HOM impedance and power requirements for ILC Crab Cavity

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Impedance and power requirements for the ILC Crab Cavity are determined by the beam offset with respect to the considered mode electrical axis. Note that this offset may be different for different modes because of the cavity field perturbations – due to e.g., a fundamental power coupler or HOM couplers – and manufacturing errors.

I. HOM impedance limitation due to resonance excitation

The crab cavity HOM impedance should be small enough to avoid single bunch effects such as

1. Distortion of the crabbing voltage along the bunch
2. Emittance dilution

In addition to the single-bunch effects, one may have resonance excitation of an HOM mode by the beam. The cavity resonance excitation is determined by the cavity HOM spectrum and the beam spectrum. Resonance excitation provides the maximal voltage which changes the beam particle transverse momentum $p_\perp$:

$$\frac{\Delta p_\perp c}{e} = \boldsymbol{U}_{kick} = \frac{\omega_0^2}{\omega^2-\omega_0^2-i\frac{\omega\omega_0}{Q}} \times k_0 x_0 I_0 \left(\frac{r_\perp}{Q}\right), \qquad (1)$$

where x_0 is the beam offset with respect to the HOM electric axis, ω_0 is the HOM frequency, ω is the beam spectrum line frequency, $k_0 = \omega_0/c$ is the wavenumber, I_0 is the average beam current, and $\left(\frac{r_\perp}{Q}\right)$ is the HOM transverse impedance in Ohm; Q is the loaded quality factor. The transverse impedance (for example, horizontal) is defined as

$$\left(\frac{r_\perp}{Q}\right) \equiv \frac{\left|\int_{-\infty}^{\infty}\left(\frac{\partial E_z(x,0,z)}{\partial x}\right)_{x=0} e^{i\omega z/c} dz\right|^2}{W k_0^2 \omega_0} \equiv \frac{U_{kick}^2}{W\omega_0} \qquad (2)$$

Here W is the energy stored in the cavity, $E_z(x,y,z)$ is the longitudinal electric field of the HOM. On resonance, the kick voltage amplitude $U_{kick} = |\boldsymbol{U}_{kick}|$ is

$$U_{kick} = k_0 x_0 I_0 Q \left(\frac{r_\perp}{Q}\right) = k_0 x_0 I_0 r_\perp, \qquad (3)$$

where $r_\perp = Q\left(\frac{r_\perp}{Q}\right)$ is the HOM shunt impedance, and Q is the HOM loaded quality factor.

In pulsed regime,

$$U_{kick} = k_0 x_0 I_p r_\perp \frac{1-e^{-t_p/\tau}}{1-e^{-1/(f_r\tau)}}, \qquad (4)$$

here I_p is the pulse beam current, t_p is the pulse width, f_r is the pulse repetition rate, and $\tau = \frac{2Q}{\omega_0}$ is the time constant. If Q is very large, and $f_r\tau >> 1$, the formula may be simplified; in this case the kick induced by the beam is determined by the average current as in CW regime (see Eq. (3)). If the time constant is much smaller than the pulse width, i.e., $\frac{t_p}{\tau} \gg 1$, the kick induced by the beam is determined by pulsed current

$$U_{kick} = k_0 x_0 I_p r_\perp. \tag{5}$$

It is the most pessimistic case, which will be used for further estimations.

1. To avoid distortion of the crabbing voltage, horizontal kick U_{kick} caused by HOM should be much smaller than the crabbing voltage U_0. In the most pessimistic case of resonance between the HOM and the beam spectrum (or when $\omega_0 = \omega$), one gets the requirement for the kick voltage amplitude

   $$U_{kick}\sigma_z k_0 = k_0^2 \sigma_z x_0 I_p r_\perp \ll U_0\, \sigma_z \omega_{RF}/c, \tag{6}$$

   and

   $$r_\perp \ll \frac{U_0\, \omega_{RF}/c}{k_0^2 x_0 I_p}. \tag{7}$$

   Here ω_{RF} is the RF frequency.

