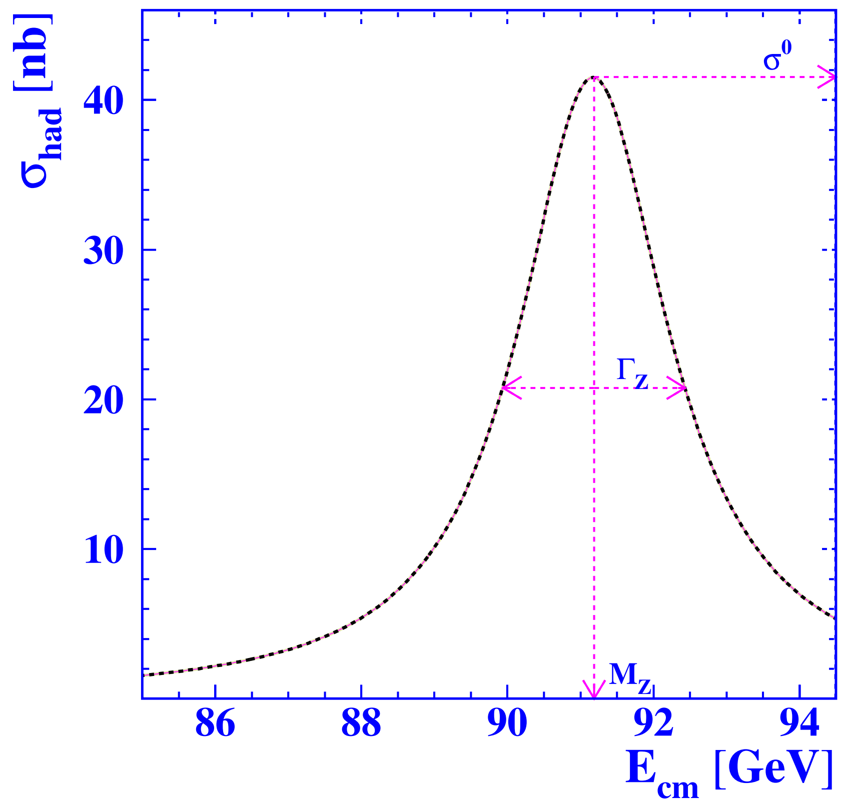
Z cross section and branching fractions

$e^+e^- \to f\bar{f}$ for $E_{\mathsf{CM}} \sim M_{\mathsf{Z}}$:

- Mass M_{Z}
- Width $\Gamma_{\mathsf{Z}} = \sum_f \Gamma_{ff}$
- Braching ratio $R_f = \Gamma_{ff}/\Gamma_{\mathsf{Z}}$
- $\sigma^0 \approx \frac{12\pi\,\Gamma_{ee}\Gamma_{ff}}{(s-M_{\mathsf{Z}}^2)^2+M_{\mathsf{Z}}^2\Gamma_{\mathsf{Z}}^2} = \frac{12\pi}{M_{\mathsf{Z}}^2}R_e R_f$

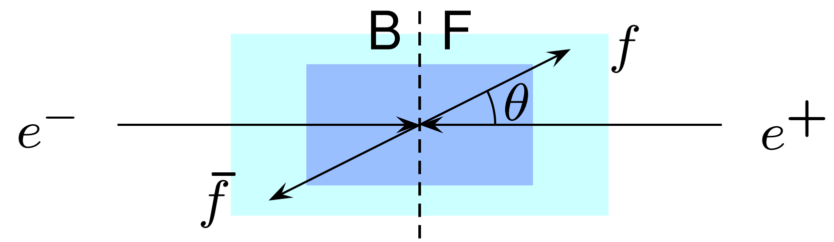
$$\Gamma_{ff} = C\left[(g_{\mathsf{L}}^f)^2 + (g_{\mathsf{R}}^f)^2\right]$$





Z-pole asymmetries

Forward-backward asymmetry:



$$A_{\mathrm{FB}} \equiv \frac{\sigma_{\mathrm{F}} - \sigma_{\mathrm{B}}}{\sigma_{\mathrm{F}} + \sigma_{\mathrm{B}}} = \frac{3}{4}\mathcal{A}_e\mathcal{A}_f$$

$$\mathcal{A}_f = \frac{2(1 - 4\sin^2\theta^f_{\mathrm{eff}})}{1 + (1 - 4\sin^2\theta^f_{\mathrm{eff}})^2} \qquad \sin^2\theta^f_{\mathrm{eff}} = \frac{g^f_R}{2|Q_f|(g^f_R - g^f_L)}$$

Left-right asymmetry:

With polarized e^- beam: $$A_{\mathrm{LR}} \equiv \frac{\sigma_{\mathrm{L}} - \sigma_{\mathrm{R}}}{\sigma_{\mathrm{L}} + \sigma_{\mathrm{R}}} = \mathcal{A}_e$$

Polarization asymmetry:

Average τ pol. in $e^+e^- \to \tau^+\tau^-$: $$\langle \mathcal{P}_\tau \rangle = -\mathcal{A}_\tau$$

Z lineshape

- Deconvolution of initial-state QED radiation:

$$\sigma[e^+e^- \to f\bar{f}] = \mathcal{R}_{\rm ini}(s,s') \otimes \sigma_{\rm hard}(s')$$

- Subtraction of γ-exchange, γ–Z interference, box contributions:

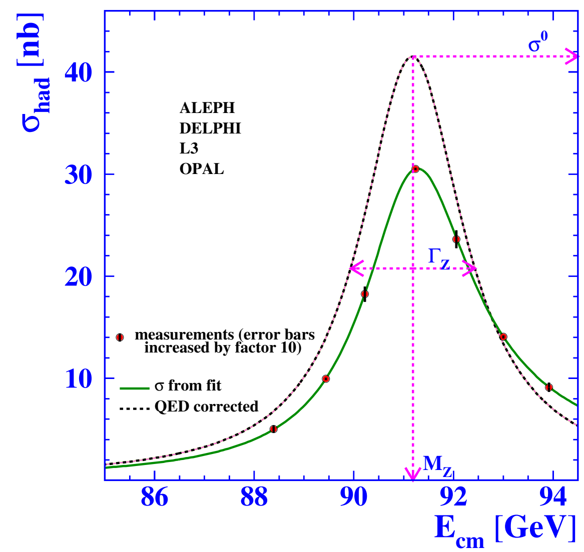
$$\sigma_{\rm hard} = \sigma_Z + \sigma_\gamma + \sigma_{\gamma Z} + \sigma_{\rm box}$$