2. To avoid emittance dilution, the transverse kick spread along the bunch caused by an HOM should be much smaller than the transverse momentum spread $\sigma_{p_\perp} c/e$, or

   $$U_{kick}\, \sigma_z k_0 = k_0^2 x_0 \sigma_z I_p r_\perp \ll \frac{\sigma_{p_\perp} c}{e}, \tag{8}$$

   because in resonance the kick is shifted versus the beam current by 90° - see (1). On the other hand, one has

   $$\frac{\sigma_{p_\perp} c}{e} = \frac{p_{||} c}{e} \sqrt{\frac{\varepsilon}{\gamma\beta}} = U \sqrt{\frac{\varepsilon}{\gamma\beta}}, \tag{9}$$

   where U is the beam energy, ε is the normalized transverse emittance, γ is the relativistic factor and β is the beta-function corresponding to the cavity position. Therefore, for both horizontal and vertical transverse shunt impedance one needs

   $$r_\perp \ll \frac{U}{k_0^2 x_0 \sigma_z I_p} \sqrt{\frac{\varepsilon}{\gamma\beta}}. \tag{10}$$

   Here $r_\perp$ is the horizontal or vertical transverse shunt impedance, x_0 is the horizontal or vertical transverse offset, and ε is the horizontal or vertical transverse normalized emittance.

   Note that if HOM frequency is far of the harmonic of operating frequency, the kick is the same along the bunch, and can be compensated by correctors.

II. *Required RF power*

RF power necessary to maintain the crabbing voltage should compensate the ohmic losses in the cavity – negligible for SC cavities – and compensate the voltage induced by the beam if the beam has an offset with respect to the electric axis of the cavity. Note that the kick voltage induced by the beam may be in phase or out of phase with the crabbing voltage depending on the sign of the offset. The maximal required RF power P for the cavity detuned from the resonance frequency by $\Delta\omega$ is (see Appendix III):

$$P = \frac{U_0^2}{4Q\left(\frac{r_\perp}{Q}\right)}\left[\left(1 + \frac{I_p Q\left(\frac{r_\perp}{Q}\right)k_0 x_0}{U_0}\right)^2 + \left(\frac{2Q\Delta\omega}{\omega_0}\right)^2\right]. \qquad (11)$$

Here $\omega_0 = \omega_{RF}$ is the RF frequency. The optimal external Q corresponding to minimal power is

$$Q_{opt} = \left[\left(\frac{1}{Q_0} + \frac{I_p\left(\frac{r_\perp}{Q}\right)k_0 x_0}{U_0}\right)^2 + \left(\frac{2\Delta\omega}{\omega_0}\right)^2\right]^{-1/2}, \qquad (12)$$

where Q_0 is the cavity unloaded quality factor. For the SRF cavity one can use simplified estimation for the loaded Q:

$$Q \approx \left[\left(\frac{I_p\left(\frac{r_\perp}{Q}\right)k_0 x_0}{U_0}\right)^2 + \left(\frac{2\Delta\omega}{\omega_0}\right)^2\right]^{-1/2}. \qquad (13)$$

If the beam offset is zero, the power is determined by the maximal cavity detuning $\Delta\omega = 2\pi\Delta f$:

$$P = \frac{U_0^2 2\Delta\omega/\omega_0}{\left(\frac{r_\perp}{Q}\right)} = 2\frac{U_0^2\Delta f/f}{\left(\frac{r_\perp}{Q}\right)}. \qquad (14)$$

The horizontal beam offset does not influence the required power if

$$x_0 \ll \frac{2\Delta\omega}{\omega_0}\cdot\frac{U_0}{I_p k_0\left(\frac{r_\perp}{Q}\right)}. \qquad (15)$$

In the opposite case, if

$$x_0 \gg \frac{2\Delta\omega}{\omega_0}\cdot\frac{U_0}{I_p k_0\left(\frac{r_\perp}{Q}\right)} \qquad (16)$$

the required power is determined by the pulsed beam current,

$$P = U_0 I_p k_0 x_0. \qquad (17)$$

Anyway, the required power is determined by the cavity, and is not the fundamental requirement.

III. *Single-bunch effects*

If the bunch has very high population, the kick caused by the bunch transverse *horizontal* wake potential may alter the crabbing kick voltage. This gives us limitation for the transverse kick-factor:

$$k_\perp \ll \frac{U_0 \sigma_z \omega_{RF}/c}{q x_0}, \quad (18)$$

where q is the bunch charge and x_0 is the beam *horizontal* offset. The transverse *vertical* wake potential should not increase the bunch emittance:

$$k_\perp \ll \frac{U}{q y_0}\sqrt{\frac{\varepsilon}{\gamma\beta}}. \quad (19)$$

Here y_0 is the beam *vertic*al offset.

IV. Cavity detuning for operation without crabbing

If the cavity is not in operation, it should be detuned properly such a way that the voltage induced by the beam does not affect the beam emittance. For detuned cavity the kick caused by the induced field is about the same for all the particles of a short bunch and is equal to the kick amplitude U_{kick}, see (1). It should be much smaller than the transverse momentum spread $\sigma_{p_\perp} c/e$. If the cavity detune $|\Delta f|$is high compared to the bandwidth, the transverse kick is equal to

$$U_{kick} \approx \frac{1}{2m} \times k_0 x_0 I_0 \left(\frac{r_\perp}{Q}\right) Q, \quad (20)$$

where$\left(\frac{r_\perp}{Q}\right)$ and Q are transverse impedance and loaded quality factor of the operating mode, k_0 is the wave number corresponding to the operation mode frequency, i.e., $k_0 = 2\pi f_{RF}/c$ and m is ratio of the cavity detune to the bandwidth, i.e.,

$$m = \frac{|f_{RF} - f|}{f_{RF}} \times Q \equiv \frac{|\Delta f|}{f_{RF}} \times Q. \quad (21)$$

Therefore, taking into account (9), one has

$$\frac{1}{2m} \times k_0 x_0 I_0 \left(\frac{r_\perp}{Q}\right) Q \ll U\sqrt{\frac{\varepsilon_x}{\gamma\beta_x}}, \quad (22)$$

or

$$m \gg \frac{\pi f_{RF} x_0 I_0 \left(\frac{r_\perp}{Q}\right) Q}{cU\sqrt{\frac{\varepsilon_x}{\gamma\beta_x}}} \; . \quad (23)$$

V. Requirements for ILC

For the current version of ILC collider we have

Beam energy	U = 250 GeV and $\gamma = 5\times10^5$
Pulsed beam current	I_p = 5.8 mA

Pulse width	t_p = 727 µs
Repetition rate	f_r = 5 Hz
Average beam current	I_{av} = 20 µA
Vertical β function at the cavity position	β_y = 15.4 km
Horizontal β function at the cavity position	β_x = 23.2 km
Bunch charge	q = 3.2 nC
Crab cavity kick voltage	U_0= 0.92 MV (2.6 GHz)
The bunch length	σ_z = 300 µm
Normalized vertical emittance	ε_y= 35 nm
Normalized horizontal emittance	ε_x= 10 µm

Suppose the HOM electric axis offset with respect to the beam is $x_0 = y_0 =$1 mm.