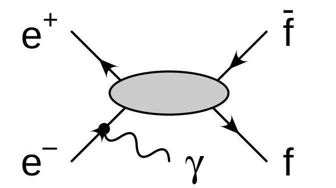
- Z-pole contribution:

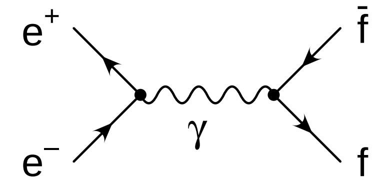
$$\sigma_Z = \frac{R}{(s-\overline{M}_Z^2)^2 + \overline{M}_Z^2\overline{\Gamma}_Z^2} + \sigma_{\rm non-res}$$

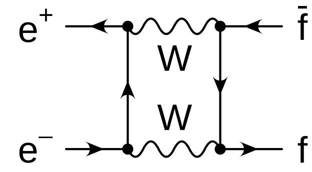
σ_γ, $\sigma_{\gamma Z}$, $\sigma_{\rm box}$, $\sigma_{\rm non-res}$ known at NLO
$\to$ need consistent pole expansion framework
$\to$ leading NNLO may be needed for FCC-ee/CEPC

LEP EWWG '05







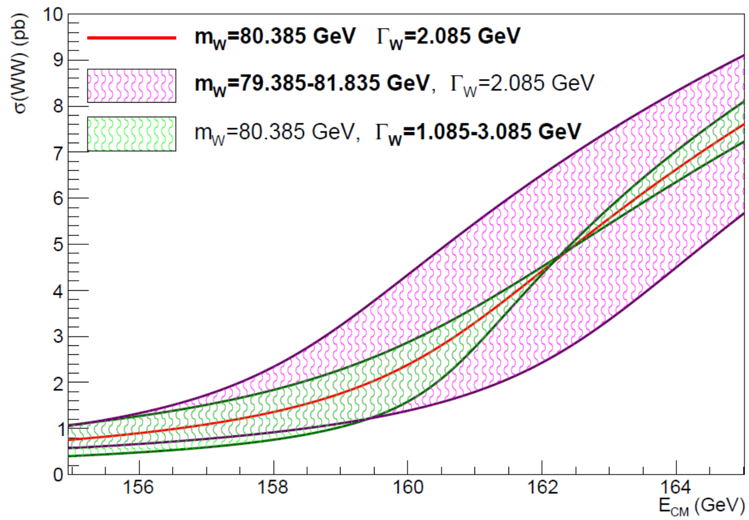


WW threshold

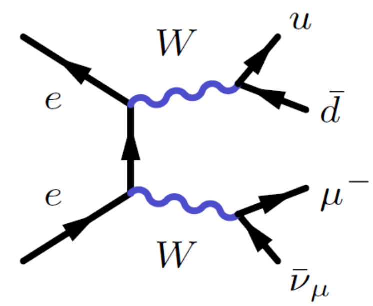
- High-precision measurement of M_W from $e^+e^- \to W^+W^-$ at threshold
- a) Corrections near threshold enhanced by $1/\beta$ and $\ln\beta$

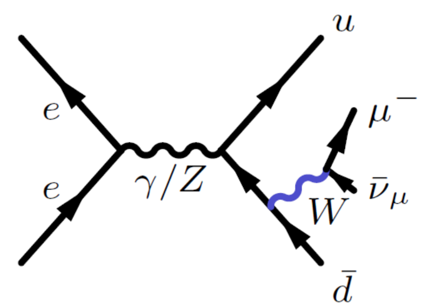
$$\beta \sim \sqrt{1 - 4\frac{M_W^2 - iM_W\Gamma_W}{s}} \sim \sqrt{\Gamma_W/M_W}$$

b) Non-resonant contributions are important



- Full $\mathcal{O}(\alpha)$ calculation of $e^+e^- \to 4f$
  Denner, Dittmaier, Roth, Wieders '05
- EFT expansion in $\alpha \sim \Gamma_W/M_W \sim \beta^2$
  Beneke, Falgari, Schwinn, Signer, Zanderighi '07
  - NLO corrections with NNLO Coulomb correction ($\propto 1/\beta^n$): $\delta_{th}M_W \sim 3$ MeV
    Actis, Beneke, Falgari, Schwinn '08
  - Adding NNLO corrections to $ee \to WW$ and $W \to f\bar{f}$ and NNLO ISR: $\delta_{th}M_W \lesssim 0.6$ MeV





Comparison of EWPOs with theory

- To probe new physics, compare EWPOs with SM theory predictions
- Need to take theory error into account:

	Current exp.	Current th.†	CEPC	FCC-ee
$M_{\mathrm{W}}{}^{*}$ [MeV]	15	4	1	1
Γ_Z [MeV]	2.3	0.4	0.5	0.1
$R_\ell = \Gamma_{\mathrm{Z}}^{\mathrm{had}}/\Gamma_{\mathrm{Z}}^{\ell}$ $[10^{-3}]$	25	5	2	1
$R_b = \Gamma_{\mathrm{Z}}^{b}/\Gamma_{\mathrm{Z}}^{\mathrm{had}}$ $[10^{-5}]$	66	10	4.3	6
$\sin^2\theta_{\mathrm{eff}}^{\ell}$ $[10^{-5}]$	16	4.5	<1	0.5

* computed from G_μ

† NNLO and leading 3/4-loop (enhanced by Y_{t} and/or N_f)

Theory and parametric uncertainties

	CEPC	perturb. error with 3-loop†	Param. error CEPC*	main source
M_W [MeV]	1	1	2.1	m_t, $\Delta\alpha$
Γ_Z [MeV]	0.5	0.15	0.15	m_t, α_s
R_b [10^{-5}]	4.3	5	< 1	
$\sin^2\theta^{\ell}_{\rm eff}$ [10^{-5}]	<1	1.5	2	m_t, $\Delta\alpha$

† **Theory scenario:** $\mathcal{O}(\alpha\alpha_s^2)$, $\mathcal{O}(N_f\alpha^2\alpha_s)$, $\mathcal{O}(N_f^2\alpha^2\alpha_s)$, leading 4-loop

(N_f^n = at least n closed fermion loops)

Parametric inputs:

***CEPC:** $\delta m_t = 600$ MeV, $\delta\alpha_s = 0.0002$, $\delta M_Z = 0.5$ MeV,
$\delta(\Delta\alpha) = 5\times 10^{-5}$