1. In this case, to avoid distortion of the crab voltage kick distribution along the bunch, one has for the *horizontal* shunt impedance of the "most dangerous mode":

$$r_\perp \ll \frac{U_0 \frac{\omega_{RF}}{c}}{k_0^2 x_0 I_p} = \frac{U_0 c f_{RF}}{2\pi f_{HOM}^2 x_0 I_p} \quad \text{and} \tag{24}$$

$$r_\perp f_{HOM}^2 \ll \frac{U_0 c f_{RF}}{2\pi x_0 I_p} = 19 \text{ GOhm·GHz}^2. \tag{25}$$

2. To exclude the HOM influence on the *vertical* emittance, one should have

$$r_\perp \ll \frac{U}{k_0^2 y_0 \sigma_z I_p}\sqrt{\frac{\varepsilon_y}{\gamma\beta_y}} \quad \text{and}$$

$$r_\perp f_{HOM}^2 \ll \frac{Uc^2}{(2\pi)^2 \sigma_z y_0 I_p}\sqrt{\frac{\varepsilon_y}{\gamma\beta_y}} = 0.7 \text{ GOhm·GHz}^2. \tag{26, 27}$$

(26)

(27)

3. To exclude the HOM influence on the *horizontal* emittance, one should have

$$r_\perp f_{HOM}^2 \ll \frac{Uc^2}{(2\pi)^2 \sigma_z x_0 I_p}\sqrt{\frac{\varepsilon_x}{\gamma\beta_x}} = 9.6 \text{ GOhm·GHz}^2 \tag{28}$$

One can see from (21) and (24) that the value of $r_\perp f_{HOM}^2$ is limited by the HOM influence on the *horizontal* emittance.

4. The horizontal kick factor necessary to avoid the crabbing voltage distortion should be

$$k_\perp \ll \frac{U_0 \sigma_z \omega_{RF}/c}{q x_0} = 4.6\times10^3 \text{ V/pC/m} \tag{29}$$

5. The kick factor necessary to avoid *horizontal* emittance dilution should be

$$k_\perp \ll \frac{U}{q x_0}\sqrt{\frac{\varepsilon_x}{\gamma\beta}} = 2.3\times10^3 \text{ V/pC/m} \tag{30}$$

6. The kick factor necessary to avoid *vertical* emittance dilution should be

$$k_\perp \ll \frac{U}{q y_0}\sqrt{\frac{\varepsilon_y}{\gamma\beta}} = 1.7\times10^{2}\ \text{V/pC/m} \tag{31}$$

Typically for the cavities operating at 1-5 GHz the kick factor has the order of < 100 V/pC/m. **It means that single-bunch effects are not a problem.**

VI. Example – QMIR

For QMIR cavity scaled from 2.8 GHz to 2.6 GHz one has:

Operation mode	$\left(\frac{r_\perp}{Q}\right)$ = 1040 Ohm (2.6 GHz)
Maximal dipole *horizontal* HOM	$\left(\frac{r_\perp}{Q}\right)$ = 10 Ohm (2.5 GHz); Q < 1×10^{5}.
Maximal dipole *vertical* HOM	$\left(\frac{r_\perp}{Q}\right)$ = 10 Ohm (4 GHz); Q < 1×10^{4}.
Horizontal kick factor	$k_\perp$ = 400 V/pC/m – no problem
Vertical kick factor	$k_\perp$ = 100 V/pC/m – no problem

1. We have requirement for *horizontal* shunt impedance:

$$r_\perp f_{HOM}^2 \ll 9.6\ \text{GOhm·GHz}^2$$

or $r_\perp$ << 1.5 GOhm. It means that for this mode Q << 1.5×10^{8}. QMIR has $Q < 1\times10^{5}$.

Ergo: a QMIR cavity well satisfies the horizontal HOM impedance requirement.

2. We have requirement for *vertical* shunt impedance:

$$r_\perp f_{HOM}^2 \ll 0.7\ \text{GOhm·GHz}^2$$

or $r_\perp$ << 600 MOhm. It means that for this mode Q << 4.4×10^{6}. QMIR has $Q < 1\times10^{4}$.

Ergo: a QMIR cavity well satisfies the vertical HOM impedance requirement.

3. Power requirements

Suppose that the maximal cavity detune is Δf = 1000 Hz (LFD, microphonics). In this case the required power will be determined by this detune if the beam offset is

$$x_0 < \frac{2\Delta\omega}{\omega_0}\cdot\frac{U_0}{I_p k_0\left(\frac{r_\perp}{Q}\right)} = 2\ \text{mm}. \tag{32}$$

It means that in this limit, e.g., when the beam offset is 1 mm, the cavity required power is practically independent of the beam current, and the average power is

$$P = 2\frac{U_0^2\Delta f/f}{\left(\frac{r_\perp}{Q}\right)} = 0.61\ \text{kW + overhead of 100\% = 1.2 kW}. \tag{33}$$

The average power is 43 W (as the duty factor is 0.36%).

The cavity loaded Q is

$$Q = 1.3\times 10^{6}, \tag{34}$$

and the cavity bandwidth δf is

$$\delta f = 2 \text{ kHz} \tag{35}$$

4. The cavity detune for operation without crabbing
According to (23) and (24), for QMIR cavity one has

$$m \gg \frac{\pi f_{RF} x_0 I_0 \left(\frac{r_\perp}{Q}\right) Q}{cU\sqrt{\frac{\varepsilon_x}{\gamma\beta_x}}} = 29 \tag{36}$$

and, therefore,

$$|f_{RF} - f| >> 58 \text{ kHz} \tag{37}$$

Therefore, in order to put the cavity out of operation, one should detune the cavity by >> 58 kHz. ~200 kHz should be OK. It gives the requirement for the tuner for QMIR cavity.

VII. Summary

From physics point of view, it makes sense to put the following parameters into specification for the ILC crab cavity:

1. *Horizontal* kick voltage $U_0 f_{RF} = 2.4$ MV·GHz
2. Requirement for *horizontal* HOM impedance $r_\perp f_{HOM}^2 \ll \frac{Uc^2}{(2\pi)^2 \sigma_z x_0 I_p}\sqrt{\frac{\varepsilon_x}{\gamma\beta_x}}$ = 9.6 GOhm·GHz^2
3. Requirement for *vertical* HOM impedance $r_\perp f_{HOM}^2 \ll \frac{Uc^2}{(2\pi)^2 \sigma_z y_0 I_p}\sqrt{\frac{\varepsilon_y}{\gamma\beta_y}}$ =0.7 GOhm·GHz^2.
4. The cavity kick factors – vertical and horizontal for ILC are not critical
5. Input pulse RF power depends on the cavity design, may be specified < 2 kW pulsed
6. Beam vertical and horizontal ffset with respect to the cavity axis < 1 mm
7. HOM electric axis offset with respect to the cavity axis < 1 mm

Appendix I

According to Wilson's theorem for a dipole mode (see Appendix II), the kick voltage U_{kick} induced by a short bunch having charge q having offset x_0 is

$$\boldsymbol{U}_{kick} = \frac{i}{2} cqk_0^2 x_0 \left(\frac{r_\perp}{Q}\right) = \frac{i}{2} q\omega_0 k_0 x_0 \left(\frac{r_\perp}{Q}\right) \tag{AI.1}$$

For a bunch train containing the similar bunch having the same offset one has on resonance, i.e., when the bunch separation T is multiple of the RF period:

$$\boldsymbol{U}_{kick} = \frac{i}{2} cqk_0^2 x_0 \left(\frac{r_\perp}{Q}\right) = \frac{i}{2} q\omega_0 k_0 x_0 \left(\frac{r_\perp}{Q}\right) \sum_{j=0}^{n} e^{-jT/\tau} = \frac{i}{2} q\omega_0 k_0 x_0 \left(\frac{r_\perp}{Q}\right) \frac{1-e^{-Tn/\tau}}{1-e^{-T/\tau}}, \tag{AI.2}$$

where $\tau = \frac{2Q}{\omega}$ is the time constant. Considering that the pulse width $t_p = nT$ and that $T \ll \tau$ one has for the kick amplitude $U_{kick} = |\boldsymbol{U}_{kick}|$:

$$U_{kick} = k_0 x_0 r_\perp I_p \left(1 - e^{-t_p/\tau}\right), \tag{AI.3}$$

where $I_p = q/T$ is the pulsed beam current.