Theory and parametric uncertainties

	FCC-ee	perturb. error with 3-loop†	Param. error FCC-ee*	main source
M_W [MeV]	1	1	0.6	$\Delta\alpha$
Γ_Z [MeV]	0.1	0.15	0.1	α_s
R_b [10^{-5}]	6	5	< 1	
$\sin^2\theta^{\ell}_{eff}$ [10^{-5}]	0.5	1.5	1	$\Delta\alpha$

† **Theory scenario:** $\mathcal{O}(\alpha\alpha_s^2)$, $\mathcal{O}(N_f\alpha^2\alpha_s)$, $\mathcal{O}(N_f^2\alpha^2\alpha_s)$, leading 4-loop

(N_f^n = at least n closed fermion loops)

Parametric inputs:

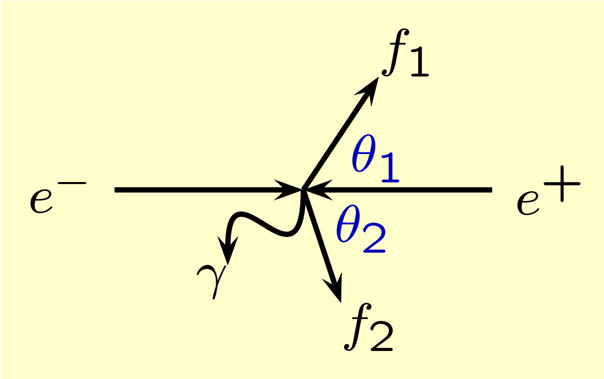
***FCC-ee:** $\delta m_t = 50$ MeV, $\delta\alpha_s = 0.0002$, $\delta M_Z = 0.1$ MeV,
$\delta(\Delta\alpha) = 3 \times 10^{-5}$

Electroweak precision at $\sqrt{s} = 250$ GeV

EWPOs accessible through **radiative return** $e^+e^- \to \gamma Z$

- γ mostly collinear with beam
- Reduction in cross-section by $\sim \frac{\alpha}{\pi} \ln \frac{s}{m_e^2} \sim 0.06$
- Precise det. of m_{ff} from measured angles:

$$m_{ff}^2 = s\,\frac{1-\beta}{1+\beta}, \quad \beta = \frac{|\sin(\theta_1+\theta_2)|}{\sin\theta_1 + \sin\theta_2}$$



- Additional backgrounds from $e^+e^- \to WW, ZZ$ that are not flat in m_{ff}

Ueno '19

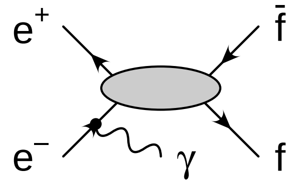
$Z\gamma$ electroweak precision: observables

Fujii et al. '19

- $A_{\mathrm{LR}} \to \sin^2\theta^{\ell}_{\mathrm{eff}}$ (limited by sys. err. on beam polarization)
- $A^{\mu,\tau,b}_{\mathrm{FB}}$ (statistics limited)
- R_ℓ, R_c, R_b (limited by sys. err. on flavor tag)
- No competitive measurements on M_Z, Γ_Z, σ^0 (need to use LEP values)

$Z\gamma$ electroweak precision: theory input

Leading effect: Soft+collinear multi-γ ISR



$$\sigma[e^+e^- \to f\bar{f}] = \mathcal{R}_{\text{ini}}(s, s') \otimes \sigma_{\text{hard}}(s')$$

Kureav, Fadin '85
Berends, Burgers, v. Neerven '88
Kniehl, Krawczyk, Kühn, Stuart '88
Beenakker, Berends, v. Neerven '89
Bardin et al. '91; Skrzypek '92
Montagna, Nicrosini, Piccinini '97

$$\mathcal{R}_{\text{ini}} = \sum_n \left(\frac{\alpha}{\pi}\right)^n \sum_{m=0}^{n} h_{nm} \ln^m\left(\frac{s}{m_e^2}\right)$$

Universal ($m{=}n$) logs known to $n = 6$, also some sub-leading terms

Ablinger, Blümlein, De Freitas, Schönwald '20

Exclusive description: MC tools

Subleading effects:

Radiative corrections to $e^+e^- \to f\bar{f}\gamma\,(+n\gamma)$

- Some corrections cancel for A_{LR}, A_{FB}, BRs
- NLO for $ee \to f\bar{f}\gamma$ + NNLO for $ee \to Z\gamma$, $Z \to f\bar{f}$ could be sufficient

W mass measurement from $e^+e^- \to WW$: Baak et al. '13

- $\ell\nu_\ell \ell'\nu_{\ell'}$: Endpoints of E_ℓ or other distributions
- $\ell\nu_\ell jj$: Kinematic reconstruction
- $jjjj$: Systematic uncertainty from color reconnection

Expected precision with $\mathcal{L}_{\text{int}} = 2$ ab^{-1} at $\sqrt{s} = 250$ GeV: $\Delta M_{\text{W}} \approx 2.5$ MeV

Theory needs: Small impact of loop corrections, but accurate decription of FSR QED effects needed

Higgs physics: SM predictions for Higgs decays

Reviews: 1404.0319, 1906.05379

hbb: [CEPC: 2.0%, FCC-ee: 0.8%]

- $\mathcal{O}(\alpha_s^4)$ QCD corrections — Baikov, Chetyrkin, Kühn '05
- $\mathcal{O}(\alpha)$ QED+EW — Dabelstein, Hollik '92; Kniehl '92
- leading $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\alpha\alpha_s)$ for large m_t — Kwiatkowski, Steinhauser '94; Butenschoen, Fugel, Kniehl '07
  $\rightarrow$ Use for error estimate

Current theory error: $\Delta_{th} < 0.4\%$

With full 2-loop: $\Delta_{th} \sim 0.2\%$

Parametric error:

$$\left.\begin{array}{l}\delta m_b = 0.030 \text{ GeV} \\ \delta\alpha_s = 0.001\end{array}\right\} \rightarrow \Delta_{par} \approx 1.4\%$$

$$\left.\begin{array}{l}\delta m_b = 0.013 \text{ GeV} \\ \delta\alpha_s = 0.0002\end{array}\right\} \rightarrow \Delta_{par} \approx 0.6\%$$

SM predictions for Higgs decays

h$\tau\tau$: [CEPC: 2.4%, FCC-ee: 1.1%]

With full 2-loop (no QCD): $\Delta_{\rm th} < 0.1\%$

Parametric error negligible

hWW*/hZZ*: [CEPC: 2.2%, FCC-ee: 0.4%]

- complete $\mathcal{O}(\alpha) + \mathcal{O}(\alpha_{\rm s})$ for $h \to 4f$ — Bredenstein, Denner, Dittmaier, Weber '06
- leading $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha\alpha_{\rm s})$ and $\mathcal{O}(\alpha\alpha_{\rm s}^2)$ for large $m_{\rm t}$ — Djouadi, Gambino, Kniehl '97; Kniehl, Veretin '12

$\rightarrow$ Small (0.2%) effect — Ma, Wang, Xu, Yang, Zhou '21

Theory error: $\Delta_{\rm th,EW} < 0.3\%$, $\Delta_{\rm th,QCD} < 0.5\%$

With NNLO final-state QCD corrections: $\Delta_{\rm th,QCD} < 0.1\%$

Parametric error:

$\delta M_{\rm H} \sim 10$ MeV $\rightarrow \Delta_{\rm par} \approx 0.1\%$

Note: Distributions affected by corrections $\rightarrow$ implementation into MC tools

SM predictions for Higgs decays

hgg: [CEPC: 2.4%, FCC-ee: 1.6%]

- $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ (in large m_t-limit) QCD corrections — Baikov, Chetyrkin '06; Schreck, Steinhauser '07
- $\mathcal{O}(\alpha)$ EW — Aglietti, Bonciani, Degrassi, Vicini '04; Degrassi, Maltoni '04

Theory error (dominated by QCD): $\Delta_{th} \approx 3\%$
With $\mathcal{O}(\alpha_s^4)$ in large m_t-limit (4-loop massless QCD diags.): $\Delta_{th} \approx 1\%$

Parametric error: $\delta\alpha_s = 0.001 \rightarrow \Delta_{par} \approx 3\%$
$\delta\alpha_s = 0.0001 \rightarrow \Delta_{par} \approx 0.3\%$

h$\gamma\gamma$: [CEPC: 3.2%, FCC-ee: 3.0%]

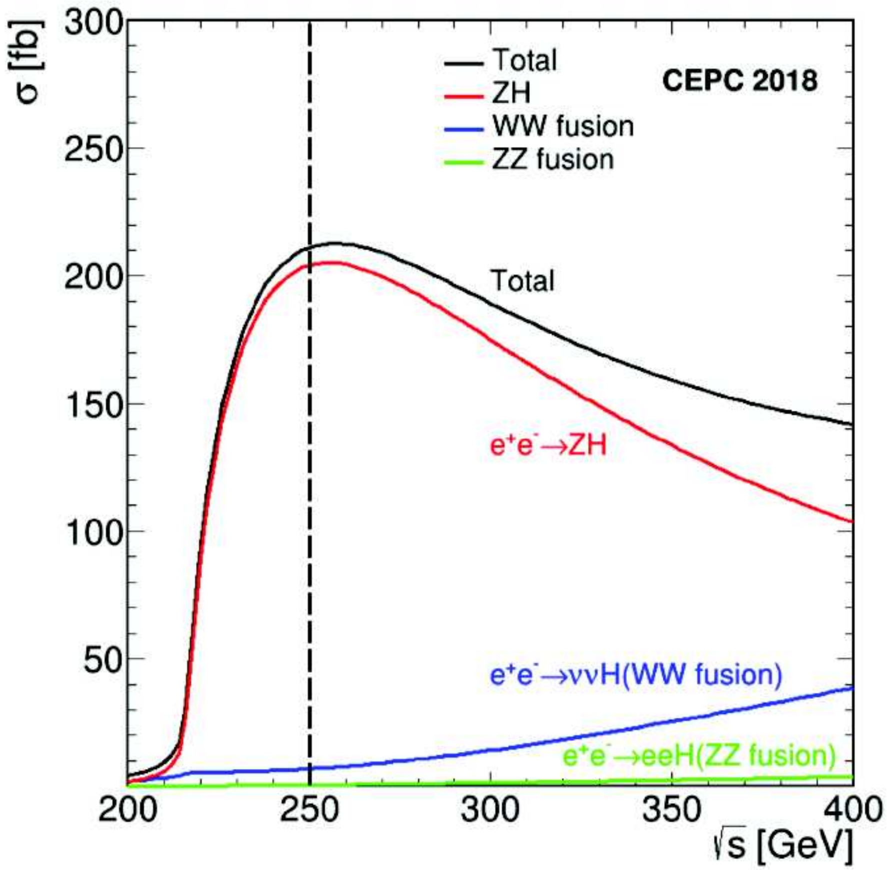
- $\mathcal{O}(\alpha_s^2)$ QCD corrections — Zheng, Wu '90; Djouadi, Spira, v.d.Bij, Zerwas '91; Dawson, Kauffman '93; Maierhöfer, Marquard '12
- $\mathcal{O}(\alpha)$ EW — Aglietti, Bonciani, Degrassi, Vicini '04; Degrassi, Maltoni '04; Actis, Passarino, Sturm, Uccirati '08

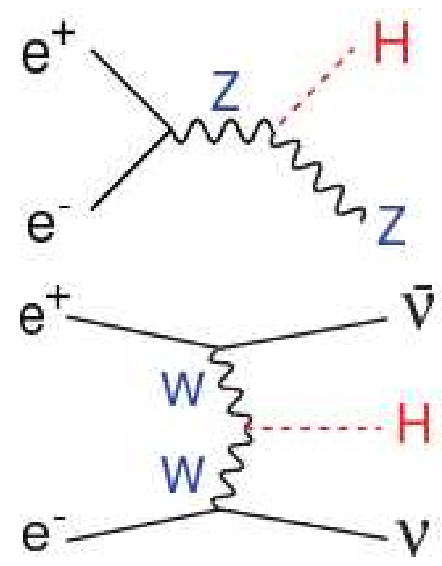
Theory error: $\Delta_{th} < 1\%$

Parametric error negligible

SM predictions for Higgs production

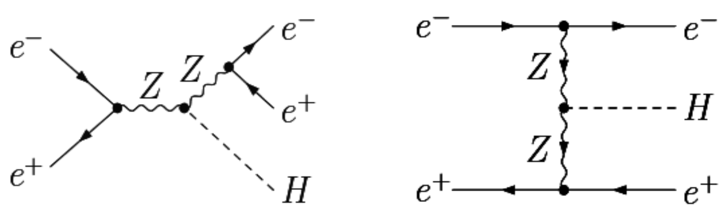
- hZ production: dominant at $\sqrt{s} \sim 240$ GeV
- WW fusion: sub-dominant but useful for constraining h width   Han, Liu, Sayre '13





hZ production: [CEPC: 0.5%, FCC-ee: 0.3%]

- $\mathcal{O}(\alpha)$ corr. to hZ production and Z decay — Kniehl '92; Denner, Küblbeck, Mertig, Böhm '92; Consoli, Lo Presti, Maiani '83; Jegerlehner '86; Akhundov, Bardin, Riemann '86
- Technology for $\mathcal{O}(\alpha)$ with off-shell Z-boson available — Boudjema et al. '04; Denner, Dittmaier, Roth, Weber '03



- Can be combined with h.o. ISR QED radiation — Greco et al. '17
- $\mathcal{O}(\alpha\alpha_s)$ corrections — Gong et al. '16; Chen, Feng, Jia, Sang '18

Theory error: $\Delta_{th} \sim O(1\%)$

With full 2-loop corrections for $ee \to HZ$: $\Delta_{th} \lesssim O(0.3\%)$

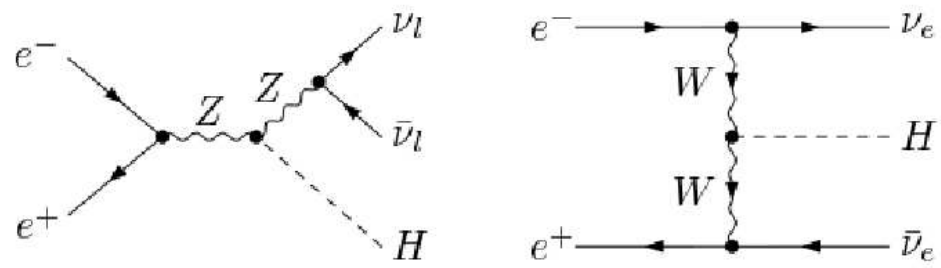
Parametric error: negligible if $\delta M_H < 100$ MeV

WW fusion:

- $\mathcal{O}(\alpha)$ corrections with h.o. ISR — Belanger et al. '02; Denner, Dittmaier, Roth, Weber '03

Theory error: $\Delta_{\text{th}} \sim O(1\%)$?

Parametric error: negligible



Full $\mathcal{O}(\alpha^2)$ calculation for 2→3 process is very challenging
→ Contributions with closed fermion loops maybe feasible

Summary

For Higgs and WW physics:

- Full NNLO for $2 \to 2$ processes
- $\mathcal{O}(\alpha_s^4)$ QCD corrections
- Matching to Monte-Carlo tools
- Also need $O(\alpha)$ (or better?) corrections for backgrounds: $e^+e^-b\bar{b}$, $\nu\bar{\nu}b\bar{b}$, etc.
  $\longrightarrow$ Technology exists, but work needed — Denner, Dittmaier, Roth, Wieders '05

For Z pole:

- 3-loop EW and mixed EW-QCD corrections for Zff vertices
- Leading 4-loop effects
- Initial-final QED effects / mergering multi-loop and Monte-Carlo

Input parameters:

- Direct determination of α_s, m_t, $\alpha(M_Z)$ at e^+e^- colliders is important
- Perturbative and non-perturbative theory uncertainties need improvement
- Lower-energy experiments can provide additional input (BELLE II, BES, ...)

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- Matching to Monte-Carlo tools
- Also need $O(\alpha)$ (or better?) corrections for backgrounds: $e^+e^-b\bar{b}$, $\nu\bar{\nu}b\bar{b}$, etc.
  $\longrightarrow$ Technology exists, but work needed Denner, Dittmaier, Roth, Wieders '05

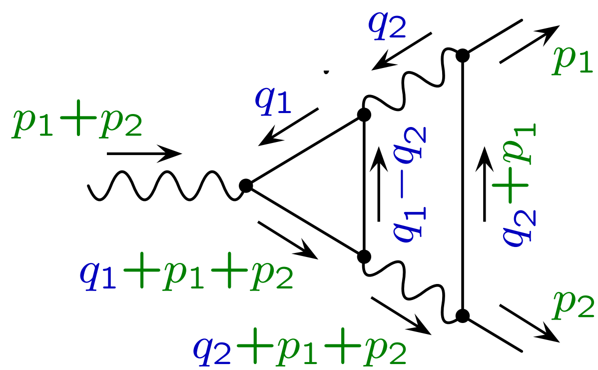
EW precision at ILC-250:

- Similar physics goals as GigaZ/TeraZ, but with reduced precision
- Theory: NNLO for $2 \to 2$ processes, ISR resummation, and ...?
- Open question: Direct determination of α_s and $\alpha(s)$

Backup slides

Calculational techniques

Experimental precision requires inclusion of **radiative corrections** in theory (1-loop, 2-loop, and partial 3-loop)



Integrals over loop momenta:

$$\int d^4q_1 d^4q_2 \; f(q_1, q_2, p_1, p_2, \ldots, m_1, m_2, \ldots)$$

Computer algebra tools:

- Generation of diagrams, $\mathcal{O}(100) - \mathcal{O}(10000)$
- Lorentz and Dirac algebra
- Integral simplification (and expansion)

Evaluation of loop integrals:

- In general not possible analytically
- Numerical methods are more general, but computing intensive
- Special numerical techniques can balance precision and evaluation time

Analytic calculations

- Mostly used for diagrams with few mass scales
- Reduce to **master integrals** with integration-by-parts and other identities

  Chetyrkin, Tkachov '81; Gehrmann, Remiddi '00; Laporta '00; ...

  Public programs:

Public programs:	`Reduze`	von Manteuffel, Studerus '12
	`FIRE`	Smirnov '13,14
	`LiteRed`	Lee '13
	`KIRA`	Maierhoefer, Usovitsch, Uwer '17

  → Large need for computing time and memory

- Evaluate master integrals with differential equations or Mellin-Barnes rep.

  Kotikov '91; Remiddi '97; Smirnov '00,01; Henn '13; ...

  → Result in terms of Goncharov polylogs / multiple polylogs

  → Some problems need iterated elliptic integrals / elliptic multiple polylogs

  Broedel, Duhr, Dulat, Trancredi '17,18
  Ablinger er al. '17

  → Even more classes of functions needed in future?

Asymptotic expansions

- Exploit large mass ratios,
  $e.g.\ M_Z^2/m_t^2 \approx 1/4$
- Evaluate coeff. integrals analytically
- Fast numerical evaluation

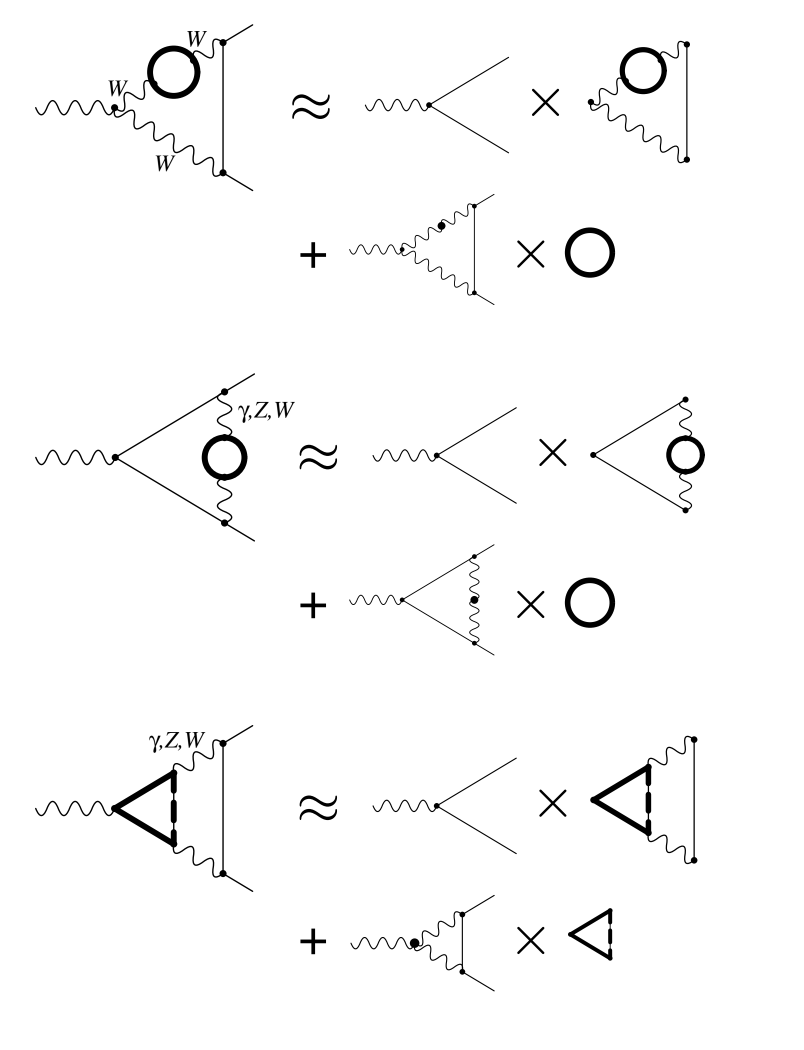
$\rightarrow$ Used in some 2/3-scale problems

$\rightarrow$ Public programs:

`exp` Harlander, Seidensticker, Steinhauser '97
`asy` Pak, Smirnov '10

$\rightarrow$ Possible limitations:

- Difficult coefficient integrals
- bad convergence



Numerical integration

Two general approches:

$\longrightarrow$ Automated treatment of UV/IR divergencies

$\longrightarrow$ No restriction on number of loops or legs

- **Sector decomposition:**

  Public programs: `(py)SecDec` Carter, Heinrich '10; Borowka et al. '12,15,17
  `FIESTA` Smirnov, Tentyukov '08; Smirnov '13,15

- **Mellin-Barnes representations:**

  Public programs: `MB/MBresolve` Czakon '06; Smirnov, Smirnov '09

  `AMBRE/MBnumerics` Gluza, Kajda, Riemann '07
  Dubovyk, Gluza, Riemann '15
  Usovitsch, Dubovyk, Riemann '18

- Diagrams with internal thresholds can cause numerical instabilities

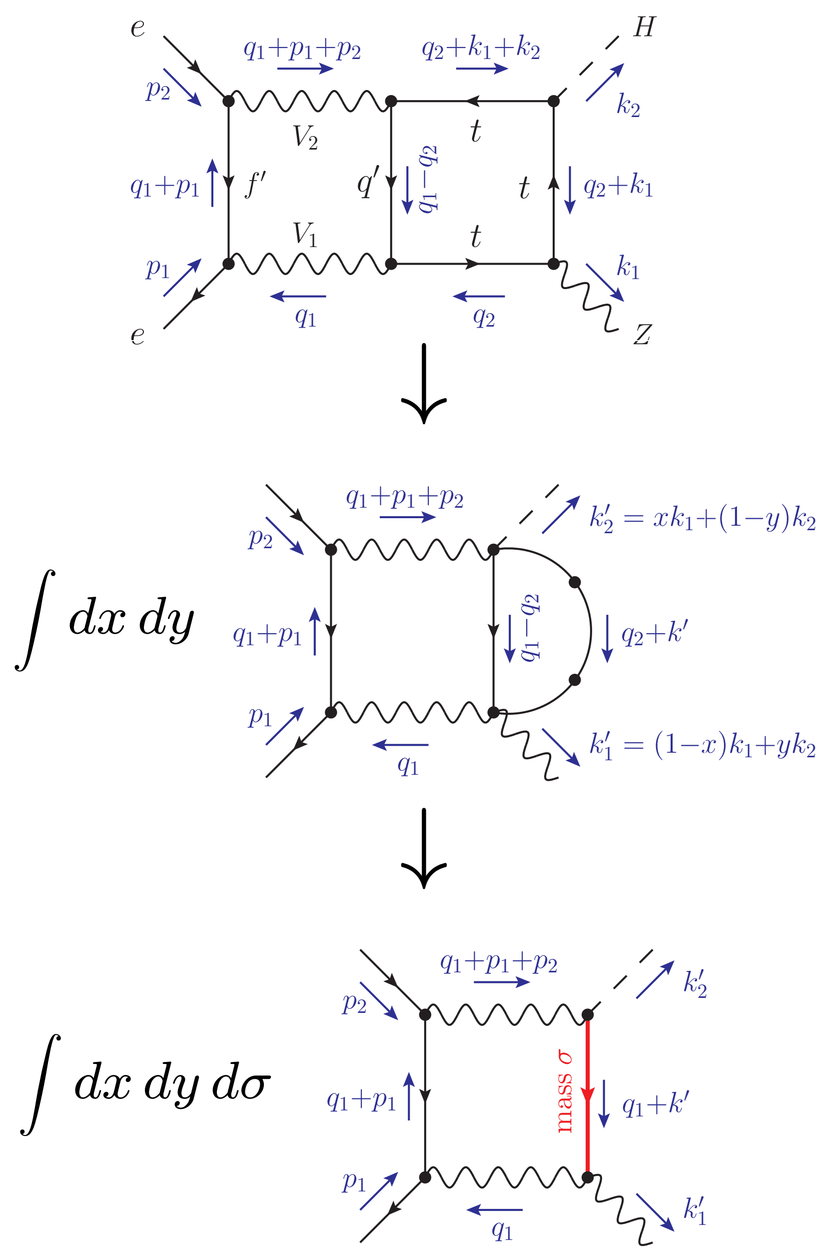
- Specialized techniques (for some type of diagrams) often improve computing time, robustness, precision (but not automated)

Specialized numerical techniques

Example: HZ double boxes

Song, Freitas '21

- Introduce Feynman parameters and disp. rel.
- Expressions for second loop from, e.g., `LoopTools`
  Hahn, Perez-Victoria '98
- 3-dim. numerical integral with adaptive Gaussian integration
- $\mathcal{O}(0.1\%)$ precision in $\mathcal{O}$(min.) on laptop



Z lineshape

- Deconvolution of initial-state QED radiation:

$$\sigma[e^+e^- \to f\bar{f}] = \mathcal{R}_{\text{ini}}(s, s') \otimes \sigma_{\text{hard}}(s')$$

- Subtraction of γ-exchange, γ–Z interference, box contributions:

$$\sigma_{\text{hard}} = \sigma_Z + \sigma_\gamma + \sigma_{\gamma Z} + \sigma_{\text{box}}$$

- Z-pole contribution:

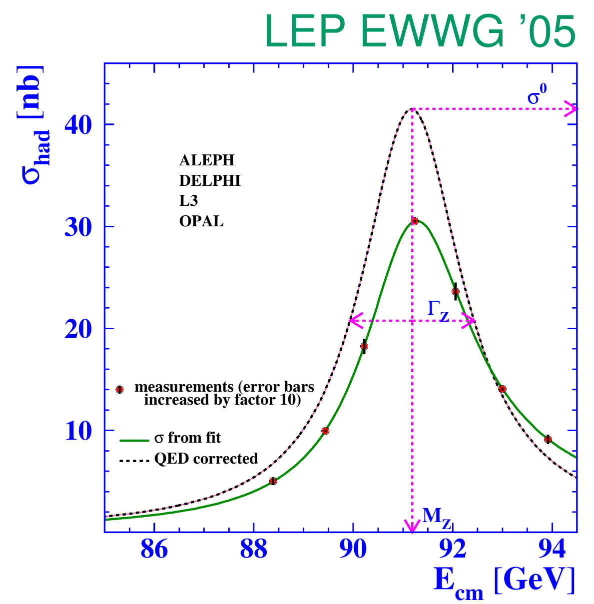
$$\sigma_Z = \frac{R}{(s - \overline{M}_Z^2)^2 + \overline{M}_Z^2 \overline{\Gamma}_Z^2} + \sigma_{\text{non-res}}$$

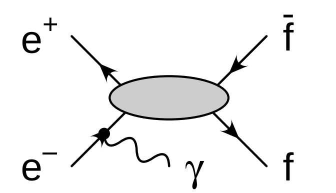
- In experimental analyses:

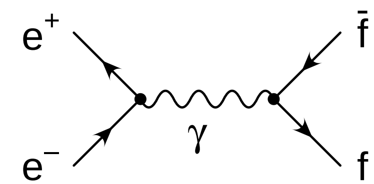
$$\sigma \sim \frac{1}{(s - M_Z^2)^2 + s^2\Gamma_Z^2/M_Z^2}$$

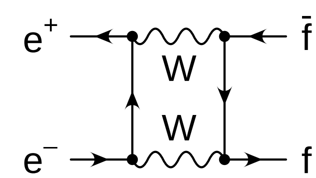
$$\overline{M}_Z = M_Z \Big/ \sqrt{1 + \Gamma_Z^2/M_Z^2} \approx M_Z - 34 \text{ MeV}$$

$$\overline{\Gamma}_Z = \Gamma_Z \Big/ \sqrt{1 + \Gamma_Z^2/M_Z^2} \approx \Gamma_Z - 0.9 \text{ MeV}$$









"Hard" matrix element

Consistent (gauge-invariant) theory setup:

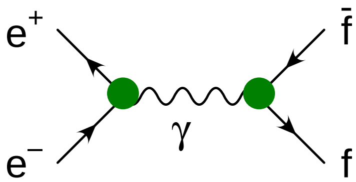
Expansion of $\mathcal{A}[e^+e^- \to \mu^+\mu^-]$ about $s_0 = M_Z^2 - iM_Z\Gamma_Z$:

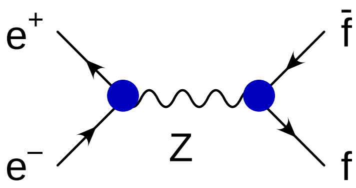
$$\mathcal{A}[e^+e^- \to f\bar{f}] = \frac{R}{s - s_0} + S + (s - s_0)T + \ldots$$

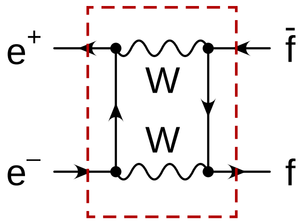
$$R = g_Z^e(s_0)\, g_Z^f(s_0)$$

$$S = \left[\frac{1}{M_Z^2} g_\gamma^e g_\gamma^f + g_Z^e g_Z^{f\,\prime} + g_Z^{e\,\prime} g_Z^f + S_{\text{box}}\right]_{s=s_0}$$

$g_V^f(s)$: effective $Vf\bar{f}$ couplings





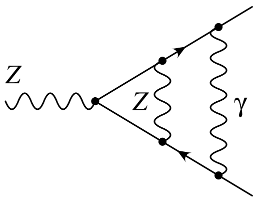


At NNLO: Need R at $\mathcal{O}(\alpha^2)$, S at $\mathcal{O}(\alpha)$, etc.

Z decay

Factorization of massive and QED/QCD FSR:

$$\overline{\Gamma}_f \approx \frac{N_c \overline{M}_Z}{12\pi} \left[\left(\mathcal{R}_V^f |g_V^f|^2 + \mathcal{R}_A^f |g_A^f|^2 \right) \frac{1}{1 + \mathrm{Re}\,\Sigma'_Z} \right]_{s=\overline{M}_Z^2}$$



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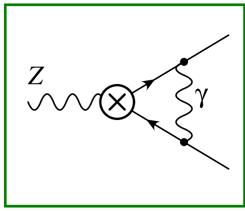
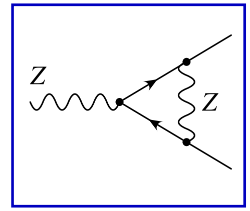
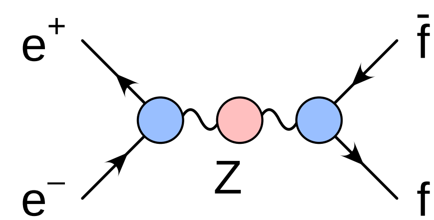


$+$ finite, with $\otimes =$


$\mathcal{R}_V^f$, $\mathcal{R}_A^f$: Final-state QED/QCD radiation;
known to $\mathcal{O}(\alpha_s^4)$, $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha\alpha_s)$

Kataev '92
Chetyrkin, Kühn, Kwiatkowski '96
Baikov, Chetyrkin, Kühn, Rittinger '12

g_V^f, g_A^f, Σ'_Z: Electroweak corrections



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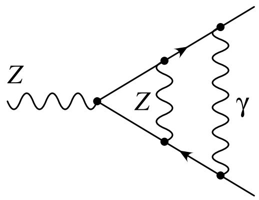
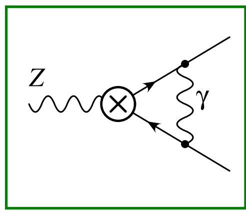
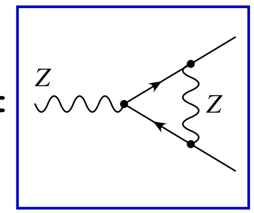
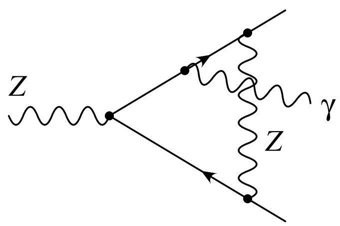


$=$ 

$+$ finite, with $\otimes =$ 

Additional non-factorizable contributions, e.g.



$\rightarrow$ Known at $\mathcal{O}(\alpha\alpha_s)$ Czarnecki, Kühn '96
Harlander, Seidensticker, Steinhauser '98

$\rightarrow$ Currently not known at $\mathcal{O}(\alpha^2)$ and beyond

$\rightarrow$ $\mathcal{O}(0.01\%)$ uncertainty on Γ_Z, σ_Z, maybe larger for A_b

$\rightarrow$ How to account for in MC simulations?

Z-pole asymmetries

Blondel scheme: (if e^- and e^+ polarization available) Blondel '88

Four independent measurements for $P_{e^+}/P_{e^-} = ++, +-, -+, --$

$$A_{\mathrm{LR}} = \sqrt{\frac{(\sigma_{++} + \sigma_{-+} - \sigma_{+-} - \sigma_{--})(-\sigma_{++} + \sigma_{-+} - \sigma_{+-} + \sigma_{--})}{(\sigma_{++} + \sigma_{-+} + \sigma_{+-} + \sigma_{--})(-\sigma_{++} + \sigma_{-+} + \sigma_{+-} - \sigma_{--})}}$$

Note: No need to know $|P_{e^\pm}|$!

Main systematic uncertainties:

- Difference of $|P|$ for $P > 0$ and $P < 0$
- Difference of $\mathcal{L}$ for $P > 0$ and $P < 0$

$\delta A_{\mathrm{LR}} \approx 10^{-4}$ $\Rightarrow$ $\delta \sin^2 \theta^{\ell}_{\mathrm{eff}} \approx 1.3 \times 10^{-5}$ Mönig, Hawkings '99

Theory calculations: Uncertainties

	Experiment	Theory error	Main source
M_W	80.379 ± 0.012 MeV	4 MeV	α^3, $\alpha^2\alpha_\mathrm{s}$
Γ_Z	2495.2 ± 2.3 MeV	0.4 MeV	α^3, $\alpha^2\alpha_\mathrm{s}$, $\alpha\alpha_\mathrm{s}^2$
R_ℓ	20.767 ± 0.025	0.005	α^3, $\alpha^2\alpha_\mathrm{s}$
R_b	0.21629 ± 0.00066	0.0001	α^3, $\alpha^2\alpha_\mathrm{s}$
$\sin^2\theta^\ell_\mathrm{eff}$	0.23153 ± 0.00016	4.5×10^{-5}	α^3, $\alpha^2\alpha_\mathrm{s}$

- Theory error estimate is not well defined, ideally $\Delta_\mathrm{th} \ll \Delta_\mathrm{exp}$
- Common methods:
  - Count prefactors (α, N_c, N_f, ...)
  - Extrapolation of perturbative series
  - Renormalization scale dependence
  - Renormalization scheme dependence

Example: Error estimation for Γ_Z

- Goemetric perturbative series

$\alpha_t = \alpha m_t^2$

$$\mathcal{O}(\alpha^3) - \mathcal{O}(\alpha_t^3) \sim \frac{\mathcal{O}(\alpha^2) - \mathcal{O}(\alpha_t^2)}{\mathcal{O}(\alpha)} \mathcal{O}(\alpha^2) \sim 0.26 \text{ MeV}$$

$$\mathcal{O}(\alpha^2\alpha_s) - \mathcal{O}(\alpha_t^2\alpha_s) \sim \frac{\mathcal{O}(\alpha^2) - \mathcal{O}(\alpha_t^2)}{\mathcal{O}(\alpha)} \mathcal{O}(\alpha\alpha_s) \sim 0.30 \text{ MeV}$$

$$\mathcal{O}(\alpha\alpha_s^2) - \mathcal{O}(\alpha_t\alpha_s^2) \sim \frac{\mathcal{O}(\alpha\alpha_s) - \mathcal{O}(\alpha_t\alpha_s)}{\mathcal{O}(\alpha)} \mathcal{O}(\alpha\alpha_s) \sim 0.23 \text{ MeV}$$

$$\mathcal{O}(\alpha\alpha_s^3) - \mathcal{O}(\alpha_t\alpha_s^3) \sim \frac{\mathcal{O}(\alpha\alpha_s) - \mathcal{O}(\alpha_t\alpha_s)}{\mathcal{O}(\alpha)} \mathcal{O}(\alpha\alpha_s^2) \sim 0.035 \text{ MeV}$$

$$\mathcal{O}(\alpha_{bos}^2) \sim \mathcal{O}(\alpha_{bos})^2 \sim 0.1 \text{ MeV}$$

- Parametric prefactors: $\mathcal{O}(\alpha_{bos}^2) \sim \Gamma_Z\alpha^2 \sim 0.1 \text{ MeV}$

$$\mathcal{O}(\alpha\alpha_s^2) - \mathcal{O}(\alpha_t\alpha_s^2) \sim \frac{\alpha n_{lq}}{\pi}\alpha_s^2 \sim 0.29 \text{ MeV}$$

Total: $\delta\Gamma_Z \approx 0.5$ MeV