For periodic pulses one has

$$U_{kick} = k_0 x_0 r_\perp I_p \left(1 - e^{-t_p/\tau}\right) \sum_{j=0}^{\infty} e^{-j/(f_r \tau)} = k_0 x_0 r_\perp I_p \frac{1-e^{-t_p/\tau}}{1-e^{-1/(f_r\tau)}}, \tag{AI.4}$$

where f_r is the pulse repetition rate.

Appendix II

Let us consider the dipole mode excitation by a single short ultra-relativistic bunch. According to the Wilson's theorem, a short bunch "sees" a half of the voltage it excites in a cavity. Thus, from the energy conservation one can find that

$$\frac{1}{2} Uq = W, \tag{A.II.1}$$

where q is the bunch charge, U is the voltage induced in the cavity, and W is the RF energy stored in the cavity after the bunch left it. Consider the bunch moving parallel to the cavity axis with an offset x_0. In this case,

$$U = A\left|\int_{-\infty}^{\infty} \mathrm{E}_z(x_0, 0, z) e^{ik_0 z} dz\right| \tag{A.II.2}$$

and

$$W = \frac{\varepsilon_0}{2} \int \left|\vec{\mathrm{E}}\right|^2 dV. \tag{A.II.3}$$

A is the eigenmode amplitude. From (A.II.1-A.II.3) one finds

$$\frac{1}{A} = q \frac{\left|\int_{-\infty}^{\infty} \mathrm{E}_z(x_0,0,z) e^{ik_0 z} dz\right|}{\varepsilon_0 \int \left|\vec{\mathrm{E}}\right|^2 dV} \tag{AII.4}$$

and the voltage along the arbitrary line parallel to the axis and having offset x is

$$U(x,0,z) = q\frac{\left|\int_{-\infty}^{\infty} \mathrm{E}_z(x,0,z)e^{ik_0 z}dz\right|\left|\int_{-\infty}^{\infty} \mathrm{E}_z(x_0,0,z)e^{ik_0 z}dz\right|}{\varepsilon_0 \int\left|\vec{\mathrm{E}}\right|^2 dV} \approx$$

$$\approx \frac{qxx_0\left|\int_{-\infty}^{\infty}\left(\frac{\partial \mathrm{E}_z(x,0,z)}{\partial x}\right)_{x=0} e^{ik_0 z}dz\right|^2}{\varepsilon_0 \int\left|\vec{\mathrm{E}}\right|^2 dV} = \frac{1}{2}xx_0\omega_0\left(\frac{r_{||}}{Q}\right)_1, \qquad \text{(AII.5)}$$

because $\frac{\partial \mathrm{E}_z(x,0,z)}{\partial x}$ in paraxial area does not depend on x for a dipole mode. From the Panofsky-Wenzel theorem one has

$$\boldsymbol{U}_{kick} = \frac{ic}{\omega_0}\frac{\partial U}{\partial x} = \frac{i}{2}cqx_0\left(\frac{r_{||}}{Q}\right)_1. \qquad \text{(AII.6)}$$

Considering that

$$\left(\frac{r_\perp}{Q}\right)_1 \equiv \left(\frac{r_{||}}{Q}\right)_1 \times \frac{1}{(k_0)^2} \qquad \text{(AII.7)}$$

one has

$$\boldsymbol{U}_{kick} = \frac{i}{2}cqk_0^2x_0\left(\frac{r_\perp}{Q}\right) = \frac{i}{2}q\omega_0 k_0 x_0\left(\frac{r_\perp}{Q}\right). \qquad \text{(AII.8)}$$

Appendix III

Let us estimate the RF power necessary to maintain the kick amplitude U_0 in a crab cavity with an operating dipole mode loaded by the beam having offset x_0 with respect to the operating mode electric axis. If the beam current is I_p, the input power P is

$$P = \frac{V_c^2}{4Q\left(\frac{R}{Q}\right)}\frac{1+\beta}{\beta}\left[\left(1+\frac{I_p\left(\frac{R}{Q}\right)Q}{V_c}\right)^2 + \left(2Q\frac{\Delta\omega}{\omega_{RF}}\right)^2\right], \qquad \text{(AIII.1)}$$

where V_c is the energy gain of the beam in eV, β is the coupling, Q is the loaded quality factor, $\frac{R}{Q}$ is the cavity impedance along the line parallel to the axis having offset x_0, $\frac{R}{Q} = \frac{V_c^2}{\omega_{RF}W}$, W is the stored energy, ω_{RF} is it's resonant frequency, $\Delta\omega = \omega_{RF} - \omega$, ω is the harmonic of the bunch repetition frequency close to the cavity resonance. Note that we consider the worst case, when V_c is in phase with the beam current, while the kick is shifted by 90°.

According to the Panofsky-Wenzel theorem, the kick voltage is

$$\boldsymbol{U}_0 = \frac{ic}{\omega_{RF}}\frac{\partial V_c}{\partial x} \approx \frac{ic}{\omega_{RF}}\frac{V_c}{x_0}, \qquad \text{(AIII.2)}$$

and the kick amplitude is

$$U_0 = \frac{c}{\omega_{RF}}\frac{V_c}{x_0} = \frac{V_c}{k_0 x_0}, \qquad \text{(AIII.3)}$$

where $k_0 = \omega_{RF}/\mathrm{c}$ is the wave number. Therefore,

$$V_c = U_0 k_0 x_0 \qquad \text{(AIII.4)}$$

On the other hand,

$$\frac{R}{Q} = \frac{V_c^2}{\omega_{RF}W} = \frac{U_0^2 (k_0 x_0)^2}{\omega_{RF}W} = \left(\frac{r_\perp}{Q}\right)(k_0 x_0)^2. \quad \text{(AIII.5)}$$

Substituting expressions for V_c and $\frac{R}{Q}$ to the formula for the power above, one has

$$P = \frac{U_0^2}{4Q\left(\frac{r_\perp}{Q}\right)} \frac{1+\beta}{\beta} \left[\left(1 + \frac{I_p Q\left(\frac{r_\perp}{Q}\right) k_0 x_0}{U_0} \right)^2 + \left(\frac{2Q\Delta\omega}{\omega_0} \right)^2 \right]. \quad \text{(AIII.6)}$$

The power P is minimal for the coupling

$$\beta_{opt} = \left[\left(1 + \frac{I_p Q_0\left(\frac{r_\perp}{Q}\right) k_0 x_0}{U_0} \right)^2 + \left(\frac{2Q_0\Delta\omega}{\omega_0} \right)^2 \right]^{1/2}, \quad \text{(AIII.7)}$$

Considering that unloaded quality factor $Q_0 = Q(1+\beta)$, one has for $\beta \gg 1$

$$P \approx \frac{U_0^2}{4Q\left(\frac{r_\perp}{Q}\right)} \left[\left(1 + \frac{I_p Q\left(\frac{r_\perp}{Q}\right) k_0 x_0}{U_0} \right)^2 + \left(\frac{2Q\Delta\omega}{\omega_0} \right)^2 \right] \quad \text{(AIII.8)}$$

and

$$Q_{opt} = \left[\left(\frac{1}{Q_0} + \frac{I_p \left(\frac{r_\perp}{Q}\right) k_0 x_0}{U_0} \right)^2 + \left(\frac{2\Delta\omega}{\omega_0} \right)^2 \right]^{-1/2}. \quad \text{(AIII.9)}$